Favor-trading with Incomplete Information: Symmetric Strategies

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(Working paper)

Abstract

We investigate whether and how individuals who stand to gain from favor-trading can best form cooperative relationships in an environment with private information about each agent's ability and willingness to do favors. For agents with a low discount factor (low types) cooperation is not incentive compatible, for agents with a high discount factor (high types) it is. Both types receive privately observed opportunities to do favors with positive probability each period. We show high types are always able to separate from low types. We focus on symmetric strategies for separation. Separation is implementable as soon as a high type receives a favor opportunity if the opportunities are independent across agents. If they are mutually exclusive, separation is only possible during a finite number of periods.

Key words: Favor-trading, incomplete information, signaling, separation, cooperation
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1 Introduction

This paper and its companion paper, Kalla [7], study whether and how individuals who stand to gain from trading favors can best form cooperative relationships in an environment with private information about each agent's ability and willingness to do favors. Previous models in the favor-trading literature focused on optimizing favor-trading relationships under complete information. This paper introduces incomplete information about player types. The central question addressed is whether cooperation can be maintained in favor-trading relationships after the introduction of non-cooperative players into the pool of potential trading partners, and if so how can the cooperative players separate themselves most efficiently from the non-cooperative types.

For the purposes of this paper favor-trading is considered to be non-monetary trade in goods, services or opportunities, and favors are assumed to be efficient. The model contains a positive measure of players with a low discount factor (*low types*) who do not find cooperation beneficial, and a positive measure of players with a high discount factor (*high types*) who do. Players receive opportunities to do favors for each other (*favor opportunities*) according to either a mutually exclusive or independent distribution, but these opportunities are private information.

As an example, consider a firm with several parallel divisions that function independently under separate managers. Suppose two new managers have been recruited to head the marketing and finance divisions, respectively. Each manager's job is to maximize productivity within her own division, but occasionally one of the managers receives a new idea or opportunity that would be beneficial for her division but even more beneficial for the neighboring division. Monetary side payments are not allowed, but reciprocation in similar favors can provide a basis for mutual gains if both managers are sufficiently patient. However, the managers do not know each other's discount factor, which in this example could be interpreted as the likelihood of staying with the firm long-term. So how should the managers proceed?

The main result in this paper is that the high type players are always able to separate themselves from low type players at the first available opportunity by using an *equality matching (EM)* mechanism if opportunities to do favors arrive independently. EM simply means that each agent waits for reciprocation for a previous favor before doing the next one. In the case of mutually exclusive favor opportunities, separation is still guaranteed for high types with probability one if one of the players is designated to do the first favor, and separation will occur as soon as the designated player receives a favor opportunity, assuming she is a high type. The designated player case is analyzed in Kalla [7].

However, strategies that rely on designating a first favor maker induce separation only half as quickly (roughly speaking) as symmetric strategies that call for the first player with opportunity to do the first favor thereby signalling her high type. In this paper we analyze symmetric strategies of this type. In equilibria based on such strategies, separation is guaranteed with probability one if favor opportunities are independent of each other, but not if they are mutually exclusive in the sense that only one, or neither, of the players may receive a favor opportunity in any single period. The paper establishes a bound on the number of periods in the mutually exclusive favor opportunity case during which the low types will never mimic the high types. An important consequence of this result is that more information (mutually exclusive favor opportunities) leads to a worse outcome.

The rest of the paper is organized as follows: Section 2 describes the model and our equilibrium concepts including how key concepts from Abdulkadiroğlu and Bagwell [1] (*AB* for short) translate to our streamlined version of their model and how other favor-trading literature relates to our model. In sections 3 and 4 we cover the case of two-sided incomplete information, that is, neither agents knows the other's type. Section 3 analyzes *symmetric separating (SS)* strategies when opportunities to do favors are mutually exclusive. Section 4 extends the analysis of SS strategies to the case of independent favor opportunities, and contrasts the results to the case of mutually exclusive opportunities. Section 5 concludes. An appendix follows and references are at the end.

2 The Model

Consider the earlier motivating example: A firm has several parallel divisions that function independently under separate managers. Suppose two new managers have been recruited to head the marketing and finance divisions, respectively. Each manager's job is to maximize productivity within her own division, but occasionally one of them receives an opportunity to help the other division at a cost to her own. The ability or opportunity to help is private information, but when possible the cost is known to be less than the benefit. Monetary side payments are not allowed, but reciprocation in similar favors can provide a basis for mutual gains if both managers are sufficiently patient. But the managers do not know how patient the other is, or how likely she is to stay with the firm long-term. To address whether and how they can form a cooperative relationship we analyze the following formal model.

Two agents, a and b, are randomly picked from a population with $\mu_o \in (0,1)$ of high types with discount factor δ^H and $1-\mu_o$ of low types with discount factor δ^L. Each agent has utility function $u(x) = x$. They play an infinitely repeated stage game with the following structure. At the beginning of each period nature allocates an opportunity to do a favor (*favor opportunity*) according to either a mutually exclusive or an independent distribution. Under a mutually exclusive distribution either agent a or b receives a favor opportunity with equal probability, $p \in (0, 1/2)$, or neither does with probability $1-2p$. Under the independent distribution each agent receives a favor opportunity with probability $p \in (0,1)$. Favor opportunities are private information. An agent who receives a favor opportunity may either keep it private and incur no cost, or do a full or partial favor of size $x, y \in (0,1]$, for agents a and b, respectively, at a cost equal to the favor size. The benefit to the recipient is ky or kx, for agents a and b, respectively, where $k > 1$. For example, if agent a does a favor of size x, flow payoffs to (a,b) are $(1-x, kx)$. Favors, including their size, are public information. The stage game is repeated in each subsequent period.

To see how favor-trading works consider the following game called *equality matching (EM)*. In EM of level $z \in (0,1]$, one agent is called *advantaged*, the other *disadvantaged*. The disadvantaged agent is said to owe the advantaged agent a favor of size z. If the disadvantaged agent does a favor of size z, she becomes advantaged and the other disadvantaged. If she does no favor, she remains disadvantaged. Favors of size other than z are not part of equilibrium play and are deterred by the threat of Nash reversion. When $z = 1$, the game is called *full equality matching*.

For the moment, consider a game of full equality matching between two high types in a complete information environment with mutually exclusive favor opportunities. Suppose agent a is disadvantaged, b advantaged. Let $(\underline{u}_{em}, \bar{u}_{em})$ denote the average discounted payoffs expected by agents (a, b), or more generally by disadvantaged and advantaged agents, respectively. Let $\sigma_{em}(\underline{u}_{em}, \bar{u}_{em}) = (\sigma^a_{em}(\underline{u}_{em}, \bar{u}_{em}), \sigma^b_{em}(\underline{u}_{em}, \bar{u}_{em}))$ denote the EM strategy profile that implements the payoff pair $(\underline{u}_{em}, \bar{u}_{em})$. Under σ_{em} the payoffs are

$$\underline{u}_{em} = p\delta^H \bar{u}_{em} + (1-p)\delta^H \underline{u}_{em}, \tag{1}$$
$$\bar{u}_{em} = p\left(1 - \delta^H + \delta^H \bar{u}_{em}\right) + p\left((1-\delta^H)k + \delta^H \underline{u}_{em}\right) + (1-2p)\delta^H \bar{u}_{em}. \tag{2}$$

The first equation consists of two events: (i) with probability p agent a receives a favor opportunity, does a full favor $(x = 1)$, and becomes the advantaged agent; that is, agent a is promised continuation payoff $\bar{u}_{em}$, (ii) with probability $(1-p)$ agent a receives no favor opportunity so her flow payoff is zero and her continuation promise remains $\underline{u}_{em}$ along with her disadvantaged status. The equation for payoff $\bar{u}_{em}$ consists of three possible events that occur with probabilities p, p and $(1-2p)$, respectively: (i) agent b receives a favor opportunity, does no favor and thereby receives a flow payoff of 1 and her continuation payoff remains $\bar{u}_{em}$ as she is still advantaged, (ii) agent a receives a favor opportunity, does a full favor $(x = 1)$ so agent b receives a flow payoff of k, but her continuation payoff drops to $\underline{u}_{em}$ because she now owes agent a the next favor, and (iii) neither agent receives a favor opportunity, so agent b's flow payoff is zero and her continuation payoff remains $\bar{u}_{em}$. The two previous equations contain two unknowns, $\underline{u}_{em}$ and $\bar{u}_{em}$, solving for these yields

$$\underline{u}_{em} = \frac{\delta^H p^2 (1+k)}{1 - \delta^H(1-2p)}, \tag{3}$$
$$\bar{u}_{em} = \frac{p\left(1 - \delta^H(1-p)\right)(1+k)}{1 - \delta^H(1-2p)}. \tag{4}$$

For the simple EM strategy profile to be a *Nash equilibrium (NE)* in each stage game, neither agent can have a profitable deviation available to them. It is trivial that the advantaged agent has no profitable deviations as she just waits for reciprocation, but does no favors. Public (observable) off-equilibrium path deviations, such as the advantaged agent doing a favor or one of the agents doing the wrong size favor, are deterred by threat of autarky (no more favors). Therefore, we only need to check that it is not profitable for the disadvantaged agent to not do favors. As usual, it is enough to consider a one-shot deviation. Agent a's discount factor has to be high enough that the incentive compatibility constraint below is satisfied.

$$\begin{aligned} ICC^a_{em}: \quad & \delta^H \bar{u}_{em} \geq 1 - \delta^H + \delta^H \underline{u}_{em} \\ \iff \quad & \bar{u}_{em} - \underline{u}_{em} \geq \left(1-\delta^H\right)/\delta^H. \end{aligned}$$

Using equations (3) and (4), ICC^a_{em} may be written as

$$\begin{aligned} & \tfrac{p\left(1-\delta^H(1-p)\right)(1+k)}{1-\delta^H(1-2p)} - \tfrac{\delta^H p^2(1+k)}{1-\delta^H(1-2p)} - \tfrac{1-\delta^H}{\delta^H} \geq 0 \\ \iff & \tfrac{1-\delta^H}{\delta^H(1-\delta^H(1-2p))}\left(\delta^H p(k-1) - \left(1-\delta^H\right)\right) \geq 0. \end{aligned}$$

It follows that δ^H must satisfy $\delta^H p(k-1) - \left(1-\delta^H\right) \geq 0$ for ICC^a_{em} to hold, or rearranging terms: $\delta^H \geq \frac{1}{1+p(k-1)} \equiv \delta^*$. We use this boundary discount factor to define high and low type agents.

$$\textbf{Condition (5):} \qquad \delta^H \geq \delta^* := \frac{1}{1+p(k-1)} > \delta^L. \tag{5}$$

It is also easy to verify that $\underline{u}_{em} = p$ for $\delta^H = \delta^*$ so the individual rationality constraints of $\bar{u}_{em}, \underline{u}_{em} \geq p$ are satisfied. Therefore, this EM strategy profile is a Nash equilibrium. In fact, we could use the stronger equilibrium concept of *public perfect equilibrium (PPE)* following Fudenberg, Levine and Maskin [4]. A strategy for agent $i \in \{a, b\}$ is public if it depends only on her current period private information, in this case whether or not she received a favor opportunity, and the public history, which consists of favors up to and including the last period. A PPE is a profile of public strategies that form a Nash equilibrium for each period and the corresponding public history.

Since the payoff pair $(\underline{u}_{em}, \bar{u}_{em})$ is enforceable (implementable), it follows by symmetry that $(\bar{u}_{em}, \underline{u}_{em})$ is also enforceable, and therefore any utility pair on the line connecting $(\underline{u}_{em}, \bar{u}_{em})$ and $(\bar{u}_{em}, \underline{u}_{em})$ is enforceable with the use of a public randomization device. AB [1] aptly call these PPE with current and continuation payoffs restricted onto a symmetric line, *symmetric self-generating line (SSGL)* equilibria. The details of SSGL and the corresponding equilibria are covered in Kalla [7]. Presently it suffices to say that AB solve for the highest such line; the *highest symmetric self-generating line (HSSGL)* and they show that condition (5) is exactly the right bound necessary to implement HSSGL equilibria. In fact, the simple EM mechanism is a HSSGL equilibrium for $\delta^H = \delta^*$. When $\delta^H > \delta^*$, we may use the additional wiggle room to obtain a higher total payoff (and thus a higher SSGL) by requiring the advantaged agent to do further small favors while she waits for reciprocation from the disadvantaged agent.[1]

Observe that for the first-best outcome both agents would have to exhibit full trust in terms of x and y. AB [1] (p. 12) call $x + y$ the *level of trust*. Agent a (b) is said to exhibit *more trust* if $x > y$ ($y > x$). If both agents exhibit full trust every period regardless of history, neither agent has any incentive to do costly favors. Thus the first-best outcome cannot be achieved. However, on the HSSGL line, the level of trust is maximized subject to the restriction that continuation payoffs are picked from the same HSSGL.

It is perhaps natural to wonder if it is incentive compatible for low types to trade smaller favors, that is, to cooperate on a lower SSGL. It is not. Decreasing the size of favors and repeating the analysis for the EM mechanism shows that discount factors above or equal to δ^* are still necessary to sustain cooperation. Furthermore, cooperation on a lower line would be less efficient. The discount factor required to support EM equilibria is independent of favor size because agents have linear utility functions. They are effectively risk-neutral with respect to the size of favors. In terms of the mathematics, the lower cost of smaller favors is directly proportional by factor one to the resulting lower continuation payoffs. A formal proof will follow after we first introduce additional

[1] AB also constructed other types of equilibria that may lead to higher or lower total payoffs than HSSGL equilibria depending on the parameter values. However, we concentrate on HSSGL equilibria because they always exist if condition (5) is satisfied, and loosely speaking outperform other types of equilibria when p is not very high.

notation. For future reference, let $\sigma_{em(z)}$ denote the EM strategy profile when the size of exchanged favors is $z \in (0,1]$. Let $\bar{u}_{em(z)}$ and $\underline{u}_{em(z)}$ denote respective continuation payoffs for advantaged and disadvantaged agent. Unless otherwise noted, we use EM to refer to matching of full favors, or simply full EM.

2.1 Summary of notation and structure

The notation that follows is necessary to formally define the equilibrium profiles we will use in the forthcoming sections, but we present it in a format intended to be useful also for reference. Payoffs are in average discounted values.

Model parameters:	
$i, j \in \{a, b\}$:	Agents, where $j \neq i$.
$\omega^i \in \{L, H\}$:	Agent i's type; L = low, H = high.
$p \in (0,1)$:	Probability agent $i \in \{a, b\}$ receives a favor opportunity.
	Opportunities are either mutually exclusive or independent.
$k > 1$:	Benefit per unit of favor.
$\delta^i \in (0,1)$:	Discount factor of agent $i \in \{a, b\}$.
$\mu_o \in (0,1)$:	Fraction of high type agents in population.
Beliefs:	
$\mu_t^i \in [0,1]$:	Agent i's belief at time t that agent j is a high type
	(see definition 1 for details).
Actions:	
$x, y \in [0,1]$:	Size of favor by agents a, b, respectively.
Payoffs:	
(u, v) :	Current payoffs to agents (a, b).
(u_o, v_o) :	Continuation payoffs to (a, b) when no one does a favor.
(u_i, v_i) :	Continuation payoffs to (a, b) when $i \in \{a, b\}$ does a favor.

Table 1: Summary of notation

Information structure: Let $t = 1, 2, \ldots$ denote the time index. Let $w_t^i = 1$ if agent i receives a favor opportunity in period t and 0 otherwise. Agent i privately observes $W_t^i = \{w_z^i\}_{z=1}^t$. Let $\tau_t = (x, y)$ denote favors $(x, y) \in (0,1]^2$ agents a and b, respectively, do in period t. If neither agent does a favor, then let $\tau_t = 0$. Both agents observe $T_t = \{\tau_z\}_{z=1}^t$. Private history of agent i and public history up to and including period t are denoted by $h_t^i = W_t^i \in \mathcal{H}_t^i$ and $H_t = T_t \in \mathcal{H}_t$, respectively. A strategy for agent i, denoted by σ^i, consists of a favor making decision, I_t^i, for each period based on i's type, her private history up to period t, and public history up to period $t-1$. More formally, $I_t^i : \{H, L\} \times \mathcal{H}_t^i \times \mathcal{H}_{t-1} \to [0,1]$ s.t. $I_t^i = 0$ when $w_t^i = 0$.

Definition 1 $\mu \equiv \mu_t \equiv \left(\mu_t^a, \mu_t^b\right)$ *where* $\mu_0^i \equiv \mu_o$ *and* $\mu_t^i : \mathcal{H}_t^i \times \mathcal{H}_{t-1} \to [0,1]$ *represents agent* i*'s belief. That is,* μ_t^i *is the probability assigned by* i *to the event that the other agent is a high type based on* i*'s private history up to period* t *and public history up to period* $t-1$.

Sometimes we drop the time index for convenience $(\mu \equiv \mu_t)$. The domain of the belief function consists of agent i's private history up to the current period and the public history up to the last period because it refers to agent i's belief at a point in period t when i has observed her private signal (her favor opportunity is 0 or 1) but not the public signal (period t favor, if any, is still pending). That is, μ_t^i captures agent i's updated belief in period t at the point in time when she has either received a favor opportunity and is deciding whether or not to do a favor, or she has received no opportunity and is waiting to see if the other agent does her a favor. For later use we also define the indicator function.

Definition 2 (Indicator function) $\mathbf{1}_{\{\arg\}} = 1$ *if* $\arg$ *is true,* 0 *otherwise.*

2.2 Strategies and equilibrium concepts

For our solution concept we use *Perfect Bayesian equilibrium (PBE)*. PBE consists of a strategy profile $\left(\sigma = \left(\sigma^a, \sigma^b\right)\right)$ and a belief system $\left(\mu = \left(\mu^a, \mu^b\right)\right)$ such that σ is sequentially rational with respect to μ and μ is consistent with σ. That is, the strategies are optimal at every stage of the game given the beliefs, and the beliefs are updated according to Bayes' rule from equilibrium strategies and observed actions. We should, strictly speaking, also specify beliefs for off-equilibrium path actions, however, these actions are deterred by the threat of autarky play, which is always an equilibrium response, so it is understood that beliefs consistent with autarky exist and would be straightforward if burdensome to specify. Therefore we generally leave out off-equilibrium path beliefs from our belief functions. But this brings us to the following definitions.

Definition 3 (Autarky strategy) *Let* σ_{aut}^i *be such that* $I_t^i = 0, \forall t$.

Definition 4 (Public on-equilibrium path histories) *Let* $\mathcal{H}_t^*$ *be the set of all public on-equilibrium path histories up to and including period* t.

For example, in a full equality matching game, any history that includes a favor by the advantaged agent or a partial favor by the disadvantaged agent would not be in $\mathcal{H}_t^*$. However, histories that include only private deviations, that is, a disadvantaged agent does not do a favor when she has the opportunity, would still be in $\mathcal{H}_t^*$. Next, let us define EM formally.

Definition 5 (Equality matching (EM)) *An EM strategy at level* $z \in (0,1]$ *for agent* i*, denoted by* $\sigma_{em(z)}^i$ *or simply* σ_{em}^i *when* $z = 1$*, is such that*

$$I_t^i = \begin{cases} z & \text{if agent } i \text{ is disadvantaged, } w_t^i = 1 \text{ and } h_{t-1} \in \mathcal{H}_{t-1}^*, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 6 (Necessary and sufficient condition for EM) $\delta^H \geq \delta^* \equiv \frac{1}{1+p(k-1)}$ *is necessary and sufficient to implement* $\sigma_{em(z)}, \forall z \in (0,1]$ *in a complete information environment.*

Proof. In appendix. ■

While EM is generally not the most efficient way to trade favors, the first-best outcome is not enforceable, and in AB's model [1], the second-best outcome may be intractable. Presumably that is why AB focused on PPE restricted to symmetric lines rather than to the whole space of feasible and individually rational payoffs. While our primary interest is to implement separation efficiently, rather than to optimize subsequent endgames, we show in Kalla [7] that a if pair of high types can separate into an EM game, they can always move to a HSSGL equilibrium either immediately or after equality matching for a sufficient number of periods to deter the low types from pooling with the high types. In section 3.1 we also use a variation of the PBE defined below.

Definition 7 (ε-perfect separating PBE) *Suppose that (σ, μ) is a PBE profile for a favor-trading game with two-sided incomplete information such that σ implements separation of two high type agents with probability $(1 - \varepsilon) \in (0, 1)$. Call (σ, μ) an ε-perfect separating PBE.*

2.3 Related literature

Möbius [12] first investigated the type of two-player favor-trading games we study (2001), albeit with complete information and continuous time. He focused on an intuitive "chips mechanism." That is, each player begins with K chips, and each time an agent does a favor, she receives a chip from the other agent. If one agent accumulates all $2K$ chips, she suspends favors until reciprocation. EM is effectively a chips game with only one chip held by the advantaged agent. Hauser and Hopenhayn [6] continue Möbius' favor-trading research by allowing partial favors (first draft in 2004). Consequently, they let the cost of favors vary based on public history of favor exchanges, notably including time passed since the last exchange. They characterize a set of Pareto optimal PPE, and show numerically that partial favors lead to significant efficiency gains over Möbius' chips mechanism. Their findings display similar characteristics to HSSGL equilibria formulated by AB [1] in discrete time (first draft in 2004). Both Hauser and Hopenhayn and AB use PPE as their solution concept and allow partial favors. Both find equilibria that call for larger favors to be followed by unlimited smaller favors until reciprocation. This is in contrast to Möbius who assumed favors were all unit, and an agent would suspend favors whenever she was owed $2K$ favors.

Both Hauser and Hopenhayn and AB discover what the former call "debt forgiveness." That is, the value of favors owed declines; debt is forgiven, unless "interest" in the form of small favors is "paid" by the advantaged agent. Therefore it seems that favor-trading in a complete information environment is robust to the model's timing structure (continuous versus discrete) and the arrival process of favors (independent versus mutually exclusive). We show that the distribution of the arrival process of favor opportunities plays an important role in the effectiveness of symmetric strategies for separation.

A notable difference between Hauser and Hopenhayn [6] and AB [1] is that AB include opportunities for immediate reciprocity with private information. However, AB show that immediate reciprocity is unnecessary for HSSGL equilibria, which is why our streamlined version of their model does not include it. Furthermore, AB describe favor opportunities as income shocks, and favors as investments. We dropped this terminology because favor-trading precludes side payments, and we felt that using monetary language to discuss the topic confused the issue. We also normalized payoffs to average discounted values for convenience.

More recent favor-trading research includes essays by Nayyar [13] (2009) that provide a discrete time version of the Hauser and Hopenhayn [6] model, and further extend some of their analysis within this setup. A very preliminary paper by Lau [9] (2010) looks at favor-trading when benefits and costs are stochastic.

Outside of the favor-trading literature, Watson [18] studied the sustainability of cooperation using a two-player infinitely-repeated prisoner's dilemma model with incomplete information about agents' types. However, in his model deviations from cooperative behavior are publicly observable, whereas in favor-trading games only cooperative actions are observable, and deviations are private. Still, both models have the broad characteristic that agents start cooperation with small stakes, but form more profitable relationships over time if each agent proves her willingness to cooperate.

2.3.1 Companion paper

In Kalla [7] we analyze the case of a patient agent facing an unknown type. We show that for this case separation into an EM game is always possible as soon as the unknown type receives a favor opportunity (immediate separation), we construct bounds on the discount factors that show when immediate separation into more efficient equilibria, in particular HSSGL equilibria, is possible, and we prove that even when these bounds are violated, high types may still reach a HSSGL endgame if they separate and play an EM game for a sufficient number of periods first to discourage the low types from mimicking them. In terms of two-sided incomplete information, the companion paper analyzes how agents can form a cooperative favor-trading relationship using a strategy that designates one of them to do the first favor, or a *designated first favor maker (DFFM)* strategy. Kalla [7] also compares DFFM strategies to the SS strategies analyzed in the present paper and shows that either one may dominate depending on the parameter values.

3 Separation with mutually exclusive favor opportunities

When incomplete information is two-sided, that is, neither agent knows the other's type, a high type agent may attempt to separate herself from the low type by doing the first favor. Denote such a favor by z_t at period t. If the initial favor necessary to trigger cooperation between high types is sufficiently large, it will deter low types from mimicking high types, providing a way for high types to separate themselves. After an initial separation, we are back to the one-sided incomplete information case analyzed in Kalla [7]. The question is, can we find a sequence of sufficiently large initial favors to deter the low types from pooling with high types, but still low enough that high types would have incentive to do the favor and risk not receiving reciprocation if the other is a low type, instead of just waiting for the other agent to go first.

3.1 Symmetric separating (SS) equilibria

By symmetric separating equilibria we mean perfect Bayesian equilibria profiles, (σ, μ), such that, σ^a defines an equivalent strategy for agent a as σ^b does for agent b, and (σ, μ) implements separation with positive probability. We do not require separation with probability one even when both agents are high types. Indeed, we will later show that SS strategies are inherently limited in this regard *when* favor opportunities are correlated. Correlated signals are informative about the other agent's type, which can result in increasingly divergent beliefs between agents over time. Once beliefs diverge sufficiently, low types cannot be deterred from pooling with high types using symmetric strategies. However, we prove that SS equilibria that implement separation with positive probability always exist.

Consider the following example of a $(1-2p)$-perfect separating PBE. Given the opportunity, a high type agent does a (small) favor of size z_1 in the first period. If one of the agents does such a favor, she becomes advantaged in the game of full equality matching that follows provided the other agent is also a high type. If neither agent does a favor in the first period, both agents follow the autarky strategy from thereon. A low type agent follows the autarky strategy regardless. Separation occurs with probability $2p$ if both agents are high types, hence the name $(1-2p)$-perfect separating PBE. This profile is described formally below.

Lemma 8 (First period SS equilibria) *Consider strategy profile (σ, μ) defined by (8)-(7) for $i \in \{a, b\}$:*

$$\sigma^i := \begin{cases} I_1^i = z_1 \mathbf{1}_{\{w_1^i = 1\}} & \text{if } \omega^i = H \text{ and } t = 1, \\ \sigma^i_{em}\left(\bar{u}_{em}, \underline{u}_{em}\right) & \text{starting at } t = 2 \text{ if } \omega^i = H \text{ and } \tau_1 = (z_1, 0), \\ \sigma^i_{em}\left(\underline{u}_{em}, \bar{u}_{em}\right) & \text{starting at } t = 2 \text{ if } \omega^i = H \text{ and } \tau_1 = (0, z_1), \\ \sigma^i_{aut} & \text{otherwise,} \end{cases} \tag{6}$$

$$\mu_t^i := \begin{cases} 0 & \text{if } h_{t-1} \notin \mathcal{H}^*_{t-1}, \\ 1 & \text{if } h_{t-1} \in \mathcal{H}^*_{t-1} \text{ and either } i \text{ received } z_1 \text{ in first period or } i \text{ did the first favor and received full reciprocation,} \\ \mu^i_{t-1} & \text{if } h_{t-1} \in \mathcal{H}^*_{t-1} \text{ and either } i \text{ received a favor opportunity this period, or no favors were done in the first period, but } i \text{ did receive a favor opportunity that period (so } \mu_t^i = \mu_o, \forall t), \\ \frac{\mu_o(1-2p)}{1-(1+\mu_o)p} & \text{if } h_{t-1} \in \mathcal{H}^*_{t-1}, \text{ no favors were done and no opportunity received by } i \text{ in first period,} \\ \frac{\mu^i_{t-1}(1-2p)}{1-\left(1+\mu^i_{t-1}\right)p} & \text{otherwise.} \end{cases} \tag{7}$$

Then (σ, μ) is a $(1-2p)$-perfect separating PBE profile for $z_1 \in [\underline{z}_1, \bar{z}_1] \neq \emptyset$, where $\underline{z}_1$ and $\bar{z}_1$ are defined by (8).

$$\underline{z}_1 = \frac{\mu_o \delta^L pk}{1-\delta^L(1-p)} \text{ and } \bar{z}_1 = \min\left\{1, \frac{\mu_o \delta^H\left(\left(1-\delta^H\right)k + \delta^H p(k-1)\right)}{(1-\delta^H)(1-\delta^H(1-2p))}\right\}. \tag{8}$$

Proof. In appendix. ■

The one period case gives us a glimpse of how the two-sided incomplete information game behaves. Namely a small favor is necessary to initiate some form of cooperation. The special characteristic of the above equilibrium is that the agents are able to go to full equality matching if both are high types and one of the two initiates cooperation during the first period. Note that as the fraction of high types decreases so does the required size of the initial favor, $\underline{z}_1$, needed to initiate cooperation. This is because both agent types become more pessimistic as μ_o decreases. The high type will be less likely to find a cooperative partner, and the low type will be less likely to find a high type to reciprocate a small initial favor with a full favor. Therefore, it seems likely that in an infinite horizon problem the initial favor necessary to initiate cooperation would decrease as agents grow more pessimistic over time. The problem, however, is that both agents do not necessarily grow pessimistic at the same rate. For example, suppose one agent, say agent a, is a low type and the other, agent b, is a high type. Then it is possible agent a receives a string of favor opportunities at the beginning of the game but does no favors. Agent b would grow more pessimistic by each period, whereas agent a would not receive any information to update her beliefs. At some point the upper bound for the initial favor z_t derived from a high type's incentive compatibility constraint would cross the lower bound derived from the low type's incentive compatibility constraint for certain histories such as described above with positive probability.

But would it be possible to lower the continuation promise along with z_t to deter the low types? For example, if the EM endgame is conducted at level z_t, we know from lemma 6 that low types would never join in while the high types could benefit from it. However, lemma 6 applies in the complete information environment with two high types, one advantaged, the other disadvantaged right from the start of the game. Without preset designations it turns out that even the high type agents would require a higher continuation promise than just z_t to start a partnership. In particular, for a game of full equality matching, a high type agent would be at most willing to do an initial favor of $1/2$ for $\delta^H = \delta^*$ and never more than $\frac{k}{k+1}$ even as $\delta^H \to 1$. The proposition below presents the exact bounds for the general case.

Proposition 9 (Bounds with complete information) *Consider a game of equality matching at level $z_2 \in (0, 1]$ between two known high type agents that follows a favor of size z_1 by either agent. The agent to do the initial favor becomes the advantaged agent in the subsequent $em(z_2)$ game. This game can be implemented with any $z_1 \in \left(0, \frac{pk\delta^H}{1+\delta^H(p(k+1)-1)} z_2\right]$ for any $\delta^H \in [\delta^*, 1)$.*

Proof. In appendix. ■

If a low type is included in the game among a continuum of high types (that is $\mu_o \approx 1$), using $z_1 < z_2$ presents a clear problem since the original bound for δ^* was derived using $z_1 = z_2$. The ICC for a low type, say agent b, to follow the autarky strategy and not to pool with the high types when $z_1 < z_2$ is as follows:

$$\begin{aligned} ICC^L_{z_1<z_2}: \quad & 1 - \delta^L + \delta^L p \geq \left(1 - \delta^L\right)(1 - z_1) + \delta^L \left(\mu^b_t v_{z_2} + (1 - \mu_o)p\right) \\ \implies \quad & \left(1 - \delta^L\right) z_1 \geq \delta^L \left(v_{z_2} - p\right), \end{aligned}$$

because $\mu_t^b = 1$ since agent b is by assumption the only low type in a continuum of high types. The continuation payoff to b if she does the favor and if agent a is a high type is

$$\begin{aligned} v_{z_2} &= p\left(1-\delta^L+\delta^L v_{z_2}\right)+p\left(\left(1-\delta^L\right)kz_2+\delta^L p\right)+(1-2p)\delta^L v_{z_2} \\ &= p+\tfrac{pk\left(1-\delta^L\right)}{1-\delta^L(1-p)}z_2. \end{aligned}$$

Substituting from above for v_{z_2} into $ICC^L_{z_1<z_2}$ and simplifying yields

$$\begin{aligned} \left(1-\delta^L\right)z_1 &\geq \delta^L\left(p+\left(1-\delta^L\right)\tfrac{pk}{1-\delta^L(1-p)}z_2-p\right) \\ \iff z_1 &\geq \delta^L\frac{pk}{1-\delta^L(1-p)}z_2 \\ \iff \delta^L &\leq \frac{1}{1+p\left(k\left(z_2/z_1\right)-1\right)}. \end{aligned}$$

Observe that the bound on δ^L is lower than $\delta^* = 1/\left(1+p(k-1)\right)$ for $z_1 < z_2$, which identifies an obvious problem with symmetric separating equilibria. Namely, we need to reduce the size of the first favor or limit the time available for separation in order to achieve separation between high types, but this will increase the incentive for low types to mimic the high types unless δ^L is sufficiently low.

Proposition 10 (Time constrained separation in SS equilibria) *Suppose (σ,μ) is a PBE profile in a game with two unknown types, and σ is such that for $t \leq T \in \mathbb{N}\cup\{\infty\}$ the first high type agent to receive a favor opportunity will do a favor of size $z_t \in [\underline{z}_t, \bar{z}_t] \subset (0,1]$, where $[\underline{z}_t, \bar{z}_t]$ is the interval of incentive compatible first favors at time t. If agent i does the first favor of size z_t, (σ,μ) implements an EM strategy for high types, starting with i advantaged, at level $m(z_t)$, where $m:[0,1]\to[0,1]$ is an increasing function. Then,*

(i) $\exists\bar{n}\in\mathbb{N}$ such that (a) $[\underline{z}_t, \bar{z}_t] \neq \emptyset$ with probability 1 for all $t \leq \bar{n}$, and (b) $[\underline{z}_t, \bar{z}_t] = \emptyset$ with positive probability for all $t > \bar{n}$. (Separation between high types is always possible with positive probability, but never with probability 1 because it has to occur within a finite time period).

(ii) In equilibrium, $\bar{z}_t$ and $m(z_t)$ satisfy the following relationship,

$$\frac{m\left(\bar{z}_t\right)}{\bar{z}_t}-1 \leq \frac{1-\delta^L(1-p)}{\mu_o\delta^L pk} \equiv M \in \left(\frac{1}{\mu_o},\infty\right), \forall t \leq \bar{n}, \tag{9}$$

and separation has to take place within a finite period of time, otherwise a low type may pool with high types with positive probability.

Proof. The details are in the appendix. From lemma 8 we know that $\bar{n} \geq 1$, but we have to show that for t large enough there exist possible histories such that a low type would be willing to do a favor that is equal to or grater than the biggest favor a high type is willing to do. We use a basic proof by contradiction. Suppose a separating symmetric equilibrium exists satisfying (σ,μ) of proposition 10 for $\bar{n}=\infty$. Since neither type can have a profitable one-shot deviation for any history that occurs with positive probability, we proceed by deriving a tight greatest lower bound for z_t from the low type's incentive compatibility constraint. To do this, we use the "worst case" scenario that given t a low type has received favor opportunities each period so far. Consequently the low type's beliefs have remained unchanged at μ_o. This provides us with condition (9), and that is why μ_o is part of the condition. Note that the condition is defined for the upper bounds of z_t and $m(z_t)$ instead of z_t and $m(z_t)$ themselves. That is because we know their upper bounds must be decreasing as agents grow more pessimistic over time, whereas the actual sequence of z_t could behave quite erratically if σ is so specified and the incentive compatibility constraints allowed. We then derive a (slack) upper bound for z_t from the high type's incentive compatibility constraint. This bound has to hold for all z_t in the range of possible equilibrium z_t, so we pick the highest one and show that condition (9) is violated when the high type grows sufficiently pessimistic, that is for t large enough. ■

Corollary 11 *For any equilibrium profile* (σ, μ) *and sequence* $\{z_t\}_{t=1}^{\infty}$ *of potential first favors consistent with proposition 10,*

$$m(z_t) = \min\left\{Mz_t + z_t^{LH}, 1\right\} \tag{10}$$

is optimal, where $z_t^{LH} := \left(1 - \delta^L(1-p)\right) \sum_{i=0}^{\infty} \left((1-p)\delta^L\right)^i z_{t+1+i}$.

Proof. This result for $m(z_t)$ is immediate because favors are efficient and $Mz_t + z_t^{LH}$ is the least upper bound on $m(z_t)$ conditional on sequence $z \equiv \{z_t\}_{t=0}^{\infty}$ as specified by (σ, μ). The least upper bound condition is from (38) in the proof of proposition 10. And z_t^{LH} is as specified by (36). ■

Equation (10) may at first seem convoluted because of the inclusion of z_t^{LH}, but that is only the case if z were an infinite sequence of strictly positive terms, which is ruled out by proposition 10. Since the strictly positive terms in z_t are finite, say ending at some $T \in \mathbb{N}$, then $z_T^{LH} = 0$, and consequently $m(z_T) = \min\{Mz_T, 1\}$. We may then calculate the rest of the least upper bounds using backward induction (in theory), although for T large this may be intractable in closed form.

Proposition 12 (Time bound on separation) *For any equilibrium profile* (σ, μ) *consistent with proposition 10 equilibrium separation has to occur within*

$$T \equiv \sup\left\{t \in \mathbb{N} : \frac{\mu_o(1-2p)^{t-1}}{\mu_o(1-2p)^{t-1} + (1-\mu_o)(1-p)^{t-1}} \geq \frac{1-\delta^H}{\delta^H BM}\right\} \text{ periods,}$$

where B *and* M *are defined by (28) and (9), respectively.*

Proof. From proposition 10 we know that separation has to occur in finite time, call this T periods at most. Then $z_s = 0$ for all $s > T$, which means that $S_T^{LH} = S_T^{HL} = S_T^{HH} = 0$ for the period T incentive compatibility constraints, which in turn means that we can write condition (41) for T as

$$z_T \leq \frac{\delta^H}{1-\delta^H} \mu_T^a B\, m(z_T)$$

$$\implies \frac{1-\delta^H}{\delta^H B} \leq \mu_T^a \frac{m(z_T)}{z_T} \leq \mu_T^a M.$$

Suppose agent a is a high type playing according to σ. Then the history must be such that she has not received a favor opportunity until now and the other agent has done no initial favor. In notation, $(h_T^a, H_{T-1}) = (\{0,0,...,0,1\}, \{0,0,...,0\}) \equiv \left(\widetilde{h}_T^a, \widetilde{H}_{T-1}\right)$. Then $\mu_T^a\left(\widetilde{h}_T^a, \widetilde{H}_{t-1}\right) = \mu_{T-1}^a$ since $w_T^a = 1$, and $\mu_{t-1}^a = \frac{\mu_o(1-2p)^{T-1}}{\mu_o(1-2p)^{T-1}+(1-\mu_o)(1-p)^{T-1}}$ by (43). Substituting this into the above incentive compatibility constraint yields,

$$\frac{1-\delta^H}{\delta^H BM} \leq \frac{\mu_o(1-2p)^{T-1}}{\mu_o(1-2p)^{T-1}+(1-\mu_o)(1-p)^{T-1}}.$$

The left-hand side of the above inequality is just a constant, so we can compute T. Also note that once T is known we can compute z_T from lemma 8, except using μ_T^a instead of μ_o and then work backward to calculate the other z_t. Furthermore, let $g := \frac{1-\delta^H}{\delta^H BM}$, then it is straightforward to show that

$$\frac{\partial g}{\partial \delta^L} = \frac{\mu_o(1-\delta^H)k(1-\delta^H(1-2p))}{\delta^H(1-\delta^L(1-p))^2(k(1-\delta^H(1-p(1-1/k)))} > 0,$$

$$\frac{\partial g}{\partial \delta^H} = -\frac{\mu_o\delta^L k\left((1-\delta^H)^2k+2(1-\delta^H)\delta^H(k-1)p+2\left(\delta^H\right)^2 p(k-1)\right)}{(\delta^H)^2(1-\delta^L(1-p))^2(k(1-\delta^H(1-p(1-1/k)))^2} < 0,$$

so g is maximized at $\delta^L = \delta^H = \delta^*$. Plugging in $\delta^L = \delta^H = \frac{1}{1+p(k-1)}$ and simplifying yields $g|_{\delta^L=\delta^H=\delta^*} = \mu_o$, so the incentive compatibility constraint is always satisfied for at least $T = 1$, which we knew already from lemma 8. ■

3.1.1 SS equilibria: Symmetric beliefs

The proofs of proposition 10 and proposition 12 relied on the mutually exclusive favor opportunities so that one agent could grow pessimistic while the other did not. It would therefore be fair to ask whether symmetric separation has to occur within a fixed time period simply because of the mutually exclusive nature of the favor opportunities and whether independent favor opportunities, which are less informative, could actually lead to a better outcome. The answer, roughly speaking, is that they would. Changing the distribution of favor opportunities from mutually exclusive to independent would change the model drastically since we would then have to worry about favor-trading during periods when both agents received a favor opportunity, which would have a major impact on the incentive compatibility constraints and on the general nature of the problem at hand. However, as a thought experiment, suppose the underlying favor opportunity distribution is kept as is, but it is assumed, contrary to fact to be symmetric; $\mu_t^a = \mu_t^b = \mu_t$, and that this is common knowledge. Then it can be shown that symmetric separation does not have to occur within a fixed time period.

Proposition 13 (SS equilibria with symmetric beliefs) *Suppose (σ, μ) is a PBE profile in a game with two unknown types. Suppose σ is such that the first agent to receive a favor opportunity will do a favor of size $z_t \in (0,1]$ to signal type if she is a high type while a low type will follow the* *autarky strategy. And suppose σ calls for separation to be followed by an equality matching game at some level $m(z_t) \in (0,1]$. Furthermore, suppose agents' beliefs are $\mu_t^a = \mu_t^b = \mu_t$, $\forall t$. Then separation can be guaranteed.*

Proof. In appendix. ■

4 Separation with independent favor opportunities

In this section, unless otherwise stated, we assume favor opportunities arrive independently across agents. In the model so far favor opportunities, w_t^a and w_t^b, were modeled as mutually exclusive. In particular, for $p \in (0, 1/2)$,

$$P\left(\left(w_t^a, w_t^b\right) = (1,0)\right) = P\left(\left(w_t^a, w_t^b\right) = (0,1)\right) = p,$$
$$P\left(\left(w_t^a, w_t^b\right) = (0,0)\right) = (1-2p)$$

Because the favor opportunities were correlated they were also informative about the other agent's type. It would be fair to ask whether or not modeling favor opportunities as mutually exclusive was a driving force behind any of the results and what impact this had on the equilibria. To investigate these questions suppose that each agent still receives a favor opportunity with probability $p \in (0,1)$, but this time the opportunities are independent of each other. For the rest of this section, for $p \in (0,1)$,

$$P\left(\left(w_t^a, w_t^b\right) = (1,1)\right) = p^2$$
$$P\left(\left(w_t^a, w_t^b\right) = (1,0)\right) = P\left(\left(w_t^a, w_t^b\right) = (0,1)\right) = p(1-p)$$
$$P\left(\left(w_t^a, w_t^b\right) = (0,0)\right) = (1-p)^2$$

Note that the total autarky payoff remains the same as before if p is kept the same $(p < 1/2)$. If neither agent does a favor the total average discounted value of the game is $2p$ as before. Similarly, the first-best outcome remains unchanged at $2pk$ if both do a favor whenever possible. So have the results changed? The answer to this question with respect to equality matching between two high type agents is no. Lemma 6 holds for both the mutually exclusive and the independent favor distribution (its proof covers both).

In subsection 3.1 we already discussed the problem of determining who is advantaged and disadvantaged endogenously. With independent favor opportunities this matter is further complicated by the possibility that both agents receive a favor opportunity in the same period. One solution is to formulate the strategy as follows: Agents are *undesignated* at the start of the game. Undesignated agents are called to do a favor of size z. If an agent does such a favor and the other agent does not, the former becomes advantaged and the latter disadvantaged in a game of full equality matching that follows. If both undesignated agents do a favor of size z, both remain undesignated and the stage game is repeated.

Lemma 14 (EM without initial designations) *Consider a complete information environment. Given $z \in (0,1]$, let $t^* := \inf\{s \in \mathbb{N} : \tau_s = (z,0)$ or $(0,z)\}$, denote the first (equilibrium) favor. Let*

(σ, μ) *be defined as follows:*

$$\sigma^i := \begin{cases} I_t^i = z\mathbf{1}_{\{w_t^i=1\}} & \textit{if } h_{t-1} \in \mathcal{H}_{t-1}^*, \omega^i = H \textit{ and } t \leq t^*, \\ \sigma_{em}^i(\underline{u}_{em}, \bar{u}_{em}) & \textit{from } t = t^*+1 \textit{ if } h_{t-1} \in \mathcal{H}_{t-1}^*, \\ & \quad \omega^i = H \textit{ and } \tau_{t^*} = (0, z), \\ \sigma_{em}^i(\bar{u}_{em}, \underline{u}_{em}) & \textit{from } t = t^*+1 \textit{ if } h_{t-1} \in \mathcal{H}_{t-1}^*, \\ & \quad \omega^i = H \textit{ and } \tau_{t^*} = (z, 0), \\ \sigma_{aut}^i & \textit{otherwise,} \end{cases} \tag{11}$$

$$\mu_t^i := \begin{cases} 0 & \textit{if } h_{t-1} \notin \mathcal{H}_{t-1}^*, \\ 1 & \textit{else if } \exists n \textit{ s.t. for } j \neq i,\ I_n^j = \begin{cases} z \textit{ if } n \leq t^* \\ 1 \textit{ if } n > t^* \end{cases} \\ & \quad \textit{(i.e. other agent has made the initial investment)} \\ \frac{\mu_{t-1}^i(1-p)}{1-\mu_{t-1}^i p} & \textit{otherwise.} \end{cases} \tag{12}$$

Then (σ, μ) *is a PBE for* $z \leq 1/2$.

Proof. In appendix. A rough outline of the proof and some results are presented below. ■

Once designations are determined, the game will be exactly the same as the one analyzed in lemma 6. In particular, payoffs and incentive compatibility constraints will be the same, so it is enough to evaluate the incentive compatibility constraint for an undesignated agent. Let $\hat{u}_{em}^z$ denote her payoff. Then we can write $\hat{u}_{em}^z$ in terms of the four possible combinations of favor opportunity events:

$$\begin{aligned} \hat{u}_{em}^z = {} & p^2\left[\left(1-\delta^H\right)(1+(k-1)z) + \delta^H \hat{u}_{em}^z\right] \\ & + p(1-p)\left[\left(1-\delta^H\right)(1-z) + \delta^H \bar{u}_{em}\right] \\ & + (1-p)p\left[\left(1-\delta^H\right)kz + \delta^H \underline{u}_{em}\right] + (1-p)^2\delta^H \hat{u}_{em}^z. \end{aligned}$$

Proof. We already know $\bar{u}_{em}$ and $\underline{u}_{em}$ from the previous lemma, so the above equation only has one unknown, solving for which yields,

$$\begin{aligned} & \hat{u}_{em}^z = p + C_1 + C_2 z, \text{ where} \\ & C_1 \equiv \tfrac{\delta^H(1-p)p^2(k-1)}{1-\delta^H+2\delta^H p(1-p)} \text{ and } C_2 \equiv \tfrac{(1-\delta^H)p(k-1)}{1-\delta^H+2\delta^H p(1-p)} \end{aligned} \tag{13}$$

The incentive compatibility constraint is

$$\begin{aligned} ICC_{em}^{nd} : {} & p\left(\left(1-\delta^H\right)(1+(k-1)z) + \delta^H \hat{u}_{em}^z\right) + (1-p)\left(\left(1-\delta^H\right)(1-z) + \delta^H \bar{u}_{em}\right) \\ & \geq p\left(\left(1-\delta^H\right)(1+kz) + \delta^H \underline{u}_{em}\right) + (1-p)\left(1-\delta^H + \delta^H \hat{u}_{em}^z\right). \end{aligned}$$

When ICC_{em}^{nd} binds we can solve for δ^H. Call the solution,

$$\underline{\delta}_z^H := \frac{2z}{\left(\begin{array}{l} 2z + p\left[p - 3z + k(1 - z - p(1 - 2z)) \right. \\ \left. + \sqrt{(p - z)^2 + k^2(1 - p - z + 2pz)^2 + 2k(1 - p - z)(z + p(1 - 2z))} \right] \end{array} \right)} \tag{14}$$

Alternatively we can solve for z given δ^H :

$$z = \min \left\{ 1, \frac{\delta^H p(p(1 - \delta^H + \delta^H p) + k(1 - p - \delta^H(1 - 3(1 - p)p)))}{(1 - \delta^H + \delta^H(1 + k)p - 2\delta^H kp^2)(1 - \delta^H(1 - 2p))} \right\}. \tag{15}$$

Plugging in $\delta^H = \delta^*$, we obtain the bound $z \leq 1/2$. Naturally this is the least upper bound because agents with higher discount factors are willing to pay a higher cost for future benefits. Finally, it is straightforward to verify that beliefs are consistent with σ and Bayesian updating when applicable, so (σ, μ) is a PBE for $z \leq 1/2$. ■

Observe that setting $z = 1/2$ and implementing the strategy profile in lemma 14 is equally efficient to designating one of the do the first favor in a full equality matching game. The former strategy initially implements only a half size favor, but with twice the likelihood as with full equality matching, so the efficiency gain over autarky in both is $p(k - 1)$. In particular,

$$\hat{u}_{em}^{1/2} = \tfrac{1}{2}p(k + 1) = p + \tfrac{1}{2}p\left(k - 1\right). \tag{16}$$

However, for $\delta^H > \delta^*$ we can implement a $z > 1/2$ determined by equation (15), and thereby obtain an efficiency gain over EM with a designated disadvantaged agent from the start. In particular, for $z = 1$, we need

$$\delta^H \geq \underline{\delta}_1^H := \frac{1}{1 - 2p + p^2\left(k + 1\right)} > \delta^*. \tag{17}$$

Note that inequality (17) cannot be satisfied for any δ^H if $p(k+1) < 2$, which roughly speaking rules out cases in which both p and k are high. However, when (17) is satisfied, the initial periods of symmetric favor strategies implement the first-best stage game outcome. The next corollary follows by symmetry from its mutually exclusive favor opportunities counterpart in Kalla [7] and is presented mainly as a formality.

Corollary 15 *Let strategy profile (σ, μ) be such that*

$$\sigma^a := \begin{cases} \sigma_{em(z)}^a\left(\underline{u}, \bar{u}\right) & \text{if } h_{t-1} \in \mathcal{H}_{t-1}^*, \omega^a = H \text{ and } t = \inf\left\{s \in \mathbb{N} : w_s^a = 1\right\}, \\ \sigma_{em}^a\left(\bar{u}, \underline{u}\right) & \text{from } t = 1 + \inf\left\{s \in \mathbb{N} : w_s^a = 1\right\} \text{ if } h_{t-1} \in \mathcal{H}_{t-1}^*, \omega^a = H, \\ \sigma_{aut}^a & \text{otherwise,} \end{cases} \tag{18}$$

$$\sigma^b := \begin{cases} \sigma_{em}^b\left(\bar{u}, \underline{u}\right) & \text{from } t = 1 + \inf\left\{s \in \mathbb{N} | \tau_s = (z, 0)\right\} \text{ if } h_{t-1} \in \mathcal{H}_{t-1}^*, \omega^b = H, \\ \sigma_{aut}^b & \text{otherwise,} \end{cases} \tag{19}$$

$$\mu_t^a := \begin{cases} 0 & \textit{if } h_{t-1} \notin \mathcal{H}_{t-1}^*, \\ 1 & \textit{if } h_{t-1} \in \mathcal{H}_{t-1}^* \textit{ and } \exists n < t \textit{ s.t. } \tau_n = (0,1), \\ \mu_o & \textit{if } h_{t-1} \in \mathcal{H}_{t-1}^* \textit{ and } \nexists n \leq t \textit{ s.t. } \tau_n = (z,0), \\ \frac{\mu_{t-1}^a(1-p)}{1-\mu_{t-1}^a p} & \textit{otherwise,} \end{cases} \tag{20}$$

$$\mu_t^b := \begin{cases} 0 & \textit{if } h_{t-1} \notin \mathcal{H}_{t-1}^*, \\ 1 & \textit{if } h_{t-1} \in \mathcal{H}_{t-1}^* \textit{ and } \exists n < t \textit{ s.t. } \tau_n = (z,0), \\ \frac{\mu_{t-1}^b(1-p)}{1-\mu_{t-1}^b p} & \textit{otherwise,} \end{cases} \tag{21}$$

$$z \in [\underline{z}, \bar{z}] := \left[\frac{\mu_o \delta^L pk}{1-\delta^L(1-p)}, \min\left\{ 1, \frac{\mu_o \delta^H p\left[\left(1-\delta^H\right)k + \delta^H p(k-1)\right]}{(1-\delta^H)(1-\delta^H(1-2p))} \right\} \right]. \tag{22}$$

Then (σ, μ) *is a PBE profile.*

Proof. Immediate from symmetry to its counterpart in Kalla [7]. ■

Proposition 16 (SS equilibria with independent favor opportunities) *Let* $t^* := \inf\{s \in \mathbb{N} : \tau_s = (z_s, z_s), (z_s, 0) \text{ or } (0, z_s)\}$ *where* $\inf\{\emptyset\} \equiv \infty$*, be the time of the first favor, and let* (σ, μ) *be defined as follows for* $i \in \{a, b\}$ *:*

$$\sigma^i := \begin{cases} I_t^i = z_t \mathbf{1}_{\{w_t^i = 1\}} & \textit{if } h_{t-1} \in \mathcal{H}_{t-1}^* \textit{ and } \omega^i = H, t < t^* \\ & \textit{(no one has done a favor yet),} \\ I_t^i = \mathbf{1}_{\{w_t^i = 1\}} & \textit{starting at } t = t^* + 1 \textit{ if } \tau_{t^*} \textit{ symmetric and} \\ & \textit{ending at } n > t^* + 1 \textit{ s.t. } \tau_{n-1} \neq (1,1) \textit{ and} \\ & \textit{only if } h_{t-1} \in \mathcal{H}_{t-1}^* \textit{ and} \omega^i = H, \\ \sigma_{em}^i(\bar{u}_{em}, \underline{u}_{em}) & \textit{starting after first non-symmetric } \tau_{t-1} \textit{ if } a \textit{ did the} \\ & \textit{favor, and only if } h_{t-1} \in \mathcal{H}_{t-1}^* \textit{ and } \omega^i = H, \\ \sigma_{em}^i(\underline{u}_{em}, \bar{u}_{em}) & \textit{starting after first non-symmetric } \tau_{t-1} \textit{ if } b \textit{ did the} \\ & \textit{favor, and only if } h_{t-1} \in \mathcal{H}_{t-1}^* \textit{ and } \omega^i = H, \\ \sigma_{aut}^i & \textit{otherwise,} \end{cases} \tag{23}$$

$$\mu_t^i := \begin{cases} 0 & \textit{if } h_{t-1} \notin H_{t-1}^*, \\ 1 & \textit{if } h_{t-1} \in H_{t-1}^*, t > t^* \textit{ and } j \neq i \textit{ did the first favor} \\ & \textit{or reciprocated,} \\ \frac{\mu_{t-1}^i(1-p)}{1-\mu_{t-1}^i p} & \textit{otherwise.} \end{cases} \tag{24}$$

Then if δ^H *satisfies condition (17), there exists a sequence* $z = \{z_s\}_{s=0}^{\infty} \in (0,1]^{\infty}$ *such that* (σ, μ) *is a separating PBE profile. If both agents are high types,* σ *will implement separation with probability one.*

Proof. In appendix. ■

5 Conclusion

In this paper we extended the analysis of favor-trading with incomplete information from Kalla [7]. We constructed and characterized symmetric strategies that achieve separation roughly twice as fast in expectation as the non-symmetric strategies for separation analyzed in Kalla [7]. If favor opportunities are mutually exclusive, SS strategies can only achieve separation during a limited number of periods before agents grow too pessimistic about the other agent's type to risk doing the first favor. This is not the case when favor opportunities are independent and therefore non-informative about the other agent's type. In that sense it turns out that in the favor-trading game analyzed in this paper, more accurate signals about the other player's type lead to a worse outcome.

6 Appendix

Proof. (Lemma 6: Necessary and sufficient condition for EM) Suppose we start the game with agent b as the advantaged agent and the level of trust is $z \in (0,1]$, that is agent a does a favor of size $x = z$ if she receives a favor opportunity, and then does no further favors until the other agent, in this case agent b, reciprocates by doing a favor of size $y = z$. It follows that in terms of average discounted payoffs

$$\begin{aligned}\underline{u}_{em(z)} &= p\left(\left(1-\delta^H\right)(1-z) + \delta^H \bar{u}_{em(z)}\right) + (1-p)\delta^H \underline{u}_{em(z)} \\ &= p\tfrac{\left(1-\delta^H\right)(1-z)+\delta^H \bar{u}_{em(z)}}{1-\delta^H(1-p)}\end{aligned}$$

regardless of whether favor opportunities are mutually exclusive or independent, while

$$\begin{aligned}\bar{u}_{em(z)} &= p^2\left(\left(1-\delta^H\right)(kz+1) + \delta^H \underline{u}_{em(z)}\right) + p(1-p)\left(\left(1-\delta^H\right)(1-z) + \delta^H \bar{u}_{em(z)}\right) \\ &+ (1-p)p\left(\left(1-\delta^H\right)kz + \delta^H \underline{u}_{em(z)}\right) + (1-p)^2\delta^H \bar{u}_{em(z)} \\ &= p^2\left(\left(1-\delta^H\right)(kz+1) + \delta^H \underline{u}_{em(z)}\right) + p\left(\left(1-\delta^H\right)(1-z) + \delta^H \bar{u}_{em(z)}\right) \\ &- p^2\left(\left(1-\delta^H\right)(1-z) + \delta^H \bar{u}_{em(z)}\right) + p\left(\left(1-\delta^H\right)kz + \delta^H \underline{u}_{em(z)}\right) \\ &- p^2\left(\left(1-\delta^H\right)kz + \delta^H \underline{u}_{em(z)}\right) + (1-2p)\delta^H \bar{u}_{em(z)} + p^2\delta^H \bar{u}_{em(z)} \\ &= p\left(\left(1-\delta^H\right)(1-z) + \delta^H \bar{u}_{em(z)}\right) + p\left(\left(1-\delta^H\right)kz + \delta^H \underline{u}_{em(z)}\right) \\ &+ (1-2p)\delta^H \bar{u}_{em(z)} \\ &= p\tfrac{\left(1-\delta^H\right)(1+kz)+\delta^H \underline{u}_{em(z)}}{1-\delta^H(1-p)}\end{aligned}$$

when favor opportunities are independent, and

$$\begin{aligned}\bar{u}_{em(z)} &= p\left(\left(1-\delta^H\right) + \delta^H \bar{u}_{em(z)}\right) \\ &+ p\left(\left(1-\delta^H\right)kz + \delta^H \underline{u}_{em(z)}\right) + (1-2p)\delta^H \bar{u}_{em(z)} \\ &= p\tfrac{\left(1-\delta^H\right)(1+kz)+\delta^H \underline{u}_{em(z)}}{1-\delta^H(1-p)}.\end{aligned}$$

when favor opportunities are mutually exclusive. That is the payoffs are the same in EM(z) regardless of the distribution.

The two equations above are in two unknowns, $\underline{u}_{em(z)}$ and $\bar{u}_{em(z)}$. Solving for these yields,

$$\underline{u}_{em(z)} = p + p\tfrac{-\left(1-\delta^H\right)+\delta^H p(k-1)}{1-\delta^H(1-2p)}z \tag{25}$$

$$= p + Az \text{ where } A \equiv p\tfrac{-\left(1-\delta^H\right)+\delta^H p(k-1)}{1-\delta^H(1-2p)} \tag{26}$$

$$\bar{u}_{em(z)} = p + p\tfrac{(1-\delta^H)k+\delta^H p(k-1)}{1-\delta^H(1-2p)}z \tag{27}$$

$$= p + Bz \text{ where } B \equiv p\tfrac{(1-\delta^H)k+\delta^H p(k-1)}{1-\delta^H(1-2p)}. \tag{28}$$

For z fixed, as δ^H ranges from δ^* to 1, $\underline{u}_{em(z)}$ ranges from p to $p+\frac{1}{2}p(k-1)z$ and $\bar{u}_{em(z)}$ from $p+p(k-1)z$ to $p+\frac{1}{2}p(k-1)z$. In particular, for any $\delta^H \in [\delta^*, 1)$,

$$\begin{aligned} \underline{u}_{em(z)} + \bar{u}_{em(z)} &= 2p + p(k-1)z, \text{ or} \\ &= p\,(k+1) \text{ for } z = 1. \end{aligned} \tag{29}$$

Agent a's incentive compatibility constraint is

$$\begin{aligned} ICC^a_{em(z)} : \left(1-\delta^H\right)(1-z) + \delta^H \bar{u}_{em(z)} &\geq \left(1-\delta^H\right) + \delta^H \underline{u}_{em(z)} \\ \iff \bar{u}_{em(z)} - \underline{u}_{em(z)} - \tfrac{\left(1-\delta^H\right)}{\delta^H} z &\geq 0, \end{aligned}$$

which using equations (25) and (27) is equivalent to

$$\begin{aligned} & p + p\tfrac{(1-\delta^H)k+\delta^H p(k-1)}{1-\delta^H(1-2p)} z - p - p\tfrac{-\left(1-\delta^H\right)+\delta^H p(k-1)}{1-\delta^H(1-2p)} z \geq \tfrac{\left(1-\delta^H\right)}{\delta^H} z \\ & \text{solving for } \delta^H \implies \tfrac{1}{1+p(k-1)} \leq \delta^H. \end{aligned}$$

Recall that $\delta^* = \dfrac{1}{1+p(k-1)}$ so $\delta^* \leq \delta^H$ is necessary and sufficient to implement any simple EM strategy profile. $x = y = 1$ ensures the greatest gains from cooperation. ■

Proof. (Lemma 8: First period SS equilibria) First, observe that the belief system simply states each agent begins the game assuming that the other is a high type with probability μ_o, and if the other agent makes an observable deviation from the equilibrium path ($h_{t-1} \notin \mathcal{H}^*_{t-1}$), she is believed to be a low type. If there has been no public deviation and the other agent has either done a favor of size z_1 in the first period, or size 1 as reciprocation in some other period ($h_{t-1} \in \mathcal{H}^*_{t-1}$ and $\exists n \leq t$ s.t. $\tau_1 = \left(I^j_1, 0\right)$ or $\left(0, I^j_1\right)$ for $j = a$ or b, respectively), she is believed to be a high type. Note that $I^j_1 > 0$ must be z_1 in the first period) or subsequently 1 otherwise it would be off-equilibrium path. Next if the agent receives a favor opportunity this period or did so in the first period without doing a favor ($h_{t-1} \in \mathcal{H}^*_{t-1}$ and $\{w^i_t = 1$ or $w^i_1 \neq \tau_1 = 0\}$), the other agent cannot do a favor or will not do a favor per σ so beliefs about her remain as they were last period. In the latter case, they would remain μ_o forever. Another possibility is that the agent received no favor opportunity and no favor during the first period $\left(h_{t-1} \in \mathcal{H}^*_{t-1}\right.$ and $\left.w^i_1 = \tau_1 = 0\right)$ so both agents revert to autarky and there will be no more informative signals on the equilibrium path. Therefore, her belief about the other agent will just be updated once from μ_o to $\frac{\mu_o(1-2p)}{1-(1+\mu_o)p}$ forever. The last possibility ("otherwise") is that the agent did an initial favor and is now waiting for reciprocation, and each period she does not receive either reciprocation or a favor opportunity, her previous belief about the other agent is updated according to Bayes' rule. It is easy to see that this belief system is consistent with strategy profile σ provided that the low types do not attempt to pool with the high types, and that the high types do not choose the autarky strategy because their favor could be wasted on a low type. Therefore, it is sufficient to check that z_1 is high enough so that a low type has no incentive to mimic a high type, and that a high type would not be better off choosing the autarky strategy. Let u^L_z denote the expected continuation payoff for a low type, say agent b, who does a favor of size z_1.

$$E\left(u_z^L|\omega^a = H\right) \equiv u = p\left(\left(1-\delta^L\right)+\delta^L u\right) + p\left(\left(1-\delta^L\right)k + \delta^L p\right) + (1-2p)\delta^L u$$
$$= p\frac{\left(1-\delta^L\right)(1+k)+\delta^L p}{1-\delta^L(1-p)}.$$

Using the above equation, it follows that

$$\begin{aligned} u_z^L &= P\left(\omega^a = L\right)p + P\left(\omega^a = H\right)E\left(u_z^L : \omega^a = H\right) \\ &= (1-\mu_o)p + \mu_o E\left(u_z^L : \omega^a = H\right) \\ &= (1-\mu_o)p + \mu_o p\frac{\left(1-\delta^L\right)(1+k)+\delta^L p}{1-\delta^L(1-p)}. \end{aligned} \tag{30}$$

The incentive compatibility constraint for a low type who has received a favor opportunity in the first period not to mimic the high type is

$$\begin{aligned} & ICC_z^L : \left(1-\delta^L\right)(1-z_1) + \delta^L u_z^L \leq \left(1-\delta^L\right) + \delta^L p \\ & \iff u_z^L - p \leq \left(1-\delta^L\right) z_1/\delta^L \\ & \iff (1-\mu_o)p + \mu_o p\frac{\left(1-\delta^L\right)(1+k)+\delta^L p}{1-\delta^L(1-p)} - p \leq \frac{1-\delta^L}{\delta^L} z_1 \text{ by (30)} \\ & \implies z_1 \geq \frac{\mu_o \delta^L pk}{1-\delta^L(1-p)} \equiv \underline{z}_1 \end{aligned} \tag{31}$$

Observe that $\underline{z}_1 = \mu_o\delta^L pk/\left(1-\delta^L(1-p)\right) < \mu_o\delta^* pk/\left(1-\delta^*(1-p)\right) = \mu_o < 1$. Next consider a high type, say agent a, who receives a favor opportunity in period 1. Let u_z^H denote the expected continuation payoff to a high type who does a favor of size z_1. Then

$$u_z^H = (1-\mu_o)p + \mu_o\bar{u}_{em} = (1-\mu_o)p + \mu_o\frac{p\left(1-\delta^H(1-p)\right)(1+k)}{1-\delta^H(1-2p)} \text{using (4)}$$

The incentive compatibility constraint for the high type who received a favor opportunity in the first period to do a favor of size z_1 is,

$$\begin{aligned} & ICC_z^H : \left(1-\delta^H\right)(1-z_1) + \delta^H u_z^H \geq \left(1-\delta^H\right) + \delta^H p \\ & \iff (1-\mu_o)p + \mu_o\frac{p\left(1-\delta^H(1-p)\right)(1+k)}{1-\delta^H(1-2p)} - p \geq \frac{1-\delta^H}{\delta^H} z_1 \\ & \implies z_1 \leq \frac{\mu_o\delta^H\left(\left(1-\delta^H\right)k + \delta^H p(k-1)\right)}{(1-\delta^H)(1-\delta^H(1-2p))} \\ & \therefore \text{let } \bar{z}_1^{em} \equiv \min\left\{1, \frac{\mu_o\delta^H\left(\left(1-\delta^H\right)k + \delta^H p(k-1)\right)}{(1-\delta^H)(1-\delta^H(1-2p))}\right\} \end{aligned} \tag{32}$$

To finish the proof it is necessary to show that $[\underline{z}_1, \bar{z}_1] \neq \emptyset$. Recall that $\underline{z}_1 < \mu_o$, so if $\bar{z}_1 = 1$, then it is immediate that $[\underline{z}_1, \bar{z}_1] \neq \emptyset$. Therefore suppose that $\bar{z}_1 < 1$, then it is enough to show that $\bar{z}_1 - \underline{z}_1 > 0$. To this end

$$\bar{z}_1 - \underline{z}_1 = \frac{\mu_o\delta^H\left(\left(1-\delta^H\right)k + \delta^H p(k-1)\right)}{(1-\delta^H)(1-\delta^H(1-2p))} - \frac{\mu_o\delta^L pk}{1-\delta^L(1-p)}$$

$$= \frac{\delta^H\left(\left(1-\delta^H\right)k+\delta^H p(k-1)\right)}{(1-\delta^H)(1-\delta^H(1-2p))} - \frac{\delta^L pk}{1-\delta^L(1-p)}$$

The expression above shows that the existence of a separating equilibrium in this game does not depend on μ_o, the fraction of high types. Still supposing $\bar{z}_1 < 1$, taking the derivative of $\bar{z}_1$ with respect to δ^H in equation (32) and rearranging yields

$$\frac{d\bar{z}_1}{d\delta^H} = \mu_o p \frac{\left(1-\delta^H\right)^2 k+2\delta^H p(k-1)\left(1-\delta^H(1-p)\right)}{(1-\delta^H)^2(1-\delta^H(1-2p))^2}$$

From the above expression it is easy to see that $d\bar{z}_1/d\delta^H > 0$ since $k-1>0$. Therefore $\bar{z}_1$ is minimized at $\delta^H = \delta^* = 1/\left(1+p(k-1)\right)$. Substituting $1/\left(1+p(k-1)\right)$ for δ^H in the expression for $\bar{z}_1$ and simplifying yields $\bar{z}_1 = \mu_o$. Therefore $[\underline{z}_1, \bar{z}_1] \neq \emptyset$. (Again recall that $\underline{z}_1 < \mu_o$). ■

Proof. (Proposition 9: Bounds with complete information) The advantaged and disadvantaged payoffs, $\bar{u}_{em(z)}$ and $\underline{u}_{em(z)}$, would be the same as before since they are clearly not dependent on any first period payment. The expected payoff for the undesignated agent would be

$$\begin{aligned}\hat{u}_{em(z)} &= p((1-\delta^H)(1-z_1)+\delta^H\bar{u}_{em(z)}) \\ &\quad + p((1-\delta^H)kz_1+\delta^H\underline{u}_{em(z)}) + (1-2p)\delta^H\hat{u}_{em(z)}.\end{aligned}$$

Solving for $\hat{u}_{em(z)}$ and substituting in for $\underline{u}_{em(z)}$ and $\bar{u}_{em(z)}$ from (26) and (28) yields

$$\hat{u}_{em(z)} = p + \frac{p(1-\delta^H)(k-1)}{1-\delta^H(1-2p)}z_1 + \frac{p^2\delta^H(k-1)}{1-\delta^H(1-2p)}z_2.$$

Per proof of lemma 6, we still need $\delta^H \geq \delta^*$ for the incentive compatibility constraints to hold during the second phase of the game that consists of equality matching at level $z_2 \in (0,1]$. The question is, what level of z_1 is required to guarantee that the incentive compatibility constraint is satisfied for the first agent with an opportunity to do a favor? As usual, it is enough to consider a one-shot deviation.

$$ICC^H_{first}: \qquad (1-\delta^H)(1-z_1)+\delta^H\bar{u}_{em(z)} \geq 1-\delta^H+\delta^H\hat{u}_{em(z)}.$$

To analyze ICC^H_{first}, let f be a function of the left-hand side minus the right-hand side,

$$\begin{aligned}f &\equiv (1-\delta^H)(1-z_1)+\delta^H\bar{u}_{em(z)} - \left(1-\delta^H\right) - \delta^H\hat{u}_{em(z)} \\ &= -z_1+\delta^H z_1+\delta^H\left(\bar{u}_{em(z)}-\hat{u}_{em(z)}\right) \\ &= -z_1+\delta^H z_1+\delta^H\Big[p+p\frac{(1-\delta^H)k+\delta^H p(k-1)}{1-\delta^H(1-2p)}z_2 \\ &\quad -\left(p+\frac{p(1-\delta^H)(k-1)}{1-\delta^H(1-2p)}z_1+\frac{p^2\delta^H(k-1)}{1-\delta^H(1-2p)}z_2\right)\Big] \\ &= -z_1+\delta^H z_1+\frac{\delta^H\left(1-\delta^H\right)}{1-\delta^H(1-2p)}p\left(kz_2-(k-1)z_1\right).\end{aligned}$$

Figure 1 provides a graphical example of the problem $f \geq 0$, for $z_2 = 1$, $p = 1/4$, $k = 5$ fixed, and $z_1 \in [0,1]$ and $\delta^H \in (\delta^*, 1)$ as variables, where $\delta^* = \frac{1}{1+p(k-1)} = \frac{1}{1+\frac{1}{4}(5-1)} = 1/2$.

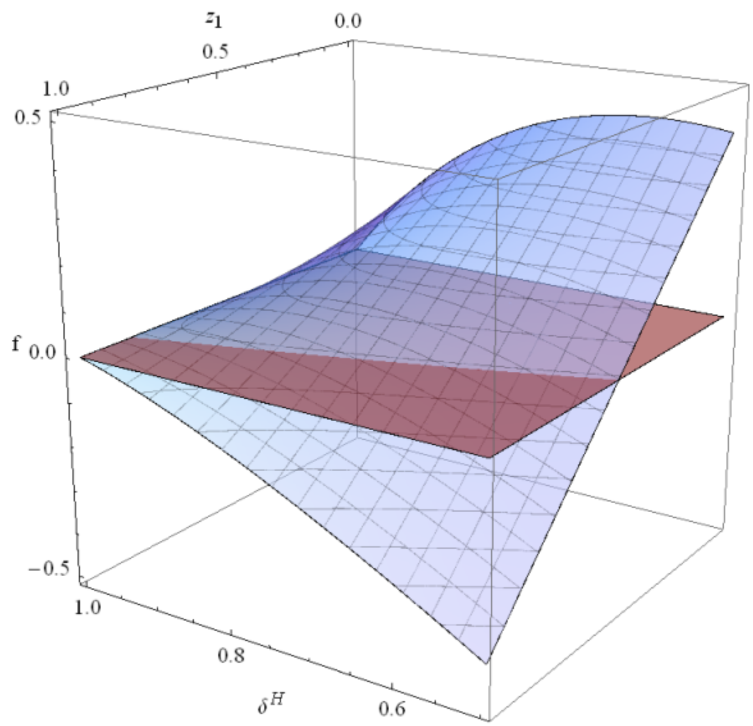

Figure 1: Constraints without designated first mover

The region above the plane and below the curved surface represents the feasible combinations of z_1 and δ^H that are incentive compatible. The $z_1 = 1$ edge of the graph also shows $f < 0$ (incentive compatibility is not satisfied), for all δ^H, in this case at least. To generalize the intuitions represented in the graph proceed by taking the first two derivatives of f with respect to δ^H,

$$\frac{df}{d\delta^H} = z_1 + \left[\frac{1-\delta^H-\delta^H\left(1-\delta^H(1-2p)\right)}{\left(1-\delta^H(1-2p)\right)^2}\right] p\left(kz_2-(k-1)z_1\right), \text{ and}$$
$$\frac{d^2 f}{d\left(\delta^H\right)^2} = -\frac{4p}{\left(1-\delta^H(1-2p)\right)^3} p\left(kz_2-(k-1)z_1\right) = -\frac{4p^2\left(kz_2-(k-1)z_1\right)}{\left(1-\delta^H(1-2p)\right)^3} < 0.$$

That is, f is strictly concave in δ^H and therefore has a minimum at either $\delta^H = \delta^*$ or $\delta^H = 1$. Evaluating these two limit points of f is a straight-forward exercise that yields the following results after the appropriate simplification,

$$\lim_{\delta^H \to 1} f = 0, \text{ and}$$

$$\lim_{\delta^H \to \delta^*} f = -z_1 + \frac{z_1}{1+p(k-1)} + \frac{\frac{1}{1+p(k-1)}\left(1-\frac{1}{1+p(k-1)}\right)}{1-\frac{1-2p}{1+p(k-1)}} p\left(kz_2 - (k-1)z_1\right)$$
$$= \frac{pk(k-1)(z_2-2z_1)}{k+1+p(k^2-1)} \geq 0 \iff z_2 \geq 2z_1.$$

Therefore a half-sized initial favor may always be used by high type agents to commence an equality matching game. To find an upper bound for z_1, say $\bar{z}_1$, we solve $f = 0$, which yields $z_1 = \frac{pk\delta^H}{1+\delta^H(p(k+1)-1)} z_2$. Observe that $\bar{z}_1$ is directly proportional to z_2, and that $\bar{z}_1 < \lim_{\delta^H \to 1} \bar{z}_1 = \frac{k}{k+1} z_2 < z_2$. In summary, a game of equality matching at level $z_2 \in (0,1]$ between high types such that neither agent is designated to go first, but instead the first agent with opportunity is required to do a favor, is incentive compatible for $z_1 \in \left(0, \frac{pk\delta^H}{1+\delta^H(p(k+1)-1)} z_2\right]$, which proves proves proposition 9. ■

Proof. (Proposition 10: Time constrained separation in SS equilibria) Suppose to the contrary of proposition 10 that for each period there exists $[\underline{z}_t, \bar{z}_t] \neq \emptyset$, $\underline{z}_t > 0$ of potential first favors, and an equilibrium strategy profile (σ, μ), such that if an agent does a favor of size z_t in period t, and that is the first favor in the game, then the other agent believes she is facing a high type with probability one. Furthermore, suppose (σ, μ) subsequently implements an EM game between high types at level $m(z_t)$, for some function $m : [0,1] \to [0,1]$, and that the initial favor maker (presumed high type) is the first advantaged agent in the EM game. And suppose (σ, μ) specifies the autarky strategy for low types.

To prove that (σ, μ) or any equivalent profile cannot be an equilibrium, suppose, without loss of generality, that agent b, is a low type. Then if b deviates from the autarky strategy, and does a favor of size z_t, she would receive a continuation payoff of p if she is facing another low type, and $v^{LH}_{z_t}$ if she is facing a high type, where

$$\begin{aligned} v^{LH}_{z_t} &= p\left(\left(1-\delta^L\right)\left(1+k\, m(z_t)\right) + \delta^L p\right) + (1-p)\delta^L v^{LH}_{z_t} \\ &= p\frac{\left(1-\delta^L\right)(1+k\, m(z_t)) + \delta^L p}{1-\delta^L(1-p)} = p + \frac{p\left(1-\delta^L\right)k}{1-\delta^L(1-p)} m(z_t), \end{aligned} \tag{33}$$

so her expected continuation payoff would be

$$\begin{aligned} v^L_{z_t} &= \left(1-\mu^b_t\right)p + \mu^b_t v^{LH}_{z_t} \\ &= \left(1-\mu^b_t\right)p + \mu^b_t p + \mu^b_t \frac{p\left(1-\delta^L\right)k}{1-\delta^L(1-p)} m(z_t) \\ &= p + \mu^b_t \frac{p\left(1-\delta^L\right)k}{1-\delta^L(1-p)} m(z_t) \end{aligned} \tag{34}$$

And if agent b does not do the favor, her continuation payoff is p if agent a is a low type, and $v^{LH}_{-z_t}$ if a is a high type, where

$$\begin{aligned} v^{LH}_{-z_t} &= p\left(1-\delta^L + \delta^L v^{LH}_{-z_{t+1}}\right) + p\left(\left(1-\delta^L\right)kz_{t+1} + \delta^L p\right) + (1-2p)\delta^L v^{LH}_{-z_{t+1}} \\ &= p\left(1-(1-p)\delta^L\right) + \underbrace{p\left(1-\delta^L\right)k}_{\equiv b} z_{t+1} + \underbrace{(1-p)\delta^L}_{\equiv a} v^{LH}_{-z_{t+1}} \end{aligned}$$

$= p(1-a) + bz_{t+1} + a\left(p(1-a) + bz_{t+2} + a\left(v^{LH}_{-z_{t+2}}\right)\right),$

iterratively expanding each $v^{LH}_{\cdot}$ and collecting like terms

produces the following geometric sums

$= p(1-a)\left(1+a+a^2+...\right) + b\left(z_{t+1} + az_{t+2} + a^2 z_{t+3} + ...\right)$

$= p + b\sum_{i=0}^{\infty} a^i z_{t+1+i} = p + p\left(1-\delta^L\right) kS^{LH},$

where $S^{LH} = \sum_{i=0}^{\infty}\left((1-p)\delta^L\right)^i z_{t+1+i}$.

Then it follows that

$$\begin{aligned} v^L_{-z_t} &= \left(1-\mu^b_t\right)p + \mu^b_t\left[p + p\left(1-\delta^L\right)kS^{LH}\right] \\ &= p + \mu^b_t p\left(1-\delta^L\right)kS^{LH} \end{aligned} \tag{35}$$

In order for agent b not to pool with high types we need z_t large enough that her incentive compatibility constraint to follow the autarky strategy instead of deviating is satisfied for all t.

$$\begin{aligned} ICC^L_z : \left(1-\delta^L\right)(1-z_t) + \delta^L v^L_{z_t} &\leq \left(1-\delta^L\right) + \delta^L v^L_{-z_t} \\ \iff z_t &\geq \tfrac{\delta^L}{1-\delta^L}\left(v^L_{z_t} - v^L_{-z_t}\right) \end{aligned}$$

Substituting in for $v^L_{z_t}$ and $v^L_{-z_t}$ from (34) and (35)yields,

$$\begin{aligned} z_t &\geq \tfrac{\delta^L}{(1-\delta^L)}\left(p + \mu^b_t \tfrac{p\left(1-\delta^L\right)k}{1-\delta^L(1-p)} m(z_t) - p - \mu^b_t p\left(1-\delta^L\right)kS^{LH}\right) \\ &= \mu^b_t \tfrac{\delta^L p\left(1-\delta^L\right)k}{1-\delta^L}\left(\tfrac{m(z_t)}{1-\delta^L(1-p)} - S^{LH}\right) \\ &= \mu^b_t \delta^L pk\left(\tfrac{m(z_t)}{1-\delta^L(1-p)} - S^{LH}\right), \\ &= \tfrac{\mu^b_t\delta^L pk}{1-\delta^L(1-p)}\left(m(z_t) - z^{LH}_t\right), \text{ where } z^{LH}_t \in \left(0, \bar{z}_{t+1}\right) \text{ is s.t.} \\ \textstyle\sum_{i=0}^{\infty}&\left((1-p)\delta^L\right)^i z_{t+1+i} = \textstyle\sum_{i=0}^{\infty}\left((1-p)\delta^L\right)^i z^{LH}_t = \tfrac{z^{LH}_t}{1-\delta^L(1-p)}. \end{aligned} \tag{36}$$

The incentive compatibility constraint needs to hold for every t and every possible history, so consider the history that maximizes the right-hand side, $h^b_t = \{1, 1, ..., 1\}$, and a public history of no favors assuming agent b follows the autarky strategy as specified by σ for low types. Then $\mu^b_t = \mu_o$ and the right-hand side of the low type's incentive compatibility constraint satisfies

$$\begin{aligned} \tfrac{\mu^b_t\delta^L pk}{1-\delta^L(1-p)}\left(m(z_t) - z^{LH}_t\right) &\leq \tfrac{\mu_o\delta^L pk}{1-\delta^L(1-p)}\left(m(z_t) - z^{LH}_t\right) \\ &= \frac{m(z_t) - z^{LH}_t}{M} \text{ by def'n of } M \text{ from (9).} \end{aligned}$$

Then z_t must satisfy

$$z_t \geq \frac{m(z_t) - z^{LH}_t}{M}, \forall t, \tag{37}$$

or else there exists a set of histories, that occur with strictly positive probability, such that a low type would could profit by mimicking a high type. Conversely, if (37) holds, the low types will never pool with high types, and given a sequence $z \equiv \{z_t\}_{t=1}^{\infty}$, condition (37) represents the greatest lower bound on the individual terms of the sequence z. Rearranging we obtain condition

$$m(z_t) \leq Mz_t + z_t^{LH}, \forall t, \tag{38}$$

for the least upper bound on $m(z_t)$ conditional on sequence z specified by (σ, μ). And independent of z,

$$\begin{aligned} & m(z_t) < Mz_t + \bar{z}_{t+1} \text{ since } z_t^{LH} \in (0, \bar{z}_{t+1}) \\ \implies & m(z_t) < Mz_t + 1, \forall z_t \text{ and } m(\bar{z}_t) < (M+1)\bar{z}_t \end{aligned} \tag{39}$$

since $\bar{z}_t$ is decreasing in t. Furthermore, if z is a decreasing sequence, (38) implies that

$$\begin{aligned} & m(z_t) \leq Mz_t + z_t \\ \implies & \frac{m(z_t)}{z_t} \leq M + 1 \end{aligned} \tag{40}$$

Observe that the above condition rules out functions of the form $m(z_t) = c \in (0,1]$ (constant bounds), for example. Given a sequence of z_t, let $t^* = \inf\left\{t \in \mathbb{N} : z_{t^*} \leq \frac{c}{M+1}\right\}$ and suppose $h_{t^*}^b = \{1, 1, ..., 1\}$, then $\mu_{t^*}^b = \mu_o$ so the low type's incentive compatibility constraint to follow the autarky strategy would be violated. In this example, the violation would occur with probability greater than p^{t^*}. Also, note that $t^* \neq \infty$ since $z_t \to 0$ as $t \to \infty$ because a high type grows more pessimistic each period that goes by without an initial favor, and therefore the largest favor she is willing to do to separate must also be decreasing.[2] If $m(z_t)$ is chosen appropriately, for example $m(z_t) = \min\{(M+1)z_t, 1\}$, then low types would have no incentive to mimic high types even if they knew they were facing one with certainty. However, we still need to consider the incentive compatibility constraints for the high type, say, without loss of generality, for agent a

$$ICC_z^H : \left(1 - \delta^H\right)(1 - z_t) + \delta^H u_{z_t}^H \geq \left(1 - \delta^H\right) + \delta^H u_{-z_t}^H,$$

where $u_{z_t}^H$ and $u_{-z_t}^H$ are the expected continuation payoffs for a high type if she does a favor of size z_t and if she does no favor, respectively. Then these payoffs can be broken down into two separate components, the payoffs when facing a high and a low type, respectively,

$$\begin{aligned} u_{z_t}^H &= (1 - \mu_t^a)\, u_{z_t}^{HL} + \mu_t^a u_{z_t}^{HH}, \text{ and} \\ u_{-z_t}^H &= (1 - \mu_t^a)\, u_{-z_t}^{HL} + \mu_t^a u_{-z_t}^{HH}. \end{aligned}$$

Clearly $u_{z_t}^{HL} = p$, the autarky payoff, since σ dictates that agent a, who did the first favor, do no

[2]The initial favors under (σ, μ) that are incentive compatible for a high type could be weakly decreasing at first (sequence of full favors), but must become strictly decreasing once the high type grows sufficiently pessimistic that her incentive compatibility constraint binds.

more favors until reciprocation is received, and that agent b, a low type in this instance, do no favors. If b is a high type, a's expected payoff is $u^{HH}_{z_t} = \bar{u}_{em(m(z_t))} = p + B\ m(z_t)$ per (28). To compute the components of $u^{H}_{-z_t}$ recall that we are considering just one-shot deviations, so next period agent a is assumed to do a favor of size z_{t+1} if she receives a favor opportunity, or she could receive a favor of size z_{t+1} from agent b if b is a high type, and so forth. Namely,

$$\begin{aligned}
u^{HL}_{-z_t} &= p\left(\left(1-\delta^H\right)(1-z_{t+1}) + \delta^H p\right) + (1-p)\delta^H u^{HL}_{-z_{t+1}} \\
&= p\left(1-(1-p)\delta^H\right) - \underbrace{p\left(1-\delta^H\right)}_{\equiv b} z_{t+1} + \underbrace{(1-p)\delta^H}_{\equiv a} u^{HL}_{-z_{t+1}} \\
&= p(1-a) - b z_{t+1} + a\left(p(1-a) - b z_{t+2} + a\left(u^{HL}_{-z_{t+2}}\right)\right),
\end{aligned}$$

iteratively expanding each $v^{HL}_{\cdot}$ and collecting like terms produces the following geometric sums

$$\begin{aligned}
&= p(1-a)\left(1+a+a^2+...\right) - b\left(z_{t+1} + a z_{t+2} + a^2 z_{t+3} + ...\right) \\
&= p\tfrac{1-a}{1-a} - b\textstyle\sum_{i=0}^{\infty} a^i z_{t+1+i} = p - p\left(1-\delta^H\right) S^{HL},
\end{aligned}$$

where $S^{HL} = \sum_{i=0}^{\infty}\left((1-p)\delta^H\right)^i z_{t+1+i}$.

Performing a similar calculation for $u^{HH}_{-z_t}$ yields

$$\begin{aligned}
u^{HH}_{-z_t} &= p\left(\left(1-\delta^H\right)(1-z_{t+1}) + \delta^H \bar{u}_{em(m(z_{t+1}))}\right) \\
&\quad + p\left(\left(1-\delta^H\right) k z_{t+1} + \delta^H \underline{u}_{em(m(z_{t+1}))}\right) + (1-2p)\delta^H u^{HH}_{-z_{t+1}} \\
&= p\left(1-\delta^H\right) + p\delta^H\left(\bar{u}_{em(m(z_{t+1}))} + \underline{u}_{em(m(z_{t+1}))}\right) \\
&\quad + p\left(1-\delta^H\right)(k-1)z_{t+1} + (1-2p)\,\delta^H u^{HH}_{-z_{t+1}} \\
&= p\left(1-\delta^H\right) + p\delta^H\left(2p + p(k-1)m(z_{t+1})\right) \\
&\quad + p\left(1-\delta^H\right)(k-1)z_{t+1} + (1-2p)\,\delta^H u^{HH}_{-z_{t+1}} \text{ by (29)} \\
&= p\left(1-\delta^H(1-2p)\right) \\
&\quad + \underbrace{p(k-1)}_{\equiv b}\underbrace{\left(p\delta^H m(z_{t+1}) + \left(1-\delta^H\right) z_{t+1}\right)}_{\equiv n(z_{t+1})} + \underbrace{(1-2p)\delta^H}_{\equiv a} u^{HH}_{-z_{t+1}} \\
&= p(1-a) + b n(z_{t+1}) \\
&\quad + a\left(p(1-a) + b\, n(z_{t+2}) + a\, u^{H}_{-z_{t+2}}\right).
\end{aligned}$$

Iteratively expanding each $u^{H}_{\cdot}$ and collecting like terms produces the following geometric sums

$$\begin{aligned}
&= a\left(1+a+a^2+...\right) + b\left(n(z_{t+1}) + a\, n(z_{t+2}) + a^2 n(z_{t+3}) + ...\right) \\
&= p\tfrac{1-a}{1-a} + b\textstyle\sum_{i=0}^{\infty} a^i n(z_{t+1+i}) = p + p(k-1)S^{HH}, \text{ where}
\end{aligned}$$

$$\begin{aligned}
S^{HH} &= \textstyle\sum_{i=0}^{\infty}\left((1-2p)\delta^H\right)^i n(z_{t+1+i}) \\
&= \textstyle\sum_{i=0}^{\infty}\left((1-2p)\delta^H\right)^i \left(p\delta^H m(z_{t+1+i}) + \left(1-\delta^H\right) z_{t+1+i}\right)
\end{aligned}$$

Putting the pieces together, we have

$$u^H_{z_t} = (1-\mu^a_t)\,p + \mu^a_t\,(p + Bm\,(z_t))\,\text{, and}$$
$$u^H_{-z_t} = (1-\mu^a_t)\left(p - p\left(1-\delta^H\right)S^{HL}\right) + \mu^a_t\left(p + p(k-1)S^{HH}\right).$$

Now rearranging ICC^H_z and plugging in the above values for $u^H_{z_t}$ and $u^H_{-z_t}$ we have

$$\begin{aligned} z_t &\le \tfrac{\delta^H}{1-\delta^H}\left(u^H_{z_t} - u^H_{-z_t}\right) \\ &= \tfrac{\delta^H}{1-\delta^H}(1-\mu^a_t)\left(p - \left(p - p\left(1-\delta^H\right)S^{HL}\right)\right) \\ &+ \tfrac{\delta^H}{1-\delta^H}\mu^a_t\left(p + B\,m\,(z_t) - \left(p + p(k-1)S^{HH}\right)\right) \\ &= (1-\mu^a_t)\,p\,\delta^H S^{HL} + \mu^a_t\tfrac{\delta^H}{1-\delta^H}\left(B\,m\,(z_t) - p\,(k-1)\,S^{HH}\right). \end{aligned} \tag{41}$$

Because $\bar{z}_t$ is decreasing,

$$p\delta^H S^{HL} = p\delta^H \textstyle\sum_{i=0}^{\infty}\left((1-p)\delta^H\right)^i z_{t+1+i} \le p\delta^H\sum_{i=0}^{\infty}\left((1-p)\delta^H\right)^i \bar{z}_t = \frac{p\delta^H}{1-\delta^H(1-p)}\bar{z}_t.$$

And because $S^{HH} > 0$,

$$z_t \le (1-\mu^a_t)\,\tfrac{p\delta^H}{1-\delta^H(1-p)}\bar{z}_t + \mu^a_t\tfrac{\delta^H B}{1-\delta^H}m\,(z_t) \le (1-\mu^a_t)\,\tfrac{p\delta^H}{1-\delta^H(1-p)}\bar{z}_t + \mu^a_t\tfrac{\delta^H B}{1-\delta^H}m\,(\bar{z}_t)\,.$$

The last inequality must hold for all t and all $z_t \in \left[\underline{z}_t, \bar{z}_t\right]$, otherwise we would have a contradiction to our initial supposition, and the proof would be over. So let $z_t = \bar{z}_t$, then the following must hold

$$\begin{aligned} \bar{z}_t &\le (1-\mu^a_t)\,\tfrac{p\delta^H}{1-\delta^H(1-p)}\bar{z}_t + \mu^a_t\tfrac{\delta^H B}{1-\delta^H}m\,(\bar{z}_t) \\ &< c\bar{z}_t + \mu^a_t d\,m\,(z_t)\ \text{ for } c \equiv \tfrac{\delta^H p}{1-\delta^H(1-p)}, d \equiv \tfrac{\delta^H B}{1-\delta^H} \\ \Longrightarrow \tfrac{1-c}{d} &< \mu^a_t\tfrac{m(z_t)}{z_t} \le \mu^a_t M \text{ by (9).} \end{aligned} \tag{42}$$

It is easy to verify that $\frac{1-c}{d} > 0$. Namely from (28),

$$\begin{aligned} &d = \tfrac{\delta^H}{1-\delta^H}B = \tfrac{\delta^H}{1-\delta^H}p\tfrac{(1-\delta^H)k+\delta^H p(k-1)}{1-\delta^H(1-2p)} > 0 \text{ and } \delta^H < 1 \\ &\Longrightarrow \delta^H p < 1-\delta^H(1-p) \Longrightarrow c = \tfrac{\delta^H p}{1-\delta^H(1-p)} < 1 \\ &\therefore \tfrac{1-c}{d} > 0. \end{aligned}$$

Furthermore, $\frac{1-c}{d}$ and M are fixed while $\mu^a_t \to 0$ in probability as $t \to \infty$ unless separation takes place, so we know that with some positive probability condition (42) will be violated producing a contradiction to our supposition that (σ, μ) was a PBE. To compute the exact time we need the following lemma.

Lemma 10a: Let s denote the number of periods agent i has received neither a favor opportunity, nor a favor from the other agent, then

$$\mu_s^i = \frac{\mu_o(1-2p)^s}{\mu_o\left(1-2p\right)^s + \left(1-\mu_o\right)\left(1-p\right)^s}. \tag{43}$$

Proof of lemma 10a: For $s = 1$, we know that

$$\mu_1^i = \frac{\mu_o(1-2p)}{1-\left(1+\mu_o\right)p} = \frac{\mu_o(1-2p)^1}{\mu_o(1-2p)^1 + \left(1-\mu_o\right)\left(1-p\right)^1}$$

Suppose (43) for some $s \in \mathbb{N}$, then

$$\begin{aligned}
\mu_{s+1}^i &= \tfrac{\mu_s^i(1-2p)}{1-(1+\mu_s^i)p} = \tfrac{\mu_s^i(1-2p)}{1-p-p\mu_s^i} \text{ per (\textbf{??} (iv))} \\
&= \frac{\frac{\mu_o(1-2p)^s}{\mu_o(1-2p)^s+(1-\mu_o)(1-p)^s}(1-2p)}{1-p-p\frac{\mu_o(1-2p)^s}{\mu_o(1-2p)^s+(1-\mu_o)(1-p)^s}} \\
&= \frac{\mu_o(1-2p)^{s+1}}{(1-p)(\mu_o(1-2p)^s+(1-\mu_o)(1-p)^s)-p\mu_o(1-2p)^s} \\
&= \frac{\mu_o(1-2p)^{s+1}}{(1-2p)\mu_o(1-2p)^s+(1-p)(1-\mu_o)(1-p)^s} \\
&= \frac{\mu_o(1-2p)^{s+1}}{\mu_o(1-2p)^{s+1}+(1-\mu_o)(1-p)^{s+1}}
\end{aligned}$$

So if (43) holds for some $s \in \mathbb{N}$, it holds for $s+1$. We know it holds for $s = 1$, therefore by induction it must hold for all $s \in \mathbb{N}$. Let $t^* := \inf\left\{t \in \mathbb{N} : \frac{\mu_o(1-2p)^{t-1}}{\mu_o(1-2p)^{t-1}+(1-\mu_o)(1-p)^{t-1}} < \frac{1-a}{bM}\right\}$ and suppose $h_{t^*}^a = \{0, ..., 0, 1\}$ and $H_{t^*} = \{0, ..., 0\}$, then $\mu_{t^*}^a = \frac{\mu_o(1-2p)^{t^*-1}}{\mu_o(1-2p)^{t^*-1}+(1-\mu_o)(1-p)^{t^*-1}} < \frac{1-a}{bM}$ by (43) and by definition of t^* so that the incentive compatibility constraint for the high type is violated by period t^* if she does not receive any favor opportunities or favors before then. Hence separation has to occur within t^*-1 or fewer periods or it will never take place. The belief function follows from Bayesian updating and must have this form given a history of no favor opportunities until now and no favors in order for μ to be consistent with σ. Thus separation cannot be guaranteed with probability 1 in any SS equilibria. ■

Proof. (Lemma 13: SS equilibria with symmetric beliefs) From ICC_z^L in the proof of proposition 10 we have that for the low type agents not to mimic the high type agents z_t must satisfy

$$z_t \geq \mu_t^b \frac{\delta^L pk}{1-\delta^L(1-p)} m(z_t),$$

and from ICC_z^H we have that for a high type agent to do the first favor, z_t must satisfy

$$\begin{aligned}
z_t &\leq \frac{\delta^H}{1-\delta^H}\left(1-\mu_t^a\right)S_1 + \frac{\delta^H}{1-\delta^H}\mu_t^a\left(Bm\left(z_t\right)-S_2\right), \text{ where} \\
S_1 &= p\left(1-\delta^H\right)\sum_{i=0}^{\infty}\left(\delta^H(1-p)\right)^i z_{t+1+i},
\end{aligned}$$

$$S_2 = p(k-1)\sum_{i=0}^{\infty}\left(\delta^H(1-2p)\right)^i\left(\left(1-\delta^H\right)z_{t+1+i} + p\delta^H m\left(z_{t+1+i}\right)\right),$$

$$\text{and } B = p\frac{(1-\delta^H)k + \delta^H p(k-1)}{1-\delta^H(1-2p)} \text{ from (28)}$$

Combining the two conditions above, rearranging and using $\mu_t^a = \mu_t^b = \mu_t$ yields the following condition that must be satisfied as t grow large for separation to be guaranteed:

$$\mu_t\tfrac{\delta^L pk}{1-\delta^L(1-p)}m(z_t) \le \tfrac{\delta^H}{1-\delta^H}\left((1-\mu_t)S_1 - \mu_t S_2\right) + \tfrac{\delta^H}{1-\delta^H}\mu_t B\, m(z_t). \tag{44}$$

First we need to show that $(1-\mu_t)S_1 - \mu_t S_2 \ge 0$ for t large enough. To this end, observe that

$$\begin{aligned}
(1-\mu_t)S_1 - \mu_t S_2 &= (1-\mu_t)p\left(1-\delta^H\right)\textstyle\sum_{i=0}^{\infty}\left(\delta^H(1-p)\right)^i z_{t+1+i} \\
&\quad - \mu_t p(k-1)\textstyle\sum_{i=0}^{\infty}\left(\delta^H(1-2p)\right)^i\left(\left(1-\delta^H\right)z_{t+1+i} + p\delta^H m\left(z_{t+1+i}\right)\right) \\
&\ge 1-\mu_t)p\left(1-\delta^H\right)\textstyle\sum_{i=0}^{\infty}\left(\delta^H(1-p)\right)^i z_{t+1+i} \\
&\quad - \mu_t p(k-1)\left(1-\delta^H(1-pM)\right)\textstyle\sum_{i=0}^{\infty}\left(\delta^H(1-p)\right)^i z_{t+1+i} \\
&\quad \text{since } \left(\delta^H(1-p)\right)^i \ge \left(\delta^H(1-2p)\right)^i, \forall i \ge 0, \\
&\quad \text{and } m\left(z_{t+1+i}\right) \le M z_t \text{ by (9)} \\
&= \left((1-\mu_t)\,p\left(1-\delta^H\right) - \mu_t p(k-1)\left(1-\delta^H(1-pM)\right)\right)S_{t+1}, \\
&\quad \text{where } S_{t+1} = \textstyle\sum_{i=0}^{\infty}\left(\delta^H(1-p)\right)^i z_{t+1+i} \\
&\ge 0 \text{ for } t \text{ large enough } \because \mu_t \to 0 \text{ as } t \to \infty \text{ while} \\
&\quad p\left(1-\delta^H\right), p\left(k-1\right)\left(1-\delta^H(1-pM)\right) > 0 \text{ are bounded.}
\end{aligned}$$

Therefore to verify that condition (44) holds it is enough to verify that

$$\mu_t\frac{\delta^L pk}{1-\delta^L(1-p)}m\left(z_t\right) \le \frac{\delta^H}{1-\delta^H}\mu_t B m\left(z_t\right)$$

or after dividing both sides by $\mu_t m\left(z_t\right)$ and substituting in $B = p\frac{(1-\delta^H)k+\delta^H p(k-1)}{1-\delta^H(1-2p)}$ that

$$\frac{\delta^L pk}{1-\delta^L(1-p)} \le \frac{\delta^H}{1-\delta^H}p\frac{(1-\delta^H)k + \delta^H p(k-1)}{1-\delta^H(1-2p)}.$$

The left-hand side is clearly increasing in δ^L and $\delta^L < \delta^* = \frac{1}{1+p(k-1)}$ by definition, so the left-hand side is bounded from above by

$$\frac{\delta^* pk}{1-\delta^*(1-p)} = \frac{\frac{1}{1+p(k-1)}pk}{1-\frac{1}{1+p(k-1)}(1-p)} = 1.$$

As for the right-hand side, let $h\left(\delta^H\right) = \frac{\delta^H}{1-\delta^H} p \frac{(1-\delta^H)k+\delta^H p(k-1)}{1-\delta^H(1-2p)}$. Then $h\left(\delta^*\right) = 1$ and

$$\frac{dh\left(\delta^H\right)}{d\delta^H} = \frac{\left(1+(k-1)p\right)^2 k}{p(k^2-1)} > 0 \text{ at } \delta^H = \delta^*, \text{ and}$$
$$\frac{d^2 h\left(\delta^H\right)}{d\left(\delta^H\right)^2} = p\left[\frac{k-1}{\left(1-\delta^H\right)^3} + \frac{k+1}{\delta^H\left(1-\delta^H(1-2p)\right)^3} - \frac{k+1}{\delta^H\left(1-\delta^H(1-2p)\right)^2}\right]$$
$$> 0 \text{ since } k > 1 \text{ and } 1-\delta^H(1-2p) \in (0,1).$$

In other words, $h\left(\delta^H\right)$ or the right-hand side of our constraint is convex and increasing at its minimum of $h\left(\delta^*\right) = 1$, which was an upper bound for the left-hand side of the constraint. Hence condition (44) is always satisfied for t large enough, and therefore a sequence of z_t would exist such that separation could be guaranteed if the agents believed that their favor opportunities were independent while the game remained exactly as before otherwise. Of course, if the favor opportunities were truly independent then the structure of the game would change significantly because the model uses discrete time and we would have to address the possibility that both agents received a favor opportunity in the same round, which would change the incentive compatibility constraints and make it difficult to properly compare the model with mutually exclusive favor opportunities. However, this thought exercise suggests that non-independent favor opportunities make it harder to separate symmetrically. ■

Proof. (Lemma 14: EM without initial designations) Once agent designations are determined, the game will be exactly as before in lemma 6. In particular, the payoffs and incentive compatibility constraints are the same as before. Therefore we can focus on the initial stage of the game before designations are determined. Let $\hat{u}^z_{em}$ denote the expected payoff for an agent with no designation. Then,

$$\begin{aligned}
\hat{u}^z_{em} &= p^2\left(\left(1-\delta^H\right)(1+(k-1)z) + \delta^H \hat{u}^z_{em}\right) \\
&\quad + p(1-p)\left(\left(1-\delta^H\right)(1-z) + \delta^H \bar{u}_{em}\right) \\
&\quad + (1-p)\left(p\left(\left(1-\delta^H\right)kz + \delta^H \underline{u}_{em}\right) + (1-p)\delta^H \hat{u}^z_{em}\right) \\
&= \frac{(1-\delta^H)p(1-z+kz)+\delta^H(p-p^2)(\underline{u}_{em}+\bar{u}_{em})}{1-\delta^H+2\delta^H p-2\delta^H p^2} \\
&= \frac{(1-\delta^H)p(1-z+kz)+\delta^H(p-p^2)p(k-1)}{1-\delta^H+2\delta^H p-2\delta^H p^2} \\
&= p + \underbrace{\frac{\delta^H(1-p)p^2(k-1)}{1-\delta^H+2\delta^H p(1-p)}}_{\equiv C_1} + \underbrace{\frac{(1-\delta^H)p(k-1)}{1-\delta^H+2\delta^H p(1-p)}}_{\equiv C_2} z. \qquad (45)
\end{aligned}$$

The incentive compatibility constraint for an undesignated agent to do a favor of size z is

$$\begin{aligned}
ICC^{nd}_{em} : \, & p\left(\left(1-\delta^H\right)(1+(k-1)\,z) + \delta^H \hat{u}^z_{em}\right) + (1-p)\left(\left(1-\delta^H\right)(1-z) + \delta^H \bar{u}_{em}\right) \\
& \geq p\left(\left(1-\delta^H\right)(1+kz) + \delta^H \underline{u}_{em}\right) + (1-p)\left(\left(1-\delta^H\right) + \delta^H \hat{u}^z_{em}\right) \\
& \iff p\left(\left(1-\delta^H\right)(-z) + \delta^H\left(\hat{u}^z_{em} - \underline{u}_{em}\right)\right) \\
& + (1-p)\left(\left(1-\delta^H\right)(-z) + \delta^H\left(\bar{u}_{em} - \hat{u}^z_{em}\right)\right) \geq 0
\end{aligned}$$

$$\iff p\delta^H\left(\hat{u}^z_{em}-\underline{u}_{em}\right)+(1-p)\,\delta^H\left(\bar{u}_{em}-\hat{u}^z_{em}\right)-\left(1-\delta^H\right)z\geq 0$$
$$\iff p\delta^H\left(\hat{u}^z_{em}-\underline{u}_{em}\right)+(1-p)\,\delta^H\left(\bar{u}_{em}-\hat{u}^z_{em}\right)-\left(1-\delta^H\right)z\geq 0$$

substituting in for $\hat{u}^z_{em}, \underline{u}_{em}$ and $\bar{u}_{em}$ from (45), (26) and (28) yields

$$p\delta^H\left(p+C_1+C_2z-p-A\right)+(1-p)\delta^H\left(p+B-p-C_1-C_2z\right)-\left(1-\delta^H\right)z\geq 0$$
$$\iff p\delta^H\left(C_1+C_2z-A\right)+(1-p)\delta^H\left(B-C_1-C_2z\right)-\left(1-\delta^H\right)z\geq 0$$

Solving for δ^H from the above constraint is complicated, but we can we solve for z and show that $z\leq 1/2$, is required when $\delta^H=\delta^*$. In particular, solving for z from the previous inequality yields

$$z\leq\frac{\delta^H p\left(p(1-\delta^H(1-p))+k\left(1-\delta^H-p+3\delta^H p-3\delta^H p^2\right)\right)}{(1-\delta^H(1-2p))(1-\delta^H(1-(1+k)p+2kp^2))}\leq 1/2 \text{ for } \delta^H=\delta^*.$$

It follows immediately that the upper bound on z is increasing with δ^H since the more patient agents are the higher cost they will pay for tomorrow's continuation promise. ■

Proof. (Proposition 16: SS equilibria with independent favor opportunities) The incentive compatibility constraint for a high type to do a favor of size z_t in period t if $H_{t-1}=\{0,0,...,0\}$ is provided below.

$$\begin{aligned}
ICC^H_{iss}\colon &\left(1-\delta^H\right)(1-z_t)+(1-\mu_t)\,\delta^H p\\
&+\mu_t\left(p\left(\left(1-\delta^H\right)kz_t+\delta^H\hat{u}_{em}\right)+(1-p)\delta^H\bar{u}_{em}\right)\\
&\geq 1-\delta^H+(1-\mu_t)\,\delta^H\hat{u}^{HL}_{-z_t}\\
&+\mu_t\left(p\left(\left(1-\delta^H\right)kz_t+\delta^H\underline{u}_{em}\right)+(1-p)\delta^H\hat{u}^H_{-z_t}\right)\\
&\Longrightarrow\frac{\delta^H}{1-\delta^H}\left(\mu_t p\left(\hat{u}_{em}-\underline{u}_{em}\right)\right.\\
&\left.+\mu_t(1-p)\left(\bar{u}_{em}-\hat{u}^{HH}_{-z_t}\right)+(1-\mu_t)\left(p-\hat{u}^{HL}_{-z_t}\right)\right)\geq z_t,
\end{aligned}\tag{46}$$

where $\mu^i_t=\mu_t$ for both agents regardless of type since neither one has done a favor yet and private favor opportunities are uninformative about the type of the other agent since favor opportunities are independent. Payoffs $\hat{u}^{HH}_{-z_t}$ and $\hat{u}^{HL}_{-z_t}$ denote the continuation values implemented by (σ,μ) for a high type agent who did not do or receive a favor in period t facing a high type and a low type, respectively. As for the low type, the following incentive compatibility constraint has to be satisfied for her not to mimic a high type.

$$\begin{aligned}
ICC^L_{iss}:&\left(1-\delta^L\right)(1-z_t)\\
&+\mu_t\left(p\left(\left(1-\delta^L\right)kz_t+\delta^L\bar{u}^L_{em}\right)+(1-p)\delta^L\bar{u}^L_{em}\right)+(1-\mu_t)\,\delta^L p\\
&\leq\left(1-\delta^L\right)+\mu_t\left(p\left(\left(1-\delta^L\right)kz_t+\delta^L p\right)+(1-p)\delta^L\hat{u}^{LH}_{-z_t}\right)+(1-\mu_t)\,\delta^L p\\
&\Longrightarrow\frac{\delta^L}{1-\delta^L}\mu_t\left(p\left(\bar{u}^L_{em}-p\right)+(1-p)\left(\bar{u}^L_{em}-\hat{u}^{LH}_{-z_t}\right)\right)\leq z_t,
\end{aligned}\tag{47}$$

where $\bar{u}^L_{em}$ denotes the expected payoff to an advantaged low type facing a high type in an EM game of full trust, and $\hat{u}^{LH}_{-z_t}$ denotes the continuation payoff a low type if she is facing a high type in period t. That is, her autarky payoff and one time small favor as soon as the high type receives a favor opportunity. Per inequalities (46) and (47), it is necessary to prove that

$$\begin{aligned}\tfrac{\mu_t\delta^H}{1-\delta^H}\left(p\left(\hat{u}_{em}-\underline{u}_{em}\right)+(1-p)\left(\bar{u}_{em}-\hat{u}^{HH}_{-z_t}\right)\right)+\tfrac{(1-\mu_t)\delta^H}{1-\delta^H}\left(p-\hat{u}^{HL}_{-z_t}\right)\\ \geq \tfrac{\mu_t\delta^L}{1-\delta^L}\left(p\left(\bar{u}^L_{em}-p\right)+(1-p)\left(\bar{u}^L_{em}-\hat{u}^{LH}_{-z_t}\right)\right),\forall t\end{aligned} \tag{48}$$

and for $z_t \in (0,1]$.First, we need to solve for $\hat{u}^{HH}_{-z_t}$, $\hat{u}^{HL}_{-z_t}$, $\bar{u}^L_{em}$ and $\hat{u}^{LH}_{-z_t}$.

$$\begin{aligned}\bar{u}^L_{em} &= p\left(1-\delta^L\right)+p\left(\left(1-\delta^L\right)\tfrac{1}{2}k+\delta^L p\right)+(1-p)\delta^L\bar{u}^L_{em}\\ &= p+\tfrac{(1-\delta^L)pk}{1-\delta^L(1-p)}.\end{aligned} \tag{49}$$

Repeating the calculation for $\hat{u}^{LH}_{-z_t}$:

$$\begin{aligned}\hat{u}^{LH}_{-z_t} &= p\left(1-\delta^L\right)+p\left(\left(1-\delta^L\right)kz_{t+1}+\delta^L p\right)+\delta^L(1-p)\,\hat{u}^{LH}_{-z_{t+1}}\\ &= \underbrace{p\left(1-\delta^L(1-p)\right)}_{\equiv a_1}+\underbrace{p\left(1-\delta^L\right)k}_{\equiv a_2}z_{t+1}+\underbrace{\delta^L(1-p)}_{\equiv d_L}\hat{u}^{LH}_{-z_{t+1}}\\ &= a_1+a_2 z_{t+1}+d_L\left(a_1+a_2 z_{t+2}+d_L\hat{u}^{LH}_{-z_{t+2}}\right)\\ &= a_1\left(1+d_L+d_L^2+...\right)+a_2\left(z_{t+1}+d_L z_{t+2}+d_L^2 z_{t+3}+...\right)\\ &= \tfrac{a_1}{1-d_L}+a_2\sum_{i=0}^{\infty}d_L^i z_{t+1+i}=\tfrac{p\left(1-\delta^L(1-p)\right)}{1-\delta^L(1-p)}+a_2\sum_{i=0}^{\infty}d_L^i z_{t+1+i}\\ &= p+\left(1-\delta^L\right)S_{iLH}\text{ for }S_{iLH}\equiv pk\sum_{i=0}^{\infty}\left(\delta^L(1-p)\right)^i z_{t+1+i}.\end{aligned} \tag{50}$$

The math remains the same for $\hat{u}^{HL}_{-z_t}$:

$$\begin{aligned}\hat{u}^{HL}_{-z_t} &= p\left(\left(1-\delta^H\right)(1-z_{t+1})+\delta^H p\right)+\delta^H(1-p)\hat{u}^{HL}_{-z_{t+1}}\\ &= \underbrace{p\left(1-\delta^H(1-p)\right)}_{\equiv b_1}-\underbrace{p\left(1-\delta^H\right)}_{\equiv b_2}z_{t+1}+\underbrace{\delta^H(1-p)}_{\equiv d_{hl}}\hat{u}^{HL}_{-z_{t+1}}\\ &= b_1-b_2 z_{t+1}+d_{hl}\left(b_1-b_2 z_{t+2}+d_{hl}u^{HL}_{-z_{t+2}}\right)\\ &+ b_1\left(1+d_{hl}+d_{hl}^2+...\right)-b_2\left(z_{t+1}+d_{hl}z_{t+2}+d_{hl}^2 z_{t+3}+...\right)\\ &= \tfrac{b_1}{1-d_{hl}}-b_2\sum_{i=0}^{\infty}d_{hl}^i z_{t+1+i}=\tfrac{p\left(1-\delta^H(1-p)\right)}{1-\delta^H(1-p)}-b_2\sum_{i=0}^{\infty}d_{hl}^i z_{t+1+i}\\ &= p-\left(1-\delta^H\right)S_{iHL}\text{ for }S_{iHL}\equiv p\sum_{i=0}^{\infty}\left(\delta^H(1-p)\right)^i z_{t+1+i}.\end{aligned} \tag{51}$$

And finally for $\hat{u}^{HH}_{-z_t}$:

$$\hat{u}^{HH}_{-z_t} = p^2\left(\left(1-\delta^H\right)(1+(k-1)z_{t+1})+\delta^H\hat{u}_{em}\right)$$

$$
\begin{aligned}
&+ p(1-p)\left(\left(1-\delta^H\right)(1-z_{t+1}) + \delta^H \bar{u}_{em}\right) \\
&+ (1-p)p\left(\left(1-\delta^H\right) k z_{t+1} + \delta^H \underline{u}_{em}\right) + (1-p)^2 \delta^H \hat{u}^{HH}_{-z_t} \\
&= p^2\left(\left(1-\delta^H\right) + \delta^H \hat{u}_{em}\right) + p(1-p)\left(\left(1-\delta^H\right) + \delta^H\left(\bar{u}_{em} + \underline{u}_{em}\right)\right) \\
&+ p^2\left(1-\delta^H\right)(k-1)z_{t+1} + p\left(1-p\right)\left(1-\delta^H\right)(k-1)z_{t+1} + (1-p)^2\delta^H \hat{u}^{HH}_{-z_t} \\
&= p\left(1-\delta^H\right) + p\delta^H\left(\bar{u}_{em} + \underline{u}_{em}\right) + p^2\delta^H\left(\hat{u}_{em} - \left(\bar{u}_{em} + \underline{u}_{em}\right)\right) \\
&+ p\left(1-\delta^H\right)(k-1)z_{t+1} + (1-p)^2\delta^H \hat{u}^{HH}_{-z_t}.
\end{aligned}
$$

Recall from (28) that $\bar{u}_{em} + \underline{u}_{em} = p\left(k+1\right)$, so

$$
\begin{aligned}
\hat{u}^{HH}_{-z_t} &= p\left(1-\delta^H\right) + p\delta^H p\left(k+1\right) + p^2\delta^H(\hat{u}_{em} - p(k+1)) \\
&+ (1-\delta^H)p(k-1)z_{t+1} + (1-p)^2\delta^H \hat{u}^{HH}_{-z_t} \\
&= \underbrace{p\left(1-\delta^H\right) + p^2\delta^H\left(\left(k+1\right)\left(1-p\right) + \hat{u}_{em}\right)}_{\equiv c_1} \\
&+ \underbrace{(1-\delta^H)p(k-1)}_{\equiv c_2} z_{t+1} + \underbrace{\delta^H(1-p)^2}_{\equiv d_{hh}} \hat{u}^H_{-z_t} \\
&= c_1 + c_2 z_{t+1} + d_{hh}\left(c_1 + c_2 z_{t+2} + d_{hh} u^{HL}_{-z_{t+2}}\right) \\
&= c_1\left(1 + d_{hh} + d_{hh}^2 + ...\right) + c_2\left(z_{t+1} + d_{hh}\, z_{t+2} + d_{hh}^2 z_{t+3} + ...\right) \\
&= \tfrac{c_1}{1-d_{hh}} + c_2 \textstyle\sum_{i=0}^{\infty} d_{hh}^i z_{t+1+i} = p + \left(1-\delta^H\right)\alpha + (1-\delta^H)S_{iHH} \qquad (52)
\end{aligned}
$$

for $S_{iHH} \equiv p(k-1)\sum_{i=0}^{\infty}\left(\delta^H(1-p)^2\right)^i z_{t+1+i}$, and

$$\alpha \equiv \frac{\frac{c_1}{1-d_{hh}} - p}{(1-\delta^H)} = \frac{\delta^H(k-1)p^2(1-\delta^H(1-p(2-(3-p)p)))}{(1-\delta^H)(1-\delta^H(1-p)^2)(1-\delta^H(1-2(1-p)p))}$$

We are now ready to return back to inequality (48) that needs to hold for the result to hold. Take all the terms to the left side and call the resultant function, Q.

$$
\begin{aligned}
Q :=& \tfrac{\mu_t\delta^H}{1-\delta^H}\left(p\left(\hat{u}_{em} - \underline{u}_{em}\right) + (1-p)\left(\bar{u}_{em} - \hat{u}^{HH}_{-z_t}\right)\right) + \tfrac{(1-\mu_t)\delta^H}{1-\delta^H}\left(p - \hat{u}^{HL}_{-z_t}\right) \\
&- \tfrac{\mu_t\delta^L}{1-\delta^L} p\left(\bar{u}^L_{em} - p\right) - \tfrac{\mu_t\delta^L}{1-\delta^L}(1-p)\left(\bar{u}^L_{em} - \hat{u}^{LH}_{-z_t}\right).
\end{aligned}
$$

Substituting in for payoffs from (26), (28), (52), (51), (49), (50) yields

$$
\begin{aligned}
Q =& \mu_t\delta^H p \tfrac{p\left(k\left(1-\delta^H(1-p)^2\right) - \delta^H p^2\right)}{(1-\delta^H(1-2p))(1-\delta^H(1-2(1-p)p))} \\
&+ \mu_t\delta^H \tfrac{p(1-p)}{2}\left(\tfrac{2(k-1)}{1-\delta^H(1-p)^2} + \tfrac{k+1}{1-\delta^H+2\delta^H p} - \tfrac{k-1}{1-\delta^H(1-2(1-p)p)}\right) \\
&- \mu_t\delta^H p(1-p)(k-1)\textstyle\sum_{i=0}^{\infty}\left(\delta^H(1-p)^2\right)^i z_{t+1+i} \\
&+ \left(1-\mu_t\right)\delta^H p \textstyle\sum_{i=0}^{\infty}\left(\delta^H(1-p)\right)^i z_{t+1+i} - \tfrac{\mu_t\delta^L pk}{1-\delta^L(1-p)} \\
&+ \mu_t\delta^L pk(1-p)\textstyle\sum_{i=0}^{\infty}\left(\delta^L(1-p)\right)^i z_{t+1+i}.
\end{aligned}
$$

Observe that for converging series there exists $z_\omega = \frac{\sum d_\omega^i z_i}{\sum d_\omega^i}$ and z_i decreasing, then $d_1 \leq d_2 \implies z_1 \geq z_2$ so we can write Q as,

$$\begin{aligned}Q = &\tfrac{\mu_t \delta^H p^2 \left(k\left(1-\delta^H(1-p)^2\right)-\delta^H p^2\right)}{(1-\delta^H(1-2p))(1-\delta^H(1-2(1-p)p))} \\ &+ \tfrac{\mu_t \delta^H p(1-p)}{2}\left(\tfrac{2(k-1)}{1-\delta^H(1-p)^2} + \tfrac{k+1}{1-\delta^H+2\delta^H p} - \tfrac{k-1}{1-\delta^H(1-2(1-p)p)}\right) \\ &- \tfrac{\mu_t \delta^L pk}{1-\delta^L(1-p)} + \tfrac{\mu_t \delta^L pk(1-p)}{1-\delta^L(1-p)} z_L - \tfrac{\mu_t \delta^H p(1-p)(k-1)}{1-\delta^H(1-p)^2}\bar{z}_H + \tfrac{(1-\mu_t)\delta^H p}{1-\delta^H(1-p)}\underline{z}_H\end{aligned}$$

where

$$\begin{aligned}z_L &= \tfrac{\sum_{i=0}^{\infty}\left(\delta^L(1-p)\right)^i z_{t+1+i}}{\sum_{i=0}^{\infty}(\delta^L(1-p))^i} \\ &= \left(1-\delta^L(1-p)\right)\textstyle\sum_{i=0}^{\infty}\left(\delta^L(1-p)\right)^i z_{t+1+i} \in (0, z_t) \qquad (53)\\ \bar{z}_H &= \left(1-\delta^H(1-p)^2\right)\textstyle\sum_{i=0}^{\infty}\left(\delta^H(1-p)^2\right)^i z_{t+1+i} \in (0, z_t) \qquad (54)\\ \underline{z}_H &= \left(1-\delta^H(1-p)\right)\textstyle\sum_{i=0}^{\infty}\left(\delta^H(1-p)\right)^i z_{t+1+i} \in (0, z_t) \qquad (55)\end{aligned}$$

observe that $\bar{z}_H > \underline{z}_H$ and $0 < z_L, \underline{z}_H, \bar{z}_H \leq 1$.

Next we show that Q is decreasing in δ^L and increasing in δ^H.

$$\frac{\partial Q}{\partial \delta^L} = -\mu_t \tfrac{pk(1-(1-p)z_L)}{(1-\delta^L(1-p))^2} < 0 \text{ since } z_L \in (0,1).$$

Since we want to prove that $Q \geq 0$ for all values of $\delta^L \in (0, \delta^*)$, and Q is decreasing in δ^L, it is enough to prove that $Q \geq 0$ for $\delta^L = \delta^*$. Next we show that Q is increasing in δ^H.

$$\begin{aligned}\tfrac{\partial Q}{\partial \delta^H} &= \frac{1}{2}p\left(-\mu_t \tfrac{(k-1)(1-2p)}{(1-\delta^H+2\delta^H p-2\delta^H p^2)^2} + \mu_t \tfrac{(k+1)}{(1-\delta^H(1-2p))^2}\right. \\ &\quad \left.+ \mu_t \tfrac{2(k-1)(1-p)(1-\bar{z}_H)}{(1-\delta^H(1-p)^2)^2} + (1-\mu_t)\tfrac{2}{(1-\delta^H(1-p))^2}\underline{z}_H\right) \\ &= \frac{1}{2}p\mu_t(\overbrace{\tfrac{k+1}{(1-\delta^H(1-2p))^2} - \tfrac{(k-1)(1-2p)}{(1-\delta^H(1-2p+2p^2))^2}}^{>0 \text{ per claim 6}} \\ &\quad + \mu_t \tfrac{2(k-1)(1-p)(1-\bar{z}_H)}{(1-\delta^H(1-p)^2)^2} + (1-\mu_t)\tfrac{2}{(1-\delta^H(1-p))^2}\underline{z}_H) > 0. \qquad (56)\end{aligned}$$

Since Q is increasing in δ^H, it is enough to prove that $Q \geq 0$ for $\delta^H = \underline{\delta}_1^H$. To that end, let $Q^* = Q$ s.t. $\delta^L = \delta^*$ and $\delta^H = \underline{\delta}_1^H = \frac{1}{1-2p+p^2(k+1)}$. Then

$$Q^* = \frac{\mu_t(1-p)(k-1)}{pk}(1-\bar{z}_H) + \frac{1-\mu_t}{p(k+1)-1}\underline{z}_H + \mu_t(1-p)z_L > 0.$$

Recall that condition (17) implies that $p(k+1) \geq 2$, so the second term of Q^* is positive and bounded, and the $\bar{z}_H, \underline{z}_H, z_L \in (0,1)$, so $Q^* > 0 \Rightarrow Q > 0$ for all $\delta^L \in (0, \delta^*)$ and $\delta^H \in [\underline{\delta}_1^H, 1)$.Furthermore, the constraint is slack, so it follows that we could find equilibria for lower

δ^H. Last, let Q^L denote the part of Q that represents ICC^L_{iss} That is,

$$Q^L = \frac{\mu_t \delta^L pk}{1-\delta^L(1-p)} - \frac{\mu_t \delta^L pk(1-p)}{1-\delta^L(1-p)} z_L.$$

Since $\frac{\partial Q}{\partial \delta^L} < 0 \implies \frac{\partial Q^L}{\partial \delta^L} > 0$, so Q^L is increasing in δ^L as is to be expected. Since $\delta^L \in (0, \delta^*)$, it follows that

$$\begin{aligned}
Q^L &\in \left(Q^L\big|_{\delta^L=0}, Q^L\big|_{\delta^L=\delta^*} \right) = (0, \mu_t(1-(1-p)z^*_L)) \\
&= \left(0, \mu_t \left(1-(1-p)\left(1-\delta^*(1-p)\right) \textstyle\sum_{i=0}^\infty \left(\delta^*(1-p)\right)^i z_{t+1+i}\right)\right) \text{ by (53)} \\
&= \left(0, \mu_t - \mu_t \tfrac{(1-p)pk}{1+p(k-1)} \textstyle\sum_{i=0}^\infty \left(\tfrac{1-p}{1+p(k-1)}\right)^i z_{t+1+i}\right) \text{ by (5)} \\
&= \left(0, \mu_t - \mu_t pk \textstyle\sum_{i=0}^\infty \tfrac{(1-p)^{i+1}}{(1+(k-1)p)^{i+1}} z_{t+1+i}\right) \\
&= \left(0, \mu_t \left(1-p+p^2\right) \tilde{z}_L\right) \text{ for } \tilde{z}_L \in (0, z_t) \subseteq (0,1] \\
&\subset \left(0, \mu_t\left(1-p+p^2\right)\right).
\end{aligned}$$

That is, the lower bound for z_t is well-defined. Next, let Q^H denote the part of Q that represents ICC^H_{iss}.

$$\begin{aligned}
Q^H = &\tfrac{\mu_t \delta^H p^2\left(k\left(1-\delta^H(1-p)^2\right)-\delta^H p^2\right)}{(1-\delta^H(1-2p))(1-\delta^H(1-2(1-p)p))} - \tfrac{\mu_t \delta^H p(1-p)(k-1)}{1-\delta^H(1-p)^2} \bar{z}_H + \tfrac{(1-\mu_t)\delta^H p}{1-\delta^H(1-p)} \underline{z}_H \\
&+ \tfrac{\mu_t \delta^H p(1-p)}{2} \left(\tfrac{2(k-1)}{1-\delta^H(1-p)^2} + \tfrac{k+1}{1-\delta^H+2\delta^H p} - \tfrac{k-1}{1-\delta^H(1-2(1-p)p)} \right).
\end{aligned}$$

Since $\frac{\partial Q}{\partial \delta^H} > 0 \implies \frac{\partial Q^H}{\partial \delta^H} > 0$, so Q^H is increasing in δ^H as is to be expected. Since $\delta^H \in \left(\underline{\delta}^H_1, 1\right)$, it follows that

$$\begin{aligned}
Q^H &\in \left(Q^H\big|_{\delta^H=\underline{\delta}^H_1}, Q^H\big|_{\delta^H=1} \right) \\
&= \Big(\mu_t \tfrac{k-1+p}{pk} - \mu_t \tfrac{(k-1)(1-p)}{pk} \bar{z}_H + \tfrac{(1-\mu_t)}{p(k+1)-1} \underline{z}_H, \\
&\quad \mu_t \left(\tfrac{k}{2} + \tfrac{(k-1)p^2}{4(1-p)(2-p)} \right) - \mu_t \tfrac{(k-1)(1-p)}{2-p} \bar{z}_H + (1-\mu_t)\underline{z}_H \Big).
\end{aligned}$$

The simplifications for z's can be found at the end,

$$\begin{aligned}
Q^H \in \Big(&\mu_t \tfrac{k-1+p}{pk} + \overbrace{\tfrac{(1-\mu_t)}{p(k+1)-1} \underline{z}_H - \mu_t \tfrac{(k-1)(1-p)}{pk} \bar{z}_H}^{\equiv(*)}, \\
&\underbrace{(1-\mu_t)\underline{z}_H - \mu_t \tfrac{(k-1)(1-p)}{2-p} \bar{z}_H}_{\equiv(**)} + \mu_t \tfrac{4k+3kp^2-6kp-p^2}{4(p-1)(p-2)} \Big).
\end{aligned} \tag{57}$$

Then there exists $\tilde{z}_{h1}, \tilde{z}_{h2} \in (0, z_t)$ such that

$$Q^H \in \left(\mu_t \tfrac{k-1+p}{pk} + \left(\tfrac{1}{p(k+1)-1} + \mu_t \tfrac{k(1-2p)-(1-p)\left(1+p\left(k^2-1\right)\right)}{pk(p(k+1)-1)}\right) \tilde{z}_{h1},\right.$$
$$\left.\mu_t \left(\frac{k}{2} + \tfrac{(k-1)p^2}{4(1-p)(2-p)}\right) + \left(1 - \tfrac{\mu_t(k(1-p)+1)}{2-p}\right) \tilde{z}_{h2}\right).$$

That is, the upper bound for z_t is well-defined. ■

Proof. (Inequality 56):

$$\begin{aligned}
& \tfrac{k+1}{(1-\delta^H(1-2p))^2} > \tfrac{(k-1)(1-2p)}{\left(1-\delta^H(1-2p+2p^2)\right)^2} \\
\iff\quad & \tfrac{\left(1-\delta^H\left(1-2p+2p^2\right)\right)^2}{(1-\delta^H(1-2p))^2} > \tfrac{(k-1)(1-2p)}{k+1} \\
\iff\quad & \tfrac{1-\delta^H(1-2p)-2\delta^H p^2}{1-\delta^H(1-2p)} > \sqrt{\tfrac{(k-1)(1-2p)}{k+1}} \\
\iff\quad & 1 - \tfrac{2\delta^H p^2}{1-\delta^H(1-2p)} > \sqrt{\tfrac{(k-1)(1-2p)}{k+1}}
\end{aligned} \tag{58}$$

The left-hand side of (58) is minimized at $\delta^H = 1$, so it is enough to verify that

$$\begin{aligned}
& 1 - p > \sqrt{\tfrac{(k-1)(1-2p)}{k+1}} \\
\Longleftarrow\ & (k+1)(1-p)^2 - (k-1)(1-2p) > 0 \\
\iff\ & p^2k - 4p + p^2 + 2 > 0 \\
\Longleftarrow\ & p^2k - 4p + p^2 + 2 \geq 2p^2 - 4p + 2 = 2(p-1)^2 > 0.
\end{aligned}$$

Therefore, the inequality used in step (56) of the proof of proposition 16 holds. ■

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