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THE INVERSE FUNCTION THEOREM

Definition. Let $U \subset \mathbb{R}^m$, $V \subset \mathbb{R}^n$ be open sets and $f : U \to V$.

I. f is said to be a *diffeomorphism* if f is a differentiable bijection and the inverse $f^{-1} : V \to U$ is also differentiable.

II. f is said to be a *local diffeomorphism* if for every $p \in U$ there exist open sets U_p and $V_{f(p)}$, with $p \in U_p \subset U$ and $f(p) \in V_{f(p)} \subset V$, such that $f|_{U_p} : U_p \to V_{f(p)}$ is a diffeomorphism.

Clearly, every diffeomorphism is a local diffeomorphism, but not conversely[1], as the following example illustrates.

Example. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}^2 \backslash \{0\}$, $f(x, y) = e^x \cdot (\cos y, \sin y)$, which takes vertical lines into co-centric circles. The function f is clearly not injective, since for any given $(x, y) \in \mathbb{R}^2$ we have that $f(x, y) = f(x, y + 2k\pi)\, \forall k \in \mathbb{Z}$. We affirm that f is a local diffeomorphism. If $(a, b) \in \mathbb{R}^2$ with $b \notin \big\{(k + \frac{1}{2})\pi \,|\, k \in \mathbb{Z}\big\}$, define $U_j := \mathbb{R} \times \big((j - \frac{1}{2})\pi, (j + \frac{1}{2})\pi\big)$, where $j \in \mathbb{Z}$ is the (unique) integer number such that $b \in \big((j - \frac{1}{2})\pi, (j + \frac{1}{2})\pi\big)$. Let $\tan_j^{-1} : \mathbb{R} \to \big((j - \frac{1}{2})\pi, (j + \frac{1}{2})\pi\big)$ denote the local inverse of the tangent function restricted to the open interval $\big((j - \frac{1}{2})\pi, (j + \frac{1}{2})\pi\big)$. The differentiable function $(\alpha, \beta) \mapsto \Big(\ln\sqrt{\alpha^2 + \beta^2}, \tan_j^{-1}(\frac{\beta}{\alpha})\Big)$, defined on the open set $V_j := \{(A, B) \in \mathbb{R}^2 \,|\, A > 0\}$ for j even, and on the open set $V_j := \{(A, B) \in \mathbb{R}^2 \,|\, A < 0\}$ for j odd, is the inverse of $f|_{U_j} : U_j \to V_j$. Likewise, if $(a, b) \in \mathbb{R}^2$ with $b \notin \{k\pi \,|\, k \in \mathbb{Z}\}$.

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[1] In the real line, the following result is a consequence of Darboux's Theorem: every local diffeomorphirsm defined on an interval is a (global) diffeomorphism. As the example illustrates, such result has no analogue in $\mathbf{R}^n$.

Inverse Function Theorem. *Let* $f : U \to \mathbb{R}^n$ *be a function of class* C^k $(k \geq 1)$ *defined on an open set* $U \subset \mathbb{R}^n$ *and let* $a \in U$. *Suppose that* $Df(a) : \mathbb{R}^n \to \mathbb{R}^n$ *is an isomorphism. Then there exist opens sets* V *and* W, *with* $a \in V \subset U$ *and* $f(a) \in W \subset \mathbb{R}^n$, *such that* $f|_V : V \to W$ *is a* C^k*-diffeomorphism of* V *onto* W.

Proof. The proof relies on the following two lemmas, whose proofs are postponed.

Lemma 1 (Perturbation of an Isomorphism). *Let* $T : \mathbf{R}^m \to \mathbf{R}^m$ *be a linear isomorphism and* $\varphi : U \to \mathbf{R}^n$ *a* λ*-contraction with* $0 \leq \lambda < 1/||T^{-1}||$ *defined on an open set* $U \subset \mathbf{R}^m$. *Let* $f : U \to \mathbf{R}^m$ *be defined by* $f(x) = T \cdot x + \varphi(x)\,,\ \forall x \in U$. *Then* f *is a homeomorphism onto the open set* $f(U)$. *Moreover, if* $U = \mathbf{R}^m$ *then* $f(U) = \mathbf{R}^m$.

Lemma 2 (Differentiability of the Inverse Homeomorphism). *Let* U *and* $V \subset \mathbf{R}^n$ *be open sets and* $f : U \to V$ *a differentiable homeomorphism. If* $a \in U$ *is such that* $Df(a) : \mathbf{R}^m \to \mathbf{R}^m$ *is invertible, then the inverse homeomorphism* $f^{-1} : V \to U$ *is differentiable at* $f(a)$.

We proceed with the proof of the Inverse Function Theorem. Without loss of generality (why?) assume $a = f(a) = 0$. Then for all $x \in U$ one has:

$$f(x) = Df(a) \cdot x + r(x)$$

where $\lim\limits_{x \to 0} \frac{r(x)}{|x|} = 0$. Thus, $r(x) = f(x) - Df(a) \cdot x$ for all $x \in U$. The function r is C^1 and $Dr(x) = Df(x) - Df(a)$. In particular, $Dr(a) = 0$. Fix some $\varepsilon > 0$ such that $\varepsilon \cdot ||[Df(a)]^{-1}|| < 1$. By the continuity of the derivative map $Dr : U \to \mathcal{L}(\mathbf{R}^m, \mathbf{R}^m)$, there exists an open ball V with $a \in V \subset U$ such that:

$$||Dr(x)|| < \varepsilon\,, \quad \forall x \in V$$

By the Mean Value Inequality,

$$|r(x) - r(y)| \leq \varepsilon \, |x - y|\,, \quad \forall x, y \in V$$

Hence, $r|_V$ is ε-Lipschitz with $\varepsilon \cdot ||[Df(a)]^{-1}|| < 1$ and therefore $f|_V$ is a perturbation of the isomorphism $Df(a)$. By Lemma 1, f maps V homeomorphically onto the open set $W = f(V)$. In addition, notice that since the derivative map $Df : U \to \mathcal{L}(\mathbf{R}^m, \mathbf{R}^m)$ is continuous and the set of invertible linear maps is open in $\mathcal{L}(\mathbf{R}^m, \mathbf{R}^m)$, we can make the nbds. V of a and W of $f(a)$ smaller, if necessary, so as to guarantee that $Df(x)$ is invertible for all $x \in V$. By Lemma 2 we conclude that the inverse homeomorphism $g = (f|_V)^{-1} : W \to V$ is differentiable. Therefore, $f|_V$ is a diffeomorphism onto W.

It remains to show that g is C^k. We proceed by induction. By the Chain Rule, one has that:

$$Dg(y) = [Df\,(g(y))]^{-1}\,, \quad \forall y \in W$$

Equivalently,

$$Dg = \texttt{Inv} \circ Df \circ g$$

where $\texttt{Inv} : GL_m \to GL_m$ is the function defined on the open set $GL_m = \{T \in \mathcal{L}(\mathbf{R}^m, \mathbf{R}^m) \,|\, T \text{ is invertible}\}$ that maps invertible linear operators into their inverse. The function $\texttt{Inv}$ is C^∞. (Prove this result an exercise. The derivative is $D\texttt{Inv}(T) \cdot V = -T^{-1} \cdot V \cdot T^{-1}$ for all $V \in \mathcal{L}(\mathbf{R}^m, \mathbf{R}^m)$ and $A \in GL_m$). Since g is differentiable and Df is C^{k-1}, Dg is a composition of differentiable functions $\Rightarrow$ Dg is differentiable $\Rightarrow$ Dg is continuous $\Rightarrow$ g is C^1. Arguing by induction, suppose we have shown that g is C^r $(1 \le r < k)$. Since $\texttt{Inv}$ is C^∞ and Df is C^k, we conclude that Dg is $C^r \Rightarrow g$ is C^{r+1}. By the Induction Principle, g is C^k. $\square$