Econ 897 - Summer 2004
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Sard's Theorem

Sard's Theorem. *Let $f : U \to \mathbb{R}^n$ be a function of class C^1 defined on an open set $U \subset \mathbb{R}^m$ and let $S \subset U$ be the set of its singular points, i.e., $S = \{x \in U \mid Df(x)$ is not surjective$\}$. Then $f(S)$ has measure zero.*

Proof. **First Case:** $m < n$. It follows directly from the following three facts, which were proved in class:

1. Every C^1 function is locally Lipschitz (by the Mean Value Inequality)
2. Every locally Lipschitz function maps zero measure sets into zero measure sets
3. Every set $X \subset \mathbb{R}^m$, when embedded in a higher dimensional space $\mathbb{R}^n$ $(m < n)$, has measure zero.

Second Case: $m > n$. This is the difficult case. The proof is beyond the scope of this course and will be skipped. **Third Case:** $m = n$. By Lindelof's Theorem it is enough to show that if C is a cube contained in U and $T = \{x \in C \mid \det Df(x) = 0\}$, then $f(T)$ has measure zero. Let $a > 0$ be the edge length of the cube C.

Since C is a compact subset of U, f is *uniformly differentiable* in C, i.e., for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x \in C, x' \in U$,

$$|x' - x| < \delta \quad \Rightarrow \quad |r_x(x')| := |f(x') - f(x) - Df(x) \cdot (x' - x)| < \varepsilon \, |x' - x|$$

Fix $\varepsilon > 0$ and let $0 < \delta < a$ be such that the above implication holds for all $x \in C, x' \in U$. Let k be an integer large enough so that $a/k < \delta \leq a/(k-1)$. Subdivide each edge of C in k sub-intervals of length a/k, we obtain a partition of C into k^m cubes C_i each of which has edge length a/k and volume a^m/k^m. If $x, y \in C_i$ then $|x-y| < \delta$ (we conveniently adopt the norm of the maximum). In each cube C_i such that $C_i \cap T \neq \emptyset$ choose some $x_i \in C_i \cap T$. For each i, the image of the derivative $Df(x_i) : \mathbb{R}^m \to \mathbb{R}^m$ is contained in a subspace E_i of dimension $m-1$. The set of all points $f(x_i)+Df(x_i)\cdot v$, $v \in \mathbf{R}^m$ is contained in the affine subspace $L_i = f(x_i)+E_i$ of dimension $m-1$. For each $x \in C_i$ one has:

$$f(x) = f(x_i) + Df(x_i) \cdot (x - x_i) + r_i(x)$$

where r_i is such that $|r_i(x)| < \varepsilon\,|x-x_i| < \varepsilon \cdot \delta$. Denote by $c = \sup_{x\in C} ||Df(x)||$. We have that for every $x \in C_i$, the point $f(x_i) + Df(x_i) \cdot (x - x_i)$ belongs to a cube of edge length $2c\delta$ in L_i. Consider the block in $\mathbb{R}^m$ that has this cube as cross section and whose height is $2\varepsilon\delta$. The volume of this block is $2^m \delta^m c^{m-1} \varepsilon = A \cdot \delta^m \cdot \varepsilon$. The image $f(T)$ is contained in at most k^m of these blocks, so that the sum of their volumes is at most $A \cdot k^m \delta^m \varepsilon \leq A\,(2a)^m \varepsilon$. Since $\varepsilon > 0$ is arbitrary, $f(T)$ has measure zero. $\square$

Transversality Theorem. *Let $\Lambda \subset \mathbf{R}^\ell$ and $U \subset \mathbf{R}^m$ be open sets and let $f : \Lambda \times U \to \mathbf{R}^n$ be a C^1-map. Let $c \in \mathbf{R}^n$ be such that for all $(\lambda, x) \in f^{-1}(c)$, $Df(\lambda, x) : \mathbf{R}^\ell \times \mathbf{R}^m \to \mathbf{R}^n$ is surjective. Then the set $\{\lambda \in \Lambda \,|\, D_x f(\lambda, x)$ is not surjective for some $x \in U$ with $f(\lambda, x) = c\}$ has measure zero.*

Proof. By Lindelof's Theorem it suffices to show that for any given $(\lambda, x) \in f^{-1}(c)$ there exist open sets L and G, with $\lambda \in L \subset \Lambda$ and $x \in G \subset U$, such that the set:

$$\Lambda_0 = \{\lambda \in L \,|\, D_x f(\lambda, x) \text{ is not surjective for some } x \in G \text{ with } f(\lambda, x) = c\}$$

has measure zero. Fix $(\lambda_0, x_0) \in \Lambda \times U$. By the Implicit Function Theorem, there exists open sets L and G, with $\lambda_0 \in L \subset \Lambda$ and $x_0 \in G \subset U$, such that $(L \times G) \cap f^{-1}(c)$ is the graph of a C^1-map defined on an open set $V \subset \mathbf{R}^{\ell+m-n}$. In particular, there is an immersion $\psi : V \to \mathbf{R}^\ell \times \mathbf{R}^m$ that maps V homeomorphically onto $(L \times G) \cap f^{-1}(c)$. Let $\pi : \mathbf{R}^\ell \times \mathbf{R}^m \to \mathbf{R}^\ell$

be the natural projection and consider the map $\pi \circ \psi : V \to \mathbf{R}^\ell$. We affirm that given any $y \in V$, if $D(\pi \circ \psi)(y) : \mathbf{R}^{\ell+m-n} \to \mathbf{R}^\ell$ is surjective then $D_x f(\psi(y))$ is surjective. In effect, if $\mathbf{R}^\ell = \pi \cdot D\psi(y) \cdot \mathbf{R}^{\ell+m-n}$ then:

$$\mathbf{R}^\ell \times \mathbf{R}^m = \{0\} \times \mathbf{R}^m \,+\, D\psi(y) \cdot \mathbf{R}^{\ell+m-n}$$

Hence,

$$\mathbf{R}^n = Df(\psi(y)) \cdot (\mathbf{R}^\ell \times \mathbf{R}^m) = D_x f(\psi(y)) \cdot \mathbf{R}^m + Df(\psi(y)) \cdot D\psi(y) \cdot \mathbf{R}^{\ell+m-n}$$

Therefore,

$$\Lambda_0 \subset \{\lambda \in L \,|\, D(\pi \circ \psi)(y) \text{ is not surjective for some } y \in (\pi \circ \psi)^{-1}(\lambda)\}$$

By Sard's Theorem, the set on the R.H.S. has measure zero. Therefore, Λ_0 also has measure zero. $\square$