ECON897 Math Camp Part II — **KyungMin (Teddy) Kim**
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Section 2 : Review of Linear Algebra

1. VECTOR SPACES

Def. A vector space (over R) consists of a set V along with two operations $+$ and $\cdot$ such that $\forall u, v, w \in V, \forall \alpha, \beta \in R$,

(i) $u + v = v + u, \forall u, v \in V$
(ii) $(u + v) + w = u + (v + w), \forall u, v, w \in V$
(iii) $\exists 0 \in V$ s.t. $\forall u \in V, u + 0 = 0 + u = u$
(iv) $\forall u \in V, \exists v \in V$ s.t. $u + v = v + u = 0$
(v) $\forall u \in V, \forall \alpha, \beta \in R, (\alpha + \beta) \cdot u = \alpha \cdot u + \beta \cdot u$
(vi) $\forall u, v \in V, \forall \alpha \in R, \alpha \cdot (u + v) = \alpha \cdot u + \alpha \cdot v$
(vii) $\forall u \in V, \forall \alpha, \beta \in R, (\alpha\beta) \cdot u = \alpha \cdot (\beta \cdot u)$
(viii) $\forall u \in V, 1 \cdot u = u$

Examples
(1) R^n
(2) $\{f : A \to R^n$, a function$\}$
(3) $P^n = \{a_0 + a_1 \cdot x + a_2 \cdot x^2 + ... + a_n \cdot x^n | a_i \in R, 0 \le i \le n + 1\}$

Def. V : a vector space, $W \subset V$ is a subspace if it is itself a vector space under the inherited operations.

To check whether W is a subspace, we only need to check that
(i) $0 \in W$
(ii) $\forall u, v \in W, u + v \in W$
(iii) $\forall u \in W, \forall \alpha \in R, \alpha \cdot u \in W$

Example : For some $\alpha \in R^n, \{x \in R^n : x = \alpha \cdot y$ for some $y \in R^n\}$

Def. V : a vector space, $S \subset V$. The span of S is the set of all (finite) linear combincations of vectors from S.

$$Span(S) = \{c_1 \cdot s_1 + ... + c_n \cdot s_n | c_1, ..., c_n \in R \text{ and } s_1, ..., s_n \in S\}$$

Examples
(1) $Span\{e_i = (0, .., 0, 1, 0, ...), 1 \le i \le n\} = R^n$
(2) $Span\{a, b \in R^2 \backslash \{0\} | \nexists c \in R \backslash \{0\}$ s.t. $c \cdot a = b\} = R^2$
(3) $Span\{1, x, x^2, ..., x^n\} = P^n$

Lemma V : a vector space, $S \subset V \Rightarrow Span(S)$ is a subspace of V.
Proof : Exercise 1

Def. A subset of a vector space is **linearly independent(LI)** if none of its elements is a (finite) linear combination of the others. Otherwise, it is **linearly dependent**.

Lemma V : a vector space, $S \subset V$. S is LI $\Leftrightarrow \forall s_1, ..., s_n \in S, s_i \neq s_j, c_1 \cdot s_1 + ... + c_n \cdot s_n = 0 \Rightarrow c_1 = ... = c_n = 0$.

2. BASIS AND DIMENSION

Def. V : a vector space, $S \subset V$. S is a basis of V if S is LI and $Span(S) = V$.

Examples
(1) $\{e_i, 1 \leq i \leq n\}$ is a basis of R^n
(2) $\{a, b \in R^2 \backslash \{0\} | \nexists c \in R \backslash \{0\}$ s.t. $c \cdot a = b\}$ is a basis of R^2
(3) $\{1, x, x^2, ..., x^n\}$ is a basis of P^n

Theorem In any vector space, a subset is a basis if and only if each vector in the space can be expressed as a linear combination of elements of the subset in a unique way.
Proof : Exercise 2

Proposition V: a vector space with a finite base, i.e., $\exists$a base $S \subset V$ with $\#S < \infty$. Then every base of V has the same number of elements.
Proof : Exercise 3

Def. Suppose V has a base S with $\#S < \infty$. Then we define $\dim V = \#S$. This is well-defined due to the previous proposition.

Examples
(1) $\dim R^n = n$
(2) $F =$ the set of polynomials with degree no greater than n. Then $\{1, x, x^2, ..., x^n\}$ becomes a base of F, so that $\dim F = n + 1$.

3. MAPS BETWEEN SPACES

Def. V, W : vector spaces. $T : V \to W$ is a linear mapping if $\forall u, v \in V, \forall \alpha$,
(i) $T(u + v) = T(u) + T(v)$
(ii) $T(\alpha \cdot u) = \alpha \cdot T(u)$

Notation
$L(V, W) = \{T : V \to W$:a linear map$\}$ = the set of all linear transformations. If $V = W$, then we write $L(V)$.

Def. For $T \in L(V, W)$, define the **norm** $\parallel T \parallel \equiv \sup_{v \in S} |Tv|$ where $S = \{v \in V : |v| \leq 1\}$.

Proposition $T \in L(R^n, R^m) \Rightarrow \parallel T \parallel < \infty$ and T is uniformly continuous.
Proof : Suppose $x = c_1 \cdot e_1 + ... + c_n \cdot e_n$ and $|x| \leq 1 (\Rightarrow |c_i| \leq 1)$. Then

$$|Tx| = |c_1 \cdot Te_1 + ... + c_n \cdot Te_n| \leq |c_1| \cdot |Te_1| + ... + |c_n| \cdot |Te_n| \leq |Te_1| + ... + |Te_n|$$

so that $\parallel T \parallel \leq |Te_1| + ... + |Te_n| < \infty$. For all $x, y \in R^n$ with $x \neq y$,

$$|Tx - Ty| = |T(x - y)| = |x - y| \cdot |T(\frac{x - y}{|x - y|})| \leq \parallel T \parallel \cdot |x - y|$$

Therefore T is uniformly continuous. **Q.E.D.**

Def. V, W : vector spaces. $f : V \to W$ is an isomorphism if f is bijective (both injective and surjective) and is a linear mapping.

Examples
(1) $f : R^n \to R^n$, $f(x) = x$
(2) $f : P^n \to R^{n+1}$ where

$$f(a_0 + a_1 x + a_2 x^2 + ... + a_n x^n) = \begin{pmatrix} a_0 \\ a_1 \\ ... \\ a_n \end{pmatrix}$$

(3) $f : C \to R^2$,$f(c) = f(c_1 + ic_2) = (c_1, c_2)$

Proposition $L(R^n, R^m)$ is isomorphic to $M_{m\times n} = M(m, n) \equiv \{A|A \text{ is } m \times n \text{ matrix}\}$.

* This proposition implies that any linear transformation from R^n to R^m can be represented by a $m \times n$ matrix.

4. THEOREM OF KERNEL AND IMAGE

Def. The kernel of a linear map $T : V \to W$ is defined as $\ker T \equiv T^{-1}(0)$.

Example
$T : R^2 \to R$, $T(x_1, x_2) = x_1 + x_2$. Then $\ker T = \{(x_1, x_2)|x_1 = -x_2\}$.

Fact
(1) $\ker T$ always contains 0.
(2) $\ker T$ is a subspace.

Def. The range of $T : V \to W$ is defined as $ranT \equiv T(V)$.

Fact : $ranT$ is a subspace.

Proposition A linear map $T : V \to W$ is injective $\Leftrightarrow \ker T = \{0\}$.
Proof : Exercise 4

Theorem of Kernel and Image V, W : finite-dimensional vector spaces, $T : V \to W$, linear. Then

$$\dim V = \dim\{\ker T\} + \dim\{ranT\}$$

Def. $T : V \to W$ is an isomorphism if T is a linear bijection.

Corollary T is an isomorphism $\Rightarrow \dim V = \dim W$
Proof : (i) T is injective $\Rightarrow \ker T = \{\emptyset\} \Rightarrow \dim\{\ker T\} = 0 \Rightarrow \dim V = \dim\{ranT\}$. (ii) T is surjective $\Rightarrow ranT = W \Rightarrow \dim\{ranT\} = \dim W$. By (i) and (ii), $\dim V = \dim W$. **Q.E.D.**

Corollary Suppose $\dim V = \dim W$. T is surjective $\Longleftrightarrow$ T is injective
Proof : T is surjective$\Leftrightarrow W = ranT \Leftrightarrow \dim W = \dim\{ranT\} \Leftrightarrow \dim\{\ker T\} = 0 \Leftrightarrow \ker T = \{0\} \Leftrightarrow T$ is injective. **Q.E.D.**

5. DETERMINANTS

Def. A $n \times n$ determinant is a function $\det : M_{n\times n} \to R$ such that

(i)	$\det(\rho_1, ..., \rho_j, ..., \rho_n) = \det(\rho_1, ..., k \cdot \rho_i + \rho_j, ..., \rho_n)$, $i \neq j$
(ii)	$\det(\rho_1, ..., \rho_i, ..., \rho_j, ..., \rho_n) = -\det(\rho_1, ..., \rho_j, ..., \rho_i, ..., \rho_n)$, $i \neq j$
(iii)	$\det(\rho_1, ..., k \cdot \rho_i, ..., \rho_n) = k \cdot \det(\rho_1, ..., \rho_i, ..., \rho_n), k \neq 0$
(iv)	$\det I = 1$

Theorem For each n, there is a unique $n \times n$ determinant function.

Properties
(1) A matrix with two identical rows (or columns) has a determinant of zero.
(2) A matrix with a zero row (or column) has a determinant of zero.
(3) A matrix is nonsingular if and only if its determinant is nonzero.
(4) The determinant of a matrix equals the determinant of its transpose.

Theorem $A : m \times m$ matrix. $\det A \neq 0 \Leftrightarrow A^{-1}$ exists $\Leftrightarrow A : R^m \to R^m$ is an isomorphism

(Useful) **Fact** : det is a continuous function.