ECON897 Math Camp Part II **KyungMin (Teddy) Kim**
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Problem Set 1

Due on Thursday, July 31 noon

You should work in groups (4-6 members in a group). One solution set for each group.

1. The best way to learn both economics and mathematics is making your hands dirty, that is, solving as many problems as possible. So, do all exercises (1-5) in Section 0.

2. Find a topological space which is not metrizable. Hint : Start from the simplest space you can imagine.

3. (Continuous Functions) Using appropriate definitions of continuity (among 3) as well as open and closed sets (topological or metric) may make your life much easier.
(1) Show that if $f : X \to R$, C^0 and $\alpha \in R$, then $\{x \in X : f(x) \geq \alpha\}$ is closed. Also, we can show $\{x \in X : f(x) \leq \alpha\}$ and $\{x \in X : f(x) = \alpha\}$ are closed, while $\{x \in X : f(x) > \alpha\}$ and $\{x \in X : f(x) < \alpha\}$ are open.
(2) $f, g : X \to Y$, C^0. $S \subset X$: dense. If $f(x) = g(x), \forall x \in S$, then $f(x) = g(x), \forall x \in X$.
(3) $f : X \to Y$, C^0 and $g : Y \to Z$, $C^0 \Rightarrow h = g \circ f : X \to Z$, C^0

4. Consider an individual firm's profit maximization problem in a Cournot competition. Suppose φ, ψ_f are continuous, $\varphi > 0$, φ is strictly decreasing and $\lim_{x\to\infty} \varphi(x) = 0$. Show that, with the assumption that $\lim_{q_f\to\infty} \psi_f(q_f)/q_f > 0, \forall f \in F$, for any given q_{-f}, there exists an optimal solution q_f^* which solves

$$\max_{q_f \geq 0} \varphi(q_f + \sum_{j \neq f} q_j)q_f - \psi_f(q_f)$$

(Hint : Use the Extreme Value Theorem in Section 1.)

5. (Differentiation) For the following exercises, do not use the general results we developed in class. You are supposed to apply the definition of differentiation in R^n.
(1) Let $f : R^n \to R$ be a function such that $\mid f(x) \mid \leq \mid x \mid^2$. Show that f is differentiable at 0.
(2) Let $f : R \to R^2$. Prove that f is differentiable at $a \in R$ if and only if f_1 and f_2 are differentiable and that in this case

$$f'(a) = \begin{pmatrix} f_1'(a) \\ f_2'(a) \end{pmatrix}$$

6. Let $U \subset R^m, V \subset R^n$ and $\phi : U \rightrightarrows V$[1].

Def. ϕ is said to be *upper hemi-continuous (uhc)* if

$$K \subset U : \text{compact} \Rightarrow \{(u, v) \in K \times V : v \in \phi(u)\} : \text{compact}$$

[1] Note that ϕ is not a function, but a correspondence, that is, for each $u \in U$, there may exist more than one $v \in V$ which satisfies $\phi(u) = v$. Example : $\phi(u) = [0, 1]$ for all $u \in U$.

or equivalently ϕ is uhc at a if

$$\begin{aligned} u^v &\in U, v^v \in \phi(u^v), v \in N, \lim_{v\to\infty} u^v = u \in U \text{ and } \phi(u) \neq \emptyset \\ &\Rightarrow \exists a \text{ subsequence}\{v^{vk}\} \subset \{v^v\} s.t. \lim_{v^k\to\infty} v^{vk} = v \in \phi(u) \end{aligned}$$

and uhc if ϕ is uhc on U.

Def. ϕ is said to be *sequentially upper hemi-continuous (suhc)* at u if

$$u^v \in U, v^v \in \phi(u^v), v \in N, \lim_{v\to\infty} u^v = u \in U, \lim_{v\to\infty} v^v = v \in R^n \Rightarrow v \in \phi(u)$$

and suhc if ϕ is suhc on U.

Show that (1) if ϕ is uhc, then ϕ is suhc, and (2) if V is compact and ϕ is suhc, then ϕ is uhc.

7. Let $X \subset R^n, Y \subset R^m$ and $f : X \to Y$.

Def. f is said to be *proper* if f is continuous and for $Y' \subset Y$ compact, $f^{-1}(Y')$ is compact, that is, a correspondence $f^{-1} : Y \Rightarrow X$ is u.h.c.

Suppose $f : X \to Y$ is proper. Show that if $X\prime \subset X$ is closed then $f(X\prime) \subset Y$ is closed.

8. Consider an exchange economy with two consumers (Ms. 1 and Mr. 2) and two goods (1 and 2). Consumer i has an initial endowment $e_i = (e_i^1, e_i^2) \in R^2_{++}$ and his or her preference is represented by a utility function $u_i(x_i) = u_i(x_i^1, x_i^2)$ where x_i^j is the amount of consumption of good j by consumer i. u_i is C^1 and $\partial u_i(x_i^1, x_i^2)/\partial x_i^j > 0, \forall j = 1, 2, \forall (x_i^1, x_i^2) \in R^2_{++}$. It also satisfies the INADA condition, that is, $\lim_{x_i^1\to 0} \partial u_i(x_i^1, x_i^2)/\partial x_i^1 = \lim_{x_i^2\to 0} \partial u_i(x_i^1, x_i^2)/\partial x_i^2 = \infty$ and $\lim_{x_i^1\to\infty} \partial u_i(x_i^1, x_i^2)/\partial x_i^1 = \lim_{x_i^2\to\infty} \partial u_i(x_i^1, x_i^2)/\partial x_i^2 = 0$.

A Walrasian (or Competitive) Equilibrium in this economy is an allocation $(x_1^*, x_2^*) \in R^4_+$ and a price vector $p^* = (p^{1*}, p^{2*}) \in R^2_+$ such that
(i) (Utility Maximization) $\forall i = 1, 2$, given $p^* = (p^{1*}, p^{2*})$,

$$x_i^* \in \arg\max_{x_i} u_i(x_i) \text{ subject to } p^* \cdot x_i \leq p^* \cdot e_i$$

(ii) (Market Clearing) $x_1^{j*} + x_2^{j*} \leq e_1^j + e_2^j, j = 1, 2$
Since utility function is strictly increasing, without loss of generality, we can normalize the price of good 1 by 1. Denote by $p \in R_{++}$ the price of good 2 (relative to that of good 1).

(1) Show that for any given $p \in R_{++}$, there exists $x_i(p), i = 1, 2$ which satisfies budget constraint and maximizes consumer's utility, that is, $\forall p \in R_{++}$, if

$$X(p) = \arg\max_{x_i} u_i(x_i) \text{ subject to } x_i \in B_i(p) = \{x_i \in R^2_+ : \ x_i^1 + px_i^2 \leq e_i^1 + pe_i^2\}$$

then $X(p) \neq \emptyset$.

(2) **Def**. $f : X \to R$ is strictly quasi-concave if $f(\alpha \cdot x + (1-\alpha) \cdot y) > \max\{f(x), f(y)\}$,$\forall x, y \in X$ with $x \neq y$ and $\forall \alpha \in (0, 1)$.
Suppose u_i is strictly quasi-concave. Show that for any $p \in R_{++}$, $X(p)$ is a singleton, that is, given prices $(1, p)$, there exists a UNIQUE $x_i(p)$ which solves consumer's utility maximization problem.

(3) Suppose for any $p \in R_{++}$, $x_i(p)$ is a unique consumption commodity which solves consumer's utility maximization problem. Derive two equations $x_i(p)$ should satisfy (by solving the Lagrangian of this problem).

(4) Define an excess demand function of good 2 by $z(p) = x_1^2(p) + x_2^2(p) - e_1^2 - e_2^2$. Show that $\lim_{p\to 0} z(p) = \infty$ and $\lim_{p\to\infty} z(p) < 0$.

(5) Suppose $x_i(\cdot)$ is continuous (In fact, you can prove this by applying (Berge's) Theorem of Maximum). Prove that there exists a price p^* which clears the market for good 2.

(6) Show that with prices $(1, p^*)$, the market for good 1 is also cleared.

You found a Walrasian equilibrium, $(1, p^*)$ and $(x_1(p^*), x_2(p^*))$!