**ECON897 Math Camp Part II** **Summer 2008**

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## Suggested Solution for Problem Set 1

**1**. (1) By simple calculation, we find that $(i \neq j)$

$$q_i^*(q_j) = \begin{cases} \frac{12-q_j}{2}, \text{ if } q_j \leq 12, \\ 0, \text{ if } q_j > 12. \end{cases}$$

It is immediate that the unique equilibrium outcome is $q_1 = q_2 = 4$.
(2) From (1), we know

$$q_2^*(q_1) = \begin{cases} \frac{12-q_1}{2}, \text{ if } q_1 \leq 12, \\ 0, \text{ if } q_1 > 12. \end{cases}$$

The problem firm 1 is facing is

$$\max_{q_1} (12 - q_1 - q_2^*(q_1))q_1.$$

The optimal solution to this problem is $q_1^* = 6$. Hence the equilibrium is $(6, q_2^*(\cdot))$ and the Stackelberg equilibrium outcome is $(6, 3)$.[1]
(3) (i) Let $p_U$ be the price that $U$ charges $D$. Then $D$'s problem is

$$\max_q \varphi(q) \cdot q - p_U \cdot q - c_D \cdot q = (\varphi(q) - p_U - c_D)q.$$

Assuming all regularity conditions, the optimal solution $q^*(p_U)$ is characterized by the following F.O.C.

$$\varphi(q^*(p_U)) - p_U - c_D + \varphi'(q^*(p_U))q^*(p_U) = 0.$$

Given $q^*(p_U)$, $U'$s problem is

$$\max_{p_U} (p_U - c_U)q^*(p_U).$$

Hence the optimality condition is

$$q^*(p_U^*) + (p_U^* - c_U)\frac{\partial q^*(p_U^*)}{\partial p_U} = 0.$$

The equilibrium outcome is $(p_U^*, q^*(p_U^*))$.
(ii) Applying $\varphi(q) = 1 - q, c_U = 1/4$ and $c_D = 0$, (roughly - assuming interior solutions-) $q^*(p_U) = (1 - p_U)/2$ and $p_U^* = 5/8$ and $q^*(p_U^*) = 3/16$.
(4) The worker's expected utility by accepting a wage offer $w$ is

$$V(w) = w + \beta[(1-\mu)V(w) + \mu U] = \frac{1}{1-\beta(1-\mu)}(w + \beta\mu U).$$

By choosing to remain unemployed, his expected utility is

$$U = b + \beta\{(1-\lambda)U + \lambda E[\max(V(w), U)]\}.$$

Similarly to the basic model, $V$ is strictly increasing, while $U$ is independent of $w$. Therefore there exists $w^*$ such that

$$w \geq w^* \Leftrightarrow V(w) \geq U.$$

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[1] You will learn the difference between equilibrium and equilibrium outcome in 703.

Therefore the reservation wage strategy is optimal. From the above equations, we get

$$\begin{aligned}
w^* &= (1-\beta)U \\
V(w) - V(w^*) &= \frac{1}{1-\beta(1-\mu)}(w - w^*) \\
(1-\beta)U &= b + \beta\lambda \int_{w^*}^{\overline{w}} [V(x) - V(w^*)] dF(x)
\end{aligned}$$

Arranging terms, we get the fundamental reservation wage equation

$$w^* = b + \frac{\beta\lambda}{1-\beta(1-\mu)} \int_{w^*}^{\overline{w}} (x - w^*) dF(x).$$

Define

$$H(w) = w - b - \frac{\beta\lambda}{1-\beta(1-\mu)} \int_{w}^{\overline{w}} (x - w) dF(x)$$

Then

$$\begin{aligned}
\frac{\partial H}{\partial w} &= 1 + \frac{\beta\lambda}{1-\beta(1-\mu)}(1 - F(w)) > 0 \\
\frac{\partial H}{\partial b} &= -1 \\
\frac{\partial H}{\partial \lambda}|_{w=w^*} &= -\frac{\beta}{1-\beta(1-\mu)} \int_{w^*}^{\overline{w}} (x - w^*) dF(x) < 0 \\
\frac{\partial H}{\partial \mu}|_{w=w^*} &= \frac{\beta^2\lambda}{1-\beta(1-\mu)} \int_{w^*}^{\overline{w}} (x - w^*) dF(x) > 0
\end{aligned}$$

Therefore $\partial w^*/\partial b > 0, \partial w^*/\partial \lambda > 0, \partial w^*/\partial \mu < 0$
(5) Applying $f(v) = 1$ to the last equation,

$$\frac{\partial b^*}{\partial v} = \frac{v - b^*(v)}{v}.$$

Guess $b^*(v) = 1/2 \cdot v$. It's immediate that this satisfies both a boundary condition $b^*(0) = 0$ and the first-order differential equation.

**2**. Consider $X = \{0, 1\}$ and its indiscrete topology $\mathcal{T} = \{\emptyset, X\}$. Suppose this topological space is metrizable with a metric $d$. By the definition of a metric function, $d(0, 0) = d(1, 1) = 0$ and $d(0, 1) = d(1, 0) > 0$. But this metric $d$ does not induce the topology $\mathcal{T}$. The only topology that can be induced by $d$ is $\mathcal{T} = \{\emptyset, \{0\}, \{1\}, X\}$.

**3**. (1) Fix a sequence $\{x_n\}$ in $\{x \in X : f(x) \geq \alpha\}$ which converges $y \in X$. Observe that $f(x_n) \geq \alpha, \forall n$, so $\lim_{n\to\infty} f(x_n) \geq \alpha$. Moreover, by the continuity of $f$, $f(x) = \lim_{n\to\infty} f(x_n) \geq \alpha$. Hence $y \in \{x \in X : f(x) \geq \alpha\}$. All other results are immediate from this. For example, $\{x \in X : f(x) < \alpha\}$ is open because $\{x \in X : f(x) \geq \alpha\}$ is closed.
(2) Fix $x \in X$. Since $S$ is dense, there exists a sequence $\{s_n\} \subset S$ which converges to $x$. By the definition of continuous functions and using the fact that $f(x) = g(x), \forall x \in S$, $f(x) = \lim_{n\to\infty} f(s_n) = \lim_{n\to\infty} g(s_n) = g(x)$.
(3) Fix an open set $O$ in $Z$. Since $g$ is continuous, $g^{-1}(O)$ is open in $Y$. Since $f$ is also continuous, $f^{-1}(g^{-1}(O)) = (g \circ f)^{-1}(O) = h^{-1}(O)$ is open in $X$. Since this is true for all open sets in $Z$, $h$ is also continuous.

**4**. Since $\lim_{x\to\infty}\varphi(x)=0$ and $\lim_{x\to\infty}\psi_f(x)/x>0$, $\exists k_f^*\in R_+$ such that $x\geq k_f^*\Rightarrow \varphi(x)-\psi_f(x)/x<0$. Therefore for $x\geq k_f^*$,

$$\varphi(0)0-\psi_f(0)=0>\varphi(x)x-\psi_f(x)=(\varphi(x)-\psi_f(x)/x)x.$$

Also for any given $q_{-f}\in R_+^{F-1}$, if $x\geq k_f^*$ then

$$0>(\varphi(x)-\psi_f(x)/x)x\geq(\varphi(x+\Sigma_{j\neq f}q_j)-\psi_f(x)/x)x,$$

because $\varphi$ is decreasing. Therefore for any given $q_{-f}$,

$$\max_{q_f\geq 0}\varphi(q_f+\Sigma_{j\neq f}q_j)q_f-\psi_f(q_f)\Leftrightarrow \max_{q_f\in[0,k_f^*]}\varphi(q_f+\Sigma_{j\neq f}q_j)q_f-\psi_f(q_f).$$

By the Extreme Value Theorem, there exists an optimal solution $q_f^*$.

**5**. (1) For all $h\in R^n$,

$$\frac{|f(h)-f(0)|}{|h|}=\frac{|f(h)|}{|h|}\leq|h|.$$

Hence

$$\lim_{h\to 0}\frac{|f(h)-f(0)|}{|h|}\leq\lim_{h\to 0}|h|=0.$$

Therefore $f$ is differentiable at 0 and its derivative is 0 when $x=0$.
(2) ($\Rightarrow$) Let

$$f'(a)=\begin{pmatrix}T_1\\T_2\end{pmatrix}.$$

Since $f$ is differentiable at $a$, for $h$,

$$\lim_{h\to 0}\frac{|f(a+h)-f(a)-f'(a)h|}{|h|}=\lim_{h\to 0}\frac{|f_1(a+h)-f_1(a)-T_1h|}{|h|}=0.$$

Hence $f_1$ is differentiable and $f_1'(a)=T_1$. Analogous proof applies to $f_2$.
($\Leftarrow$) Observe that

$$\begin{aligned}|f(a+h)-f(a)-\begin{pmatrix}f_1'(a)\\f_2'(a)\end{pmatrix}h| &= \left|\begin{pmatrix}f_1(a+h)-f_1(a)-f_1'(a)h\\f_2(a+h)-f_2(a)-f_2'(a)h\end{pmatrix}\right|\\ &\leq \max\{|f_1(a+h)-f_1(a)-f_1'(a)h|,|f_2(a+h)-f_2(a)-f_2'(a)h|\}.\end{aligned}$$

Since $\lim_{h\to 0}|f_i(a+h)-f_i(a)-f_i'(a)h|/|h|=0, i=1,2$,

$$\lim_{h\to 0}|f(a+h)-f(a)-\begin{pmatrix}f_1'(a)\\f_2'(a)\end{pmatrix}h|/|h|=0.$$

Therefore $f$ is also differentiable and

$$f'(a)=\begin{pmatrix}f_1'(a)\\f_2'(a)\end{pmatrix}.$$

**6**. (1) Suppose $u^\nu\in U, v^\nu\in\phi(u^\nu), \nu\in N, \lim_{\upsilon\to\infty}u^\upsilon=u\in U, \lim_{\upsilon\to\infty}v^\upsilon=v\in R^n$. WTS $v\in\phi(u)$. Since $\phi$ is uhc, there exists a subsequence $<v^{\nu_k}>\to v'\in\phi(u)$. Since we supposed $\lim_{\upsilon\to\infty}v^\upsilon=v\in R^n$, we have $v=\lim_{\upsilon\to\infty}v^\upsilon=\lim_{\upsilon_k\to\infty}v^{\nu_k}=v'\in\phi(u)$.

(2) Suppose $u^\nu \in U, v^\nu \in \phi(u^\nu), \nu \in N, \lim_{\upsilon\to\infty} u^\upsilon = u \in U$ and $\phi(u) \neq \emptyset$. Since $V$ is compact, there exists a subsequence $< v^{\upsilon_k} >\to v \in V$. Also, since $\phi$ is suhc, we know $\lim_{\upsilon_k\to\infty} v^{\upsilon_k} = v \in \phi(\lim_{\upsilon_k\to\infty} u^{\upsilon_k}) = \phi(u)$.

**7**. Take a sequence $< y_n >\subset f(X')$ which converges to $y \in Y$. WTS $y \in f(X')$. First notice that $Y' = \{y_n, n \in N\} \cup \{y\}$ is compact. Since $f$ is proper, we know that $f^{-1}(Y')$ is compact. For each $n \in N$, take $x_n \in f^{-1}(y_n)$ and consider the sequence $< x_n >\subset f^{-1}(Y')$. Since $f^{-1}(Y')$ is compact, there exists a subsequence $< x_{n_k} >\to x \in f^{-1}(Y') \subset X$. Also since $X$ is closed, $x \in X'$. From the continuity of $f$, we conclude $y = \lim_{n_k\to\infty} y_{n_k} = \lim_{n_k\to\infty} f(x_{n_k}) = f(x) \in f(X')$.

**8**. (1) For any $p > 0$, $B_i(p)$ is compact (since it is closed and bounded). Also $u_i$ is continuous. Therefore by the EVT, there exists an optimal solution $x(p)$.
(2) Suppose for some $p > 0$, there exist two different solutions, $x_i$ and $x_i'$. Then consider $x_i'' = \alpha x_i + (1-\alpha)x_i'$ for some $\alpha \in (0,1)$. It is easy to check $x_i'' \in B_i(p)$. Also $u_i(x_i'') = u_i(\alpha x_i + (1-\alpha)x_i') > \max\{u_i(x_i), u_i(x_i')\} = u_i(x_i) = u_i(x_i')$. This is a contradiction to the assumption that $x_i$ and $x_i'$ are optimal solutions.
(3) Since we have the INADA condition, all optimal solutions are interior as long as $p > 0$. Applying usual Lagrangian method, we get

$$\begin{aligned}\frac{\partial u_i(x_i)/\partial x_i^1}{\partial u_i(x_i)/\partial x_i^2} &= \frac{1}{p} \text{ — } (*)\\ x_i^1(p) + p x_i^2(p) &= e_i^1 + p e_i^2 \text{ — } (**)\end{aligned}$$

(4) As $p \to 0$, for (*) to be true, $\partial u_i(x_i)/\partial x_i^2 \to 0$ (because of (**), $\partial u_i(x_i)/\partial x_i^1 < \infty$), which leads to $x_i^2(p) \to \infty$. Therefore $\lim_{p\to 0} z(p) = \infty$. Similarly, as $p \to \infty$, $\partial u_i(x_i)/\partial x_i^2 \to \infty$, which implies $x_i^2(p) \to 0$. Therefore $\lim_{p\to\infty} z(p) < 0$.
(5) Notice that $z$ is continuous. Since $\lim_{p\to 0} z(p) = \infty$ and $\lim_{p\to\infty} z(p) < 0$, by the Intermediate Value Theorem, there exists $p^* \in (0,\infty)$ which satisfies $z(p^*) = 0$. At this price, $x_1^2(p^*) + x_2^2(p^*) = e_1^2 + e_2^2$, which means the market is cleared.

(6) (From the Walras' law, this it obvious). Combining (**) and $x_1^2(p^*) + x_2^2(p^*) = e_1^2 + e_2^2$, we immediately have $x_1^1(p^*) + x_2^1(p^*) = e_1^1 + e_2^1$.