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Suggested Solution for Problem Set 2

1. $f(x) = f(y)$. Then by the Mean Value Theorem, there exists $z \in (x, y)$ such that $0 = f(x) - f(y) = f'(z)(x - y)$. Since $f'(z) \neq 0$, $x - y = 0 \Rightarrow x = y$. (The Mean Value Theorem doesn't require continuously differentiability!)

(b) For any $(x, y) \in R^2$,

$$\det Df(x,y) = \det \begin{pmatrix} \partial_1 f_1(x,y) & \partial_1 f_2(x,y) \\ \partial_2 f_1(x,y) & \partial_2 f_2(x,y) \end{pmatrix} = \det \begin{pmatrix} e^x \cos y & e^x \sin y \\ -e^x \sin y & e^x \cos y \end{pmatrix} = e^{2x} > 0$$

Notice that $f(x, y) = f(x, y + 2\pi)$. Therefore f is not injective.

2. Let $X^{\upsilon C} = X^{\upsilon-1} \backslash X^{\upsilon}, \upsilon \geq 1$.
Fact : $X^2 \subset X^1$: open in X^1, $X^1 \subset X^0$: open in $X^0 \Rightarrow X^2$: open in X^0. Proof is easy.

Based on this fact, we can show that X^{υ} is open in $X^0, 1 \leq \upsilon \leq \overline{\upsilon}$. $X^0 \backslash X^{\upsilon} = \cup_{\upsilon \geq 1} X^{\upsilon C}$: closed. In addition, we can show that $\cup_{\upsilon \geq 1} X^{\upsilon C}$ has zero measure because it is a finite union of zero measure sets[1]. Consequently

$$X^{\overline{\upsilon}} = X^{\overline{\upsilon}-1} \backslash X^{\overline{\upsilon}C} = X^{\overline{\upsilon}-2} \backslash X^{\overline{\upsilon}-1C} \backslash X^{\overline{\upsilon}C} ... = X^0 \backslash \cup_{\upsilon \geq 1} X^{\upsilon C}$$

is open and has full measure.

3.

$$\begin{aligned} \int_0^\infty p y^{p-1} P(Y &> y) dy = \int_0^\infty \int_\Omega p y^{p-1} 1_{(Y>y)} dF dy \\ &= \int_\Omega \int_0^\infty p y^{p-1} 1_{(Y>y)} dy dF \\ &= \int_\Omega \int_0^Y p y^{p-1} dy dF \\ &= \int_\Omega Y^p dF \end{aligned}$$

4. Take $\eta' = (\xi', \theta') \in \pi^{-1}(\theta')$. Since $\det D_\xi \Phi(\eta') \neq 0$, by the Implicit Function Theorem, there are a neighborhood of θ', say, Θ', a neighborhood of η', say, H' and C^1 mapping, say, $\phi : \Theta' \to H'$ s.t. for $\theta \in \Theta', \eta \in H' \cap M$ iff $\eta = \phi(\theta)$. Notice that $\{H' : \eta' \in \pi^{-1}(\theta')\}$ is an open covering of $\pi^{-1}(\theta')$. But since π is proper and $\{\theta'\}$ is compact, $\pi^{-1}(\theta')$ is compact, so that, for some $\eta'_i, i = 1, 2, ..., n' < \infty, \pi^{-1}(\theta') \subset \cup_i H'_i$, that is, $\pi^{-1}(\theta') = \{\eta'_i, i = 1, 2, ..., n'\}$.

5. ($\Leftarrow$) Consider the following probability measure:

$$r^\theta([a, b]) = \frac{\mu}{1-\mu} 1_{\{1-\theta \in [a,b]\}} + \frac{1-2\mu}{1-\mu}(b - a).$$

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[1] Note that, when we define zero measure, we used closed retangle and its volumn, which is not relative to the mother space, in R^n.

That is,

$$r^{\theta}=\left\{\begin{array}{c}1-\theta \text{ with probability } \mu/\left(1-\mu\right)\\ \text{uniform distribution over } [0,1] \text{ with probability } \left(1-2\mu\right)/\left(1-\mu\right).\end{array}\right.$$

Then

$$\begin{aligned}\int_0^1 r^{\theta_1}\left([a,b]\right)d\theta_1 &= \frac{\mu}{1-\mu}\left(b-a\right)+\frac{1-2\mu}{1-\mu}\left(b-a\right)=\left(b-a\right)\\ \int_0^1 \theta_2 r^{\theta_2}\left([a,b]\right)d\theta_2 &= \left(b-a\right)\left[\frac{\mu}{1-\mu}\frac{2-\left(a+b\right)}{2}+\frac{1-2\mu}{1-\mu}\frac{1}{2}\right]\end{aligned}$$

$$E_{\mu,R}\left[\theta|m\in[a,b]\right]=\mu\frac{a+b}{2}+\left(1-\mu\right)\left(\frac{\mu}{1-\mu}\frac{2-\left(a+b\right)}{2}+\frac{1-2\mu}{1-\mu}\frac{1}{2}\right)=\frac{1}{2}.$$

($\Rightarrow$) Let $\overline{\theta}$ be the value such that

$$\frac{\mu\left(1/2-0\right)}{\mu\left(1/2-0\right)+\left(1-\mu\right)\left(1-\overline{\theta}\right)}\frac{0+1/2}{2}+\frac{\left(1-\mu\right)\left(1-\overline{\theta}\right)}{\mu\left(1/2-0\right)+\left(1-\mu\right)\left(1-\overline{\theta}\right)}\frac{\overline{\theta}+1}{2}=\frac{1}{2}.$$

Arranging terms,

$$\left(1-\overline{\theta}\right)\overline{\theta}=\frac{\mu}{4\left(1-\mu\right)}.$$

$\overline{\theta}$ is well-defined only for $\mu\leq 1/2$ and $\overline{\theta}\geq 1/2$ if it is well-defined. Now let $\underline{\theta}$ be the value such that

$$\frac{\mu\left(1-1/2\right)}{\mu\left(1-1/2\right)+\left(1-\mu\right)\left(\underline{\theta}-0\right)}\frac{1/2+1}{2}+\frac{\left(1-\mu\right)\left(\underline{\theta}-0\right)}{\mu\left(1-1/2\right)+\left(1-\mu\right)\left(\underline{\theta}-0\right)}\frac{\underline{\theta}}{2}=\frac{1}{2}.$$

Arranging terms,

$$\underline{\theta}^2-\underline{\theta}+\frac{\mu}{4\left(1-\mu\right)}=0.$$

Again, this value is well-defined only for $\mu\leq 1/2$ and $\underline{\theta}\leq 1/2$.
Suppose there exists a collection of probability measures that satisfies the condition. Then by construction,

$$\int_0^1 r^{\theta}\left([0,1/2]\right)d\theta\geq 1-\overline{\theta}$$

and

$$\int_0^1 r^{\theta}\left([1/2,1]\right)d\theta\geq \underline{\theta}.$$

Since

$$1=\int_0^1 r^{\theta}\left([0,1]\right)d\theta=\int_0^1 r^{\theta}\left([0,1/2]\right)d\theta+\int_0^1 r^{\theta}\left([1/2,1]\right)d\theta\geq 1-\left(\overline{\theta}-\underline{\theta}\right),$$

for there to exist such a collection, $\overline{\theta}$ and $\underline{\theta}$ should be well-defined. This establishes the result.