Suggested Solution for Quiz 1

1. $x, y \in X$. By triangular inequality,

$$|f(x) - f(y)| = |d(x,a) - d(y,a)| \leq d(x,y) = |x - y|$$

For any $\varepsilon > 0$, let $\delta = \varepsilon$. Then $|x - y| < \delta$ implies $|f(x) - f(y)| < \varepsilon$.

2. See Section 1. Connectedness

3. (1) See Section 1. Compactness

(2) Consider the following metric.

$$d(x,y) = \begin{cases} 0 \text{ if } x = y \\ 1 \text{ if } x \neq y \end{cases}$$

Then $B_1(a) = \{a\}$. This set is immediately compact, because whenever there exists an open covering, we can pick up an open set that contains a.

(3) We use the same metric in (2). Then $\overline{B}_1(a) = X$. Consider the following open covering: $\mathcal{U} = \{N_{1/2}(x), x \in X\}$. Obviously, this open covering does not have any finite subcovering. Hence with the specified metric, $\overline{B}_1(a)$ is not compact.

4. (1) Notice that $U(p', 0) = \underline{v} < U(p, 1) < U(p', 1)$. The second inequality comes from (ii). Since $U(p', \cdot)$ is continuous, by the Intermediate Value Theorem, there exists $\mu^p(p')$ such that $U(p, 1) = U(p', \mu^p(p'))$. Furthermore, it is unique because of (iv).

(2) Ilwoo provided a counterexample to this problem. To get the result, we need more restrictions on U.