I = Initial flux of galaxy
P = PSF
F = Observed flux of galaxy

$$F(x, y) = \iint I(u, v)\, P(x-u, y-v)\, \mathrm{d}u\, \mathrm{d}v \qquad \textbf{(1)}$$

$$\mathrm{I}_{xx} = \iint x^2\, I(x, y)\, \mathrm{d}x\, \mathrm{d}y \qquad \textbf{(2)}$$

$$P_{xx} = \iint x^2\, P(x, y)\, \mathrm{d}x\, \mathrm{d}y \qquad \textbf{(3)}$$

$$F_{xx} = \iint x^2\, F(x, y)\, \mathrm{d}x\, \mathrm{d}y \qquad \textbf{(4)}$$

$$F_{xx} = \iint x^2 \left(\iint I(u, v)\, P(x-u, y-v)\, \mathrm{d}u\, \mathrm{d}v \right) \mathrm{d}x\, \mathrm{d}y \qquad \textbf{(5)}$$

$$F_{xx} = \iint I(u, v) \left(\iint x^2\, P(x-u, y-v)\, \mathrm{d}x\, \mathrm{d}y \right) \mathrm{d}u\, \mathrm{d}v \qquad \textbf{(6)}$$

Make substitution h = x-u, k = y-v

$$F_{xx} = \iint I(u, v) \left(\iint (h+u)^2\, P(h, k)\, \mathrm{d}h\, \mathrm{d}k \right) \mathrm{d}u\, \mathrm{d}v \qquad \textbf{(7)}$$

$$F_{xx} = \iint I(u, v) \left[\iint h^2\, P(h, k)\, \mathrm{d}h\, \mathrm{d}k + 2\,u \left(\iint h\, P(h, k)\, \mathrm{d}h\, \mathrm{d}k \right) + u^2 \left(\iint P(h, k)\, \mathrm{d}h\, \mathrm{d}k \right) \right] \mathrm{d}u\, \mathrm{d}v \qquad \textbf{(8)}$$

We assume that I and P are normalized and properly centroided, so this simplifies to

$$F_{xx} = \iint I(u, v) \left[P_{xx} + 2\,u\,0 + u^2\,1 \right] \mathrm{d}u\, \mathrm{d}v \qquad \textbf{(9)}$$

$$F_{xx} = \iint I(u, v) \left[P_{xx} + u^2 \right] \mathrm{d}u\, \mathrm{d}v \qquad \textbf{(10)}$$

$$F_{xx} = P_{xx} \left(\iint I(u, v)\, \mathrm{d}u\, \mathrm{d}v \right) + \iint I(u, v)\, u^2\, \mathrm{d}u\, \mathrm{d}v \qquad \textbf{(11)}$$

$$F_{xx} = P_{xx} + \mathrm{I}_{xx} \qquad \textbf{(12)}$$

The same procedure follows for the other two relations as well:

$$F_{yy} = P_{yy} + \mathrm{I}_{yy} \qquad \textbf{(13)}$$

$$F_{xy} = P_{xy} + \mathrm{I}_{xy} \qquad \textbf{(14)}$$

2.(a) Distances in the flat 0.3/0.7 Universe:

$$\chi\left(\begin{matrix} z=0.3 \\ z=1.0 \end{matrix}\right) = c/H_0 \cdot \begin{matrix} 0.279 = \chi_L \text{ (comoving ang diam dist)} \\ 0.77 = \chi_S \end{matrix}$$

$$\left[\text{Recall } \chi = \int_0^z \frac{c\,dz}{H(z)} = \frac{c}{H_0}\int_0^z dz \left[\Omega_\Lambda + \Omega_M (1+z)^3\right]^{-1/2} \right]$$

Now: $$\Sigma_{crit} = \frac{c^2}{4\pi G} \cdot \frac{D_S}{D_L D_{LS}}$$

$$= \frac{cH_0}{4\pi G} \cdot \frac{(1+z_L)\chi_S}{\chi_L(\chi_S - \chi_L)}$$

$$= \frac{3\times10^8 \text{ m/s} \cdot (3.1\times10^{17}\text{ s})^{-1}}{4\pi \cdot (6.67\times10^{-11}\text{ kg}^{-1}\text{m}^3\text{s}^{-2})} \cdot \frac{1.3\times 0.77}{0.28(0.77-0.28)}$$

$$= 8.4 \frac{\text{kg}}{\text{m}^2} = \boxed{0.84 \text{ g/cm}^2}.$$

(b) This is a little tricky since SIS mass/potential integrals can diverge.

Projected density: $$\Sigma(\rho) = \int_{-\infty}^{\infty} dz \frac{\sigma^2}{2\pi G(\rho^2+z^2)} = \frac{\sigma^2}{2\pi G\rho} \cdot \int_{-\infty}^{\infty} \frac{du}{1+u^2} = \frac{\sigma^2}{2G\rho}$$

Lensing potential will satisfy

$$\nabla_\theta^2 \psi = 2\Sigma/\Sigma_{crit} = 2\frac{D_L D_{LS}}{D_S} \cdot \frac{4\pi G}{c^2} \cdot \frac{\sigma^2}{2G\rho} = \frac{4\pi D_{LS} D_L}{D_S} \cdot \frac{\sigma^2}{c^2} \cdot \frac{1}{\rho}.$$

Put $\rho = D_L \cdot \theta$, note also that $\nabla_\theta^2 \psi = \frac{1}{\theta}\frac{\partial}{\partial\theta}\left(\theta \frac{\partial\psi}{\partial\theta}\right)$ for circ. symmetry.

$$\frac{1}{\theta}\frac{\partial}{\partial\theta}\left(\theta\frac{\partial\psi}{\partial\theta}\right) = \frac{1}{\theta}\cdot\left[\frac{4\pi D_{LS}}{D_S}\frac{\sigma^2}{c^2}\right]$$

2(b) Potential solution is $\psi = \left[\frac{4\pi D_{LS}}{D_S} \cdot \frac{\sigma^3}{c^2}\right] \cdot \Theta$

Look at shear at Θ_x, $\Theta_y = 0$ - must be γ_1 component only.

$$\gamma_1 = \frac{1}{2}\left(\frac{\partial^2}{\partial\Theta_x^2} - \frac{\partial^2}{\partial\Theta_y^2}\right)\psi = \boxed{\frac{4\pi\sigma^2}{c^2} \cdot \frac{D_{LS}}{D_S} \cdot \frac{1}{\Theta}}$$

For $\frac{D_{LS}}{D_S} = \frac{0.77-0.28}{0.77}$, $\sigma = 1000$ km/s, this is

$$\gamma = \frac{8.9\times10^{-3}}{\Theta} = 0.31\times\left(\frac{1'}{\Theta}\right).$$

The deflection angle (apparent) is $\nabla_\Theta \psi$ which is constant at $\alpha = \frac{4\pi\sigma^2}{c^2} \cdot \frac{D_{LS}}{D_S} = 8.9\times10^{-5}$ rad $= 18.''3$.

The Einstein ring will appear at radius $\Theta = 18.''3$ since such rays land at origin of source plane.

2 (c) Uncertainty in mean tangential shear is $\frac{\sigma_\gamma}{\sqrt{N}}$ for N sources in annulus.

$$N = 2\pi\Theta\Delta\Theta \cdot n, \quad n = 20/\text{arcmin}^2 = 2.4\times10^8/\text{sr}$$

If $\Theta, \Delta\Theta$ are in arcmin:

$$\frac{S}{N} = \frac{4\pi\sigma^2}{c^2}\frac{D_{LS}}{D_S} \cdot \frac{1}{\Theta} \Big/ \left(\sigma_\gamma \big/ \sqrt{2\pi\Theta\Delta\Theta n}\right)$$

$$= \frac{\sqrt{8\pi n}\,\sigma^2}{\sigma_\gamma c^2} \cdot \frac{D_{LS}}{D_S} \sqrt{\frac{\Delta\Theta}{\Theta}} = 1.8\times\sqrt{\Delta(\ln\Theta)}$$

[illegible]

2(c) Each radial bin gives

$$S/N = \sqrt{32\pi^3 n} \cdot \frac{\sigma^2}{c^2} \cdot \frac{1}{\sigma_\gamma} \cdot \frac{D_{LS}}{D_S} \cdot \sqrt{\frac{\Delta\theta}{\theta}}$$

$$= 11.6 \times \sqrt{\Delta(\ln\theta)} .$$

Each e-fold range in θ gives $S/N \hat{=} 12$ on the sheer signal.

2(d). What's range of θ that we can use?

$\theta_{max} = 30'$

$\theta_{min} = 1'$ since we need $\gamma = \frac{0.3}{\theta} \leq 0.3$

$$\Rightarrow \Delta\ln\theta = \ln 30 = 3.4$$

So total S/N is $11.6 \times \sqrt{3.4} = \boxed{21.4}$

This means that amplitude of the sheer is measured to 1 part in 21.4, or 4.7%. Since $\gamma \propto \sigma^2$, the value of σ is measured to $\pm 4.7\%/2 = \pm 2.3\% = \boxed{\pm 23 \text{ km/s.}}$

As an aside: The virial mass of a cluster is mass within R_{VIR} - which is often defined as radius within which

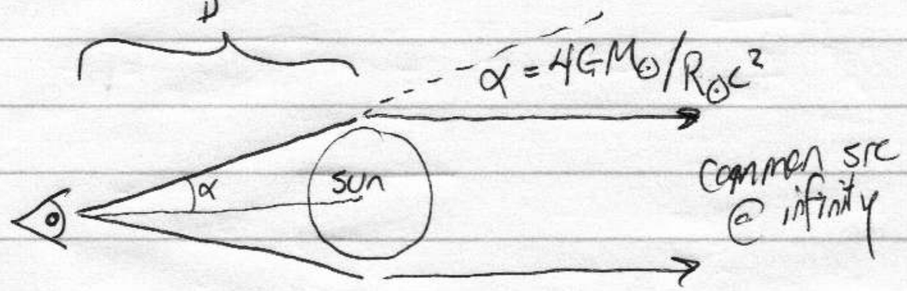
$$\frac{M(<R_{vir})}{\frac{4}{3}\pi R_{vir}^3} = 200 \cdot \rho_{crit}$$

For SIS: $M(<R) = 4\pi \int_0^R r^2 dr \cdot \frac{\sigma^2}{2\pi G r^2} = \frac{2\sigma^2 R}{G}$ $\boxed{\approx 7\times10^{14} M_\odot}$

$$\Rightarrow \frac{3\sigma^2}{2\pi G R_{VIR}^2} = 200\,\rho_{crit} \Rightarrow R_{VIR}^2 = \frac{3\sigma^2}{400\pi G \rho_{crit}} \quad !!$$

$$\Rightarrow M(<R) = \sigma^3 \cdot \sqrt{\frac{3}{100\pi G^3 \rho_{crit}}} = \sigma^3 \cdot \sqrt{\frac{3}{100\pi G^3} \cdot \frac{8\pi G}{3H^2}} = \frac{\sigma^3}{GH} \cdot \frac{\sqrt{2}}{5}$$

Optional problem #3:



To get 2 lines of sight near limb of Sun to emerge parallel or converging requires

$$R_\odot/D \leq \alpha = \frac{4GM_\odot}{R_\odot c^2} = 1.''75$$

$$\Rightarrow D \geq \frac{R_\odot}{\alpha} \cong \frac{1\,\text{AU}/200}{1.7''} \approx \frac{1\,\text{pc}}{340}$$ - not far at all.

Precise: $D \geq \frac{R_\odot^2 c^2}{4GM_\odot} = 8.3 \times 10^{13}\,\text{m} = 0.0027\,\text{pc}.$

$(M_\odot = 2 \times 10^{30}\,\text{kg},\ R_\odot = 7 \times 10^{8}\,\text{m})$