SIAM Review
Vol. 16, No. 2, April 1974

MARKOVIAN LEARNING PROCESSES*

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Abstract. Real-valued Markov processes with steps of the form $\Delta X_n = \theta_i(\lambda_i - X_n)$ or $\Delta X_n = \delta_i$, $i = 1, \cdots, v$, arise frequently in the theory of human and animal learning. This paper first surveys results pertaining to the asymptotic behavior of such processes as $n \to \infty$. Next, general theorems are given that cover limiting behavior as $n \to \infty$ and, simultaneously, $\theta_i \to 0$ or $\delta_i \to 0$. Finally, an example of a two-dimensional learning process X_n is presented.

1. Introduction. During the last 25 years, psychological theorists have learned how to construct stochastic models for human and animal learning. The simplicity of these models reflects, to some extent, the simplicity of the learning experiments conducted in most psychological laboratories. Such experiments typically involve a sequence of trials under similar experimental conditions. It is natural to regard the subject's behavior or response, R_n, on trial n as a random variable. Stochastic learning models assume that the distribution of R_n is determined by another random variable, X_n, which represents the subject's state of learning on trial n. The value of X_n changes from trial to trial as a function of the subject's response, its relation to prevailing stimulus conditions, and the ensuing outcome or payoff.

In typical models, the responses $R_0, R_1, R_2, \cdots$ will exhibit complicated statistical interdependence, but the state sequence $X_0, X_1, X_2, \cdots$ will be Markovian with stationary transition probabilities. Thus mathematical investigations of learning models tend to focus on X_n. It turns out that such state sequences and their generalizations have considerable mathematical interest. The fact that, as often as not, their state spaces are continuous, lends a special flavor to their study.

A comprehensive presentation of the Markov process theories that relate to learning models is given in [14]. This paper attempts to convey something of the substance and spirit of this area by surveying briefly a variety of typical results. Some important open problems are also pointed out. In the interest of simplicity, only real-valued processes X_n are considered, except in the final section.[1]

In the remainder of this section we will describe two special learning models. These examples will make clear exactly what sorts of processes we have in mind.

Consider first a *prediction experiment*. On each trial, a human subject predicts whether or not a lamp on a panel before him will flash. Flashes on different trials are independent events with identical probability π. The *linear model* characterizes a subject by his probability X_n of predicting a flash on trial n. If he predicts a flash and it occurs, X_n increases to

$$X_{n+1} = (1 - \theta_1)X_n + \theta_1,$$

where $0 \leqq \theta_1 < 1$. The probability of this occurrence is $X_n\pi$. If he predicts a flash

* Received by the editors February 20, 1973, and in revised form May 7, 1973.

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[1] The reader who wishes to sample a broad range of current mathematical theories of learning and memory should consult [1].

and it does not occur, X_n decreases to

$$X_{n+1} = (1 - \theta_2)X_n.$$

This has probability $X_n(1 - \pi)$. An unpredicted flash raises X_n to

$$X_{n+1} = (1 - \theta_3)X_n + \theta_3,$$

while a predicted nonflash drops X_n to

$$X_{n+1} = (1 - \theta_4)X_n.$$

These events have probabilities $(1 - X_n)\pi$ and $(1 - X_n)(1 - \pi)$. The parameters $\theta_i \in [0, 1)$ control the rates of learning on trials with various responses and outcomes. Models of this type were introduced by Estes [7] and by Bush and Mosteller [2].

In a comparable *detection experiment*, a flash, if it occurs at all, is so faint that it cannot be seen reliably. The subject's task is to report whether or not a flash occurs at the beginning of a trial. After his report, he is told whether or not he was correct. Our model assumes that the subject has a real-valued internal response Y_n to the stimulus condition on trial n. If the flash (signal) occurs, Y_n has cumulative distribution function F_S; otherwise (noise) it has distribution F_N (e.g., $F_N(x) = F_S(x + \mu)$, $\mu > 0$). We assume that F_S and F_N are continuous and strictly increasing. The subject reports a flash if and only if Y_n exceeds a criterion X_n. This criterion varies from trial to trial, decreasing when the signal occurs, and increasing when it does not. More specifically, X_n is a Markov process with stationary transition probabilities determined by the equations

$$\Delta X_n = \begin{cases} \delta_1 \leqq 0 & \text{with probability} \quad \pi(1 - F_S(X_n)), \\ \delta_2 \leqq 0 & \text{with probability} \quad \pi F_S(X_n), \\ \delta_3 \geqq 0 & \text{with probability} \quad (1 - \pi)(1 - F_N(X_n)), \\ \delta_4 \geqq 0 & \text{with probability} \quad (1 - \pi)F_N(X_n). \end{cases}$$

The four lines correspond, respectively, to correct detections, missed signals, false alarms, and correct nondetections. The constants δ_i are learning rate parameters, and, as in the last paragraph, π is the probability of a flash. The special case of this model with learning only on errors ($\delta_1 = \delta_4 = 0$) and with equal downward and upward steps ($\delta_2 = -\delta_3$) was introduced by Kac [9]. The above generalization is due to Dorfman and Biderman [5], who tested it empirically. We will refer to it as the Kac–Dorfman–Biderman or KDB model. An alternative generalization of Kac's model, to which the results reviewed in this paper are equally applicable, was proposed by Thomas [16].

Section 2 treats the asymptotic behavior, as $n \to \infty$, of processes X_n like those arising in the linear and KDB models. Section 3 considers the limiting behavior of such processes when their learning rate parameters θ_i and δ_i approach 0, so that learning is slow. Section 4 describes a model for discrimination learning in which the process X_n is two-dimensional.

Though this paper concentrates on state sequences X_n, § 2.3 presents a rather striking result concerning the response sequence R_n in Kac's model. The proportion P_N of reports of signals in N trials converges almost surely to the probability π that a signal occurs. Such *probability matching* has been observed experimentally [4].

2. Asymptotic behavior of X_n. We will describe the asymptotic behavior of *linear processes* and *additive processes*, which generalize, respectively, the processes X_n generated by the linear and KDB models of the last section. The forms of transition from X_n to X_{n+1} (e.g., $\Delta X_n = \delta_i$) are retained, but any finite number $\nu \geqq 2$ of such transitions is permitted, and the probability $p_i(X_n)$ of the ith transition is subject only to mild regularity conditions.

We denote the transition kernel of X_n by K,

$$K(x, A) = P(X_{n+1} \in A | X_n = x),$$

and the corresponding transition operator on bounded measurable real-valued functions by U,

$$Uf(x) = \int K(x, dy) f(y) = E(f(X_{n+1}) | X_n = x).$$

The iterates $K^{(m)}$ and U^m of K and U satisfy

$$K^{(m)}(x, A) = P(X_{n+m} \in A | X_n = x)$$

and

$$U^m f(x) = \int K^{(m)}(x, dy) f(y) = E(f(X_{n+m}) | X_n = x).$$

2.1. Linear processes. In this paper we use the term *linear process* to denote a Markov process X_n with state space $[0, 1]$ governed by transition equations of the form

$$X_{n+1} = (1 - \theta_i) X_n + \theta_i \lambda_i \quad \text{with probability } p_i(X_n),$$

where $0 \leqq \theta_i < 1$, $\lambda_i = 0$ or 1, and $1 \leqq i \leqq \nu$. Thus

$$Uf(x) = \sum_{i=1}^{\nu} f((1 - \theta_i)x + \theta_i \lambda_i) p_i(x).$$

It is assumed that $p_i(x) > 0$ for $0 < x < 1$, and that p_i is Lipschitz, i.e.,

$$m(p_i) = \sup_{x \neq y} \frac{|p_i(x) - p_i(y)|}{|x - y|} < \infty.$$

Of course, $\sum_i p_i(x) \equiv 1$. Finally, we assume that there is an index i such that $\theta_i > 0$ and $p_i(0) > 0$, and there is an index j such that $\theta_j > 0$ and $p_j(1) > 0$.

The set L of Lipschitz functions on $[0, 1]$ is a Banach space with respect to the norm

$$\|f\| = |f| + m(f),$$

where $|\cdot|$ is the supremum norm.

THEOREM 2.1. *Under these hypotheses the sequence U^n converges uniformly to an operator U^∞:*

$$\lim_{n \to \infty} \|U^n - U^\infty\| = 0.$$

The only possible absorbing states are 0 and 1, and it is easy to decide whether or not one of these states is absorbing. The main results for the case of k absorbing states are presented under "Case k" of Theorem 2.2.

THEOREM 2.2.

Case 1. *If j is the only absorbing state, then $X_n \to j$ a.s. as $n \to \infty$.*

Case 2. *If both* 0 *and* 1 *are absorbing, then*

$$X_n \to X_\infty \in \{0, 1\}$$

a.s. as $n \to \infty$. Let $h(x)$ be the probability that $X_\infty = 1$ when $X_0 = x$,

$$h(x) = P_x(X_\infty = 1).$$

Then h is Lipschitz, $0 < h(x) < 1$ for $0 < x < 1$, and h is the only bounded measurable solution of $Uh = h$ with $h(0^+) = h(0) = 0$ and $h(1^-) = h(1) = 1$.

Case 0. *If there are no absorbing states, then*

$$\limsup_{n\to\infty} X_n = 1 \quad \text{and} \quad \liminf_{n\to\infty} X_n = 0 \tag{2.1}$$

a.s. There is a unique stationary probability $K^\infty(\cdot)$, and, for any x,

$$K^{(n)}(x, \cdot) \to K^\infty(\cdot)$$

weakly as $n \to \infty$. If f is continuous,

$$A_N = \frac{1}{N}\sum_{n=0}^{N-1} f(X_n) \to \int f\, dK^\infty = a \tag{2.2}$$

a.s. as $N \to \infty$. If f is Lipschitz, then $\sqrt{N}(A_N - a)$ is asymptotically normally distributed with mean 0 *and variance $\sigma^2 \geqq 0$ as $N \to \infty$.*

Theorems 2.1 and 2.2 apply to the linear learning model of § 1 if, for example, $0 < \pi < 1$, $\theta_1 > 0$, and $\theta_4 > 0$ ($p_1(x) = x\pi$, so $p_1(1) > 0$, and $p_4(x) = (1 - x)(1 - \pi)$, so $p_4(0) > 0$). The state 1 is absorbing if and only if $\theta_2 = 0$, while 0 is absorbing if and only if $\theta_3 = 0$.

Proofs. If $f \in L$,

$$Uf(x) - Uf(y) = \sum_i (f(x_i) - f(y_i))p_i(x) + \sum_i f(y_i)(p_i(x) - p_i(y)),$$

where $x_i = (1 - \theta_i)x + \theta_i\lambda_i$. Thus

$$|Uf(x) - Uf(y)| \leqq \left[m(f)\sum_i (1 - \theta_i)p_i(x) + |f|\sum_i m(p_i)\right]|x - y|,$$

so that

$$m(Uf) \leqq rm(f) + R|f|, \tag{2.3}$$

where

$$r = \max_x \sum_i (1 - \theta_i)p_i(x)$$

and

$$R = \sum_i m(p_i).$$

Under our assumptions, $R < \infty$ and $r < 1$, so (2.3) means that U is a Doeblin–Fortet operator [14, Def. 2.1.2]. Consequently, since the state space [0, 1] is compact, X_n is a compact Markov process [14, Def. 3.3.1]. The theory of such processes, which draws on the work of Ionescu Tulcea and Marinescu, Jamison, and Rosenblatt, is developed in §§ 3.3–3.6 of [14]. The remainder of the proof is heavily dependent on that development.

Let $\sigma_n(x)$ be the support of $K^{(n)}(x, \cdot)$, and write $\sigma_n(x) \to y$ if

$$d(\sigma_n(x), y) = \inf_{z \in \sigma_n(x)} |z - y| \to 0$$

as $n \to \infty$. We will need the following elementary lemmas.

LEMMA 2.1. *If j is not absorbing, then $\sigma_n(x) \to 1 - j$ for all $0 \leqq x \leqq 1$.*

LEMMA 2.2. *If j is absorbing, then $\sigma_n(x) \to j$ for all $x \neq 1 - j$.*

Proofs of lemmas. Suppose, for instance, that 1 is not absorbing. Then there is an i such that $p_i(1) > 0$, $\theta_i > 0$, and $\lambda_i = 0$. Let $x^{(0)} = x$ and $x^{(n+1)} = (1 - \theta_i)x^{(n)}$. Since $\theta_i < 1$ and $p_i(x) > 0$ for $x > 0$, it follows by induction that $x^{(n)} \in \sigma_n(x) \cap (0, 1]$ for $x > 0$ and $n \geqq 0$. But $x^{(n)} \to 0$, so $\sigma_n(x) \to 0$ as $n \to \infty$. If 0 is absorbing, $\sigma_n(x) = \{0\}$ for all n. If 0 is not absorbing, there is an $x > 0$ in $\sigma_1(0)$, so $x^{(n)} \in \sigma_{n+1}(0)$. This completes the proof of Lemma 2.1. To prove Lemma 2.2, we recall our assumption that there is an i such that $p_i(0) > 0$ and $\theta_i > 0$. If 0 is absorbing, λ_i is necessarily 0. Since $p_i(x) > 0$ for $x < 1$, it follows that $x^{(n)} \in \sigma_n(x)$. Thus $\sigma_n(x) \to 0$ and the proof of Lemma 2.2 is complete.

If $1 - j$ is the only absorbing state, Lemma 2.1 shows that $\sigma_n(x)$ approaches it for all $0 \leqq x \leqq 1$. If both 0 and 1 are absorbing, Lemma 2.2 shows that $\sigma_n(x)$ approaches one or the other of these states for any x. It follows from Theorem 3.6.2 of [14] that X_n is an absorbing process in Cases 1 and 2. In particular, X_n converges a.s. to the absorbing states. If neither 0 nor 1 is absorbing, then, by Lemma 2.1, $\sigma_n(x)$ approaches 1 (and 0) as $n \to \infty$, for any $0 \leqq x \leqq 1$. As a consequence of Theorem 3.6.1 of [14], X_n is regular in this case. It follows that $K^{(n)}(x, \cdot)$ converges to a unique stationary probability. Regardless of the number of absorbing states, Theorem 2.1 (called *aperiodicity* in [14]) applies.

The fact that h is Lipschitz follows from Theorem 3.3.2 of [14]. Since $h(1^-) = h(1) = 1$, there is a $\delta \in (0, 1)$ such that $h(y) > 0$ for $y > 1 - \delta$. Let $0 < x < 1$. As a consequence of Lemma 2.2, $d(\sigma_n(x), 1) < \delta$ for n sufficiently large, hence

$$h(x) = U^n h(x) = E_x(h(X_n)) > 0.$$

The proof that $h(x) < 1$ is similar.

If H is a bounded measurable solution of $UH = H$ with $H(0) = H(0^+) = 0$ and $H(1) = H(1^-) = 1$, then

$$\begin{aligned} H(x) &= E_x(H(X_n)) \\ &\to E_x(I_{X_n \to 1}) = h(x), \end{aligned}$$

where I_A is the indicator variable of the event A. Thus $H = h$.

The strong law of large numbers (2.2) is contained in Theorem 3.4.4 of [14], and the central limit theorem for A_N follows from Theorem 5.2.3 of [14].

In Case 0, the support F of $K^\infty(\cdot)$ is stochastically as well as topologically closed [14, Thm. 3.4.2]. Hence, by Lemma 2.1, 0 and 1 are in F. For any $1 > \delta > 0$, let f be continuous with $f(x) = 0$ for $x \leqq 1 - \delta$ and $f(x) > 0$ for $x > 1 - \delta$. By (2.2),

$$A_N \to \int f\, dK^\infty > 0.$$

This implies that

$$\limsup_{n\to\infty} X_n \geqq 1 - \delta$$

a.s. Since δ is arbitrary,

$$\limsup_{n\to\infty} X_n = 1.$$

The other part of (2.1) is proved similarly.

2.2. Additive processes. We will call a Markov process X_n with state space $(-\infty, \infty)$ an *additive process* if it is governed by transition equations of the form

$$X_{n+1} = X_n + \delta_i \quad \text{with probability } p_i(X_n),$$

where δ_i is a real constant for $1 \leqq i \leqq \nu$. The transition operator for such a process is

$$Uf(x) = \sum_{i=1}^{\nu} f(x + \delta_i)p_i(x).$$

We assume that p_i is continuous and $p_i(x) > 0$ for all x.

Considerably more is known about linear processes than about additive processes. Theorem 2.3 gives analogues for the latter of some of the results in Theorem 2.2. Let

$$W(x) = E(\Delta X_n | X_n = x) = \sum_i \delta_i p_i(x),$$

$$W^*(\infty) = \limsup_{x\to\infty} W(x), \quad \text{and} \quad W_*(\infty) = \liminf_{x\to\infty} W(x).$$

$W^*(-\infty)$ and $W_*(-\infty)$ are defined similarly. We call the boundary ∞ *attractive* if $W_*(\infty) > 0$ and *reflective* if $W^*(\infty) < 0$. Similarly, $-\infty$ is attractive if $W^*(-\infty) < 0$ and reflective if $W_*(-\infty) > 0$. "Case k" in Theorem 2.3 treats k attractive boundaries.

THEOREM 2.3.

Case 1. *If one boundary is attractive and the other is reflective, then X_n converges to the attractive boundary a.s.*

Case 2. *If both boundaries are attractive,*

$$X_n \to X_\infty \in \{\infty, -\infty\}$$

a.s. Let

$$h(x) = P_x(X_\infty = \infty).$$

Then $0 < h(x) < 1$ for all x, and h is the only bounded measurable solution to $Uh = h$ for which $h(-\infty) = 0$ and $h(\infty) = 1$.

Case 0. *If both boundaries are reflective, then*

$$\limsup_{n\to\infty} X_n = \infty \quad and \quad \liminf_{n\to\infty} X_n = -\infty$$

a.s. If $|\lambda|$ *is sufficiently small, there is a constant* $B(\lambda)$ *such that*

$$\limsup_{n\to\infty} E_x(e^{\lambda X_n}) \leqq B(\lambda) \tag{2.4}$$

for all x. *There is at least one stationary probability, and all stationary probabilities* μ *satisfy*

$$\int_{-\infty}^{\infty} e^{\lambda y}\mu(dy) \leqq B(\lambda).$$

Proof. The uniqueness of h in Case 2 is established in the same way as the analogous property in Theorem 2.2. Proofs of the remaining assertions can be constructed along the lines of proofs of comparable statements in Chapter 14 of [14]. The only proofs that will be given here are those of (2.4) and the existence of a stationary probability in Case 0.

Expanding

$$F(x,\lambda) = E(e^{\lambda\Delta X_n}|X_n = x)$$

about $\lambda = 0$, we get

$$F(x,\lambda) = 1 + \lambda W(x) + \lambda^2 g(x,\lambda),$$

where

$$g(x,\lambda) \leqq 2^{-1} \max_{1\leqq i\leqq \nu} |\delta_i|^2\, e^{|\delta_i|} = G$$

for $|\lambda| \leqq 1$. Let X be so large that $W(x) \leqq W^*(\infty)/2$ for $x \geqq X$. If $x \geqq X$ and $0 < \lambda < |W^*(\infty)|/4G$, then

$$F(x,\lambda) \leqq 1 + 4^{-1}\lambda W^*(\infty) = C(\lambda) < 1.$$

Consequently

$$\begin{aligned} E(e^{\lambda X_{n+1}}|X_n) &= F(X_n,\lambda)\, e^{\lambda X_n} \\ &\leqq C(\lambda)\, e^{\lambda X_n} \end{aligned}$$

for $X_n \geqq X$. If $X_n \leqq X$, we have

$$E(e^{\lambda X_{n+1}}|X_n) \leqq \max_i e^{\lambda(X+\delta_i)} = D(\lambda).$$

Thus

$$E(e^{\lambda X_{n+1}}|X_n) \leqq C(\lambda)\, e^{\lambda X_n} + D(\lambda)$$

a.s. and

$$E(e^{\lambda X_{n+1}}) \leqq C(\lambda)E(e^{\lambda X_n}) + D(\lambda).$$

Equation (2.4) with $B(\lambda) = D(\lambda)/(1 - C(\lambda))$ follows. The case $\lambda < 0$ can be handled similarly. Note that this proof requires neither positivity nor continuity of p_i.

In fact, it applies to any real-valued Markov process with bounded increments and two reflective boundaries.

Choose any real x. It follows from (2.4) that

$$E_x(\cosh \lambda X_n) = \int K^{(n)}(x, dy) \cosh \lambda y$$

is bounded in n for any λ with $|\lambda|$ sufficiently small. Thus the sequence

$$\mu_N = \frac{1}{N} \sum_{n=0}^{N-1} K^{(n)}(x, \cdot)$$

has a subsequence that converges weakly to a probability μ. Let T be the operator on probabilities defined by

$$T\omega(A) = \int \omega(dy) K(y, A).$$

Clearly $T\mu_N - \mu_N \to 0$. Assuming that p_i is continuous, U preserves continuity. Consequently, if f is bounded and continuous,

$$\int f\, dT\mu_N = \int Uf\, d\mu_N$$

$$\to \int Uf\, d\mu = \int f\, dT\mu,$$

i.e., $T\mu_N \to T\mu$ along the subsequence. Thus $T\mu = \mu$, so μ is stationary. This completes the proof.

Theorem 2.3 says nothing about the behavior of X_n when $W(\infty)$ or $W(-\infty)$ is 0. Useful information about such cases can be obtained by the methods of Lamperti [10].

We turn now to the question of uniqueness of the stationary probability in Case 0. If the steps δ_i are all integral multiples of one number δ ($\delta_i = j_i \delta$), there is a continuum of distinct stationary probabilities, for the stationary probability μ constructed in the proof of Theorem 2.3 is concentrated on the lattice $x + \delta Z$, where Z is the set of all integers, and x can be chosen arbitrarily. Whenever the δ_i are not commensurable in this sense, the support of any stationary probability is the entire line, and it seems likely that uniqueness obtains. Proof or disproof of this conjecture is a conspicuous open problem. The main positive result obtained to date applies to the process

$$\Delta X_n = \begin{cases} \delta_1 < 0 & \text{with probability} \quad p_1(X_n), \\ \delta_2 > 0 & \text{with probability} \quad p_2(X_n), \end{cases}$$

for which $\nu = 2$.

THEOREM 2.4. *If both boundaries are reflective, p_1 is nondecreasing, and δ_1/δ_2 is irrational, then there is a unique stationary probability.*

This result differs from Theorem 14.4.2 of [14] only in that the latter theorem assumes $p_1(\infty) = 1$ and $p_1(-\infty) = 0$. These conditions imply that both boundaries are reflective. The proof of Theorem 14.4.2 of [14] applies without change to Theorem 2.4.

Naturally, the results of this section can be applied to the KDB model. Suppose that $0 < \pi < 1$. If the subject learns only when he is correct ($\delta_1 < 0$, $\delta_4 > 0, \delta_2 = \delta_3 = 0$), then $W(\infty) = \delta_4(1 - \pi) > 0$ and $W(-\infty) = \delta_1\pi < 0$. Thus both boundaries are attractive, and $X_n \to \pm\infty$ a.s. by Theorem 2.3. If the subject learns only on errors ($\delta_2 < 0$, $\delta_3 > 0$, $\delta_1 = \delta_4 = 0$), then $W(\infty) = \delta_2\pi < 0$ and $W(-\infty) = \delta_3(1 - \pi) > 0$. Thus both boundaries are reflective, and Case 0 of Theorem 2.3 applies. It follows from Theorem 2.4, via a brief argument given in [14, p. 254], that the process X_n has a unique stationary probability when δ_2/δ_3 is irrational.

2.3. Probability matching. Continuing our analysis of the KDB model with learning only on errors, suppose that the corrections after false alarms and missed signals are equal in magnitude though opposite in direction. Thus $|\delta_2| = \delta_3 = \delta$, as in Kac's paper [9]. Let R_n be a random variable that is 1 or 0, depending on whether or not the subject says a flash occurred on trial n, and let

$$P_N = \frac{1}{N}\sum_{n=0}^{N-1} R_n$$

be his proportion of "yes" responses over the first N trials of the experiment.

THEOREM 2.5. *P_N converges to π a.s. and $\sqrt{N}(P_N - \pi)$ is asymptotically normally distributed with mean* 0 *and variance $\pi(1 - \pi)$ as $N \to \infty$.*

There is an analogue of Theorem 2.5 for the linear model with all θ_i's equal. In that case, the proportion P_N of predictions of flashes converges to π and $\sqrt{N}(P_N - \pi)$ is asymptotically normal, but the asymptotic variance exceeds $\pi(1 - \pi)$ [14, p. 180 and (12.3.23)]. Deviations of P_N from π in prediction experiments tend to be larger than those predicted by the linear model, but averages of P_N over several subjects are typically very close to π [8].

Proof. Let S_n be 1 or 0, depending on whether or not the light flashes on trial n. Clearly

$$\begin{aligned}\Delta X_n &= -\delta S_n(1 - R_n) + \delta(1 - S_n)R_n\\ &= -\delta S_n + \delta S_n R_n + \delta(1 - S_n)R_n\\ &= -\delta S_n + \delta R_n.\end{aligned}$$

Thus

$$R_n = S_n + \Delta X_n/\delta.$$

Averaging this equation over $0 \leqq n \leqq N - 1$, we obtain

$$P_N = V_N + (X_N - X_0)/\delta N, \tag{2.5}$$

and thus

$$\sqrt{N}(P_N - \pi) = \sqrt{N}(V_N - \pi) + (X_N - X_0)/\delta\sqrt{N}, \tag{2.6}$$

where

$$V_N = \frac{1}{N}\sum_{n=0}^{N-1} S_n.$$

Suppose for the moment that $X_0 = x$ a.s. It was noted in the proof of Theorem 2.3 that $E_x(\cosh \lambda X_n)$ is bounded if $|\lambda|$ is sufficiently small. Hence $E_x(X_n^4)$ is also bounded. Consequently

$$E_x\left(\sum_{n=1}^{\infty} X_n^4/n^2\right) = \sum_{n=1}^{\infty} E_x(X_n^4)/n^2 < \infty,$$

$$\sum_{n=1}^{\infty} X_n^4/n^2 < \infty \quad \text{a.s.,}$$

and

$$X_n/\sqrt{n} \to 0 \quad \text{a.s.} \tag{2.7}$$

Since (2.7) holds for any initial state x, it holds for any distribution of X_0. Thus the terms on the extreme right in (2.5) and (2.6) converge a.s. to 0. But S_n is an independent Bernoulli sequence, so the theorem certainly holds with V_N in place of P_N. In view of (2.5) and (2.6), the proof is complete.

Theorem 2.5 assumes that the changes δ_1 and δ_4 in X_n after correct responses are zero. The theorem and its proof extend immediately to the case where these quantities have equal positive magnitudes ($|\delta_1| = \delta_4 = \delta^* \geqq 0$), as long as δ^* is sufficiently small that both boundaries are reflective. Let $\sigma = \delta^* + \delta$ and $q = \delta^*/\sigma$. A simple calculation shows that

$$W(\infty) = \sigma(q - \pi) \quad \text{and} \quad W(-\infty) = \sigma(1 - \pi - q)$$

Thus both boundaries are reflective if and only if $q < \min(\pi, 1 - \pi)$. In that case,

$$P_N \to P_\infty = \pi + (2\pi - 1)q/(1 - 2q)$$

a.s., and $\sqrt{N}(P_N - P_\infty)$ is asymptotically normally distributed with mean 0 and variance $\pi(1 - \pi)/(1 - 2q)^2$.

Suppose that $\pi > 1/2$. Then $P_\infty = P_\infty(q)$ increases from $P_\infty(0) = \pi$ to $P_\infty(1 - \pi) = 1$. The above argument does not apply to $q = 1 - \pi$, since $W(-\infty) = 0$ in that case. However, it can be shown that P_N converges in probability to 1. When $1 - \pi < q < \pi$, $-\infty$ is attractive and ∞ is reflective, hence $X_n \to -\infty$ a.s. by Case 1 of Theorem 2.3. It follows easily that $P_N \to 1$ a.s. as $N \to \infty$. Finally, when $q > \pi$, both boundaries are attractive, and P_N converges to 0 or 1, depending on whether $X_n \to \infty$ or $X_n \to -\infty$.

3. Limits as ΔX_n approaches 0. In order to study the behavior of linear and additive processes when ΔX_n is small, we suppose that the step size parameters θ_i and δ_i are proportional to a single parameter $\theta > 0$. Thus

$$\theta_i = \theta\eta_i \quad \text{or} \quad \delta_i = \theta\zeta_i, \tag{3.1}$$

where ζ_i is a real constant and η_i is a nonnegative constant. This leads to a family X_n^θ of Markov processes. We seek the limiting distribution of X_n^θ as $\theta \to 0$ and $n \to \infty$ in such a way that $n\theta$ or $n\theta^2$ converges.

The discreteness of the possible transitions in linear and additive processes is a substantial expository convenience, but it actually does not play a fundamental role in Theorems 2.1–2.3. Discreteness plays no role at all in the theorems of this section. This is indicative of the fact that the scope of these theorems is considerably broader than the types of processes considered in the last section.

3.1. A normal convergence theorem. For each $\theta \in (0, 1]$, let $X_0^\theta, X_1^\theta, X_2^\theta, \cdots$ be a Markov process with stationary transition probabilities in a closed but perhaps unbounded real interval I. Suppose that

$$E(\Delta X_n^\theta | X_n^\theta = x) = \theta w(x) + O(\theta^2), \tag{3.2}$$

$$E((\Delta X_n^\theta)^2 | X_n^\theta = x) = \theta^2 a(x) + o(\theta^2), \tag{3.3}$$

$$E(|\Delta X_n^\theta|^3 | X_n^\theta = x) = O(\theta^3), \tag{3.4}$$

where the orders of magnitude are uniform over x, e.g.,

$$\sup_{x \in I} |o(\theta^2)|/\theta^2 \to 0$$

as $\theta \to 0$. Assume that a is Lipschitz, and that w has a bounded Lipschitz derivative.

THEOREM 3.1.(a) *For any* $T < \infty$,

$$\operatorname{var}(X_n^\theta | X_0^\theta = x) = O(\theta), \tag{3.5}$$

uniformly over $x \in I$ *and* $n \leqq T/\theta$.

(b) *For any* $x \in I$, *the differential equation*

$$f'(t) = w(f(t)) \tag{3.6}$$

has a unique solution $f(t) = f(t, x)$ *with* $f(0) = x$. *We have*

$$E_x(X_n^\theta) = f(n\theta) + O(\theta) \tag{3.7}$$

uniformly over $x \in I$ *and* $n \leqq T/\theta$.

(c) *Let* $s(y) = a(y) - w(y)^2$. *The differential equation*

$$g'(t) = 2w'(f(t))g(t) + s(f(t))$$

has a unique solution $g(t) = g(t, x)$ *with* $g(0) = 0$. *Let* $X_0^\theta = x$ *a.s. for all* θ. *Then*

$$(X_n^\theta - f(n\theta))/\sqrt{\theta} \sim \mathcal{N}(0, g(t)) \tag{3.8}$$

as $\theta \to 0$ *and* $n\theta \to t < \infty$. *Here* $\sim$ *indicates weak convergence of distributions, and* $\mathcal{N}(0, g(t))$ *is the normal distribution with mean* 0 *and variance* $g(t)$.

Theorem 3.1 is a special case of Theorem 8.1.1 of [14].

THEOREM 3.2. *If* I *is bounded,* w *has a unique root* λ, *and* $w'(\lambda) < 0$, *then* (i) *the estimates* (3.5) *and* (3.7) *hold uniformly over* $x \in I$ *and* $n \geqq 0$. *In addition,* (ii) $g(\infty) = s(\lambda)/2|w'(\lambda)|$ *and* (3.8) *obtains as* $\theta \to 0$ *and* $n\theta \to t = \infty$.

Statement (i) means that the terms $O(\theta)$ in (3.5) and (3.7) satisfy $|O(\theta)| \leqq k\theta$, where k does not depend on $x \in I$ or $n \geqq 0$. A result of this type was announced previously [13, Thm. 3.2], but the proof is published here for the first time. The proof of (ii) is rather lengthy, and will be given elsewhere.

To apply Theorem 3.1 to linear and additive processes, we need only introduce the parameter θ via (3.1) and assume that p_i has a bounded Lipschitz derivative. In the linear model of § 1, p_i is linear. If F_S and F_N in the KDB model have bounded

Lipschitz derivatives, then p_i does also. For linear processes,

$$w(x) = \sum_{i=1}^{\nu} \eta_i(\lambda_i - x)p_i(x),$$

$$a(x) = \sum_{i=1}^{\nu} \eta_i^2(\lambda_i - x)^2 p_i(x),$$

and for additive processes,

$$w(x) = \sum_{i=1}^{\nu} \zeta_i p_i(x),$$

$$a(x) = \sum_{i=1}^{\nu} \zeta_i^2 p_i(x).$$

In both cases, the errors $O(\theta^2)$ and $o(\theta^2)$ in (3.2) and (3.3) vanish.

Combining parts (a) and (b) of Theorem 3.1, we see that $X_n^\theta \sim f(t)$ as $\theta \to 0$ and $n\theta \to t$. Kac [9] gave two heuristic derivations of this result for his model. He described $f(t)$ via the equation

$$\int_x^{f(t)} w(y)^{-1}\, dy = t,$$

which follows from (3.6).

In the linear prediction model, w is quadratic, and, if neither 0 nor 1 is absorbing, then $w(0) > 0$ and $w(1) < 0$. It follows that w has a unique root $\lambda \in (0, 1)$, and $w'(\lambda) < 0$. Since $I = [0, 1]$ is bounded, the hypotheses of Theorem 3.2 are satisfied. If 1 is absorbing but 0 is not, these hypotheses are satisfied if and only if $w'(1) < 0$.

The state space $I = (-\infty, \infty)$ of additive processes is not bounded, and there is at present no analogue of Theorem 3.2 for these processes. A generalization of Theorem 3.2 that covers additive processes would be an important contribution.

Proof of Theorem 3.2(i). The full force of (3.3) and (3.4) is not needed for the proof of (i). All that is needed is

$$E_x((\Delta X_n)^2) = O(\theta^2) \tag{3.9}$$

uniformly over n and x, which follows from either (3.3) or (3.4).

We begin with some analytical observations. Let

$$R(x, y) = \begin{cases} (w(x) - w(y))/(x - y) & \text{if } x \neq y \\ w'(y) & \text{if } x = y \end{cases}$$

$$= \int_0^1 w'(px + qy)\, dp.$$

Since w' is continuous and I is compact, R is uniformly continuous over $I \times I$. But $R(x, \lambda) < 0$ for all $x \in I$, so

$$-\alpha = \max_{x \in I} R(x, \lambda) < 0. \tag{3.10}$$

Let δ be so small that

$$|R(x, y) - R(x, \lambda)| \leqq \alpha/2$$

for all $x \in I$ if $|y - \lambda| \leqq \delta$. Then

$$R(x, y) \leqq -\alpha/2 \tag{3.11}$$

for all $x \in I$ and $|y - \lambda| \leqq \delta$.

Let $I = [A, B]$. Since λ is the unique root of w and $w'(\lambda) < 0$, we must have $w(A) \geqq 0$ and $w(B) \leqq 0$. From these conditions and the fact that w is Lipschitz, it follows that (3.6) has one and only one solution with initial value x. Clearly

$$\frac{d}{dx}(f(t) - \lambda)^2 = 2(f(t) - \lambda)^2 R(f(t), \lambda)$$

$$\leqq -2\alpha(f(t) - \lambda)^2$$

by (3.10), so

$$|f(t) - \lambda| \leqq |x - \lambda| \, e^{-\alpha t} \leqq (B - A) \, e^{-\alpha t}. \tag{3.12}$$

The equality

$$f''(t) = w'(f(t))w(f(t))$$

implies

$$|f''(t)| \leqq |w'||w|.$$

Thus, if $v_n = v_n(\theta, x) = f(n\theta, x)$, we have

$$\Delta v_n = \theta w(v_n) + O(\theta^2) \tag{3.13}$$

uniformly over $x \in I$ and $n \geqq 0$.

We can now proceed to the core of the proof. We suppose that $X_0^\theta = x$ a.s. and henceforth suppress θ superscripts. Let

$$a_n = E[(X_n - v_n)^2].$$

Note first that

$$a_{n+1} = a_n + 2E[(\Delta X_n - \Delta v_n)(X_n - v_n)] + E[(\Delta X_n - \Delta v_n)^2]. \tag{3.14}$$

Now

$$E[(\Delta X_n - \Delta v_n)^2] \leqq 2E[(\Delta X_n)^2] + 2(\Delta v_n)^2 \leqq k\theta^2 \tag{3.15}$$

by (3.9). Here and below k denotes a constant that does not depend on θ, n, or x. Also,

$$E[(\Delta X_n - \Delta v_n)(X_n - v_n)] = E[(E(\Delta X_n|X_n) - \Delta v_n)(X_n - v_n)]$$

$$= \theta E[(w(X_n) - w(v_n))(X_n - v_n)] + O(\theta^2)$$

(by (3.2) and (3.13))

$$= \theta E[R(X_n, v_n)(X_n - v_n)^2] + O(\theta^2)$$

$$\leqq \theta a_n \begin{cases} |R| \\ -\alpha/2 \end{cases} + k\theta^2, \tag{3.16}$$

where the first estimate is valid in general, and, by (3.11), the second applies when $|v_n - \lambda| \leqq \delta$. In view of (3.12), the latter condition is satisfied if $n\theta \geqq c$, where $(B - A)e^{-\alpha c} \leqq \delta$. Combining (3.14), (3.15), and (3.16), we obtain

$$a_{n+1} \leqq a_n \begin{cases} (1 + 2|R|\theta) \\ (1 - \theta\alpha) \end{cases} + k\theta^2. \tag{3.17}$$

Iterating the upper inequality and noting that $a_0 = 0$, we get

$$\begin{aligned} a_n &\leqq \theta[(1 + 2|R|\theta)^n - 1]k/2|R| \\ &\leqq \theta[e^{2|R|(c+1)} - 1]k/2|R| = k\theta \end{aligned} \tag{3.18}$$

if $n \leqq (c + 1)/\theta$. Iterating the lower inequality, we obtain

$$\begin{aligned} a_n &\leqq (1 - \alpha\theta)^{n-i} a_i + \theta[1 - (1 - \alpha\theta)^{n-i}]k/\alpha \\ &\leqq a_i + k\theta \end{aligned} \tag{3.19}$$

if $n \geqq i \geqq c/\theta$. If i is the smallest integer not less than c/θ, then $i < (c + 1)/\theta$, so that $a_i \leqq k\theta$ by (3.18). Consequently (3.19) yields $a_n \leqq k\theta$ for $n \geqq c/\theta$. By (3.18) the same estimate is valid for $n \leqq c/\theta$. We conclude that

$$E_x[(X_n - v_n)^2] \leqq k\theta \tag{3.20}$$

for all $x \in I$ and $n \geqq 0$, from which (3.5) follows immediately.

We turn now to the proof of (3.7). It follows from (3.2) that

$$E(X_{n+1}) = E(X_n) + \theta E(w(X_n)) + O(\theta^2).$$

Thus, by (3.13),

$$d_{n+1} = d_n + \theta E(w(X_n) - w(v_n)) + O(\theta^2),$$

where $d_n = E(X_n) - v_n$. Since w' is Lipschitz,

$$w(X_n) - w(v_n) = (X_n - v_n)w'(v_n) + O(|X_n - v_n|^2).$$

Therefore

$$d_{n+1} = (1 + \theta w'(v_n))d_n + O(\theta^2)$$

by (3.20). Thus

$$|d_{n+1}| \leqq |d_n| \begin{cases} (1 + \theta|w'|) \\ (1 - \theta\alpha/2) \end{cases} + k\theta^2,$$

where the last line applies if $n\theta$ is sufficiently large. Since this inequality is of the same form as (3.17), which implies $a_n \leqq k\theta$, we conclude that $|d_n| \leqq k\theta$. This completes the proof.

3.2. Convergence to a diffusion. Consider the linear model of § 1 with no learning after incorrect predictions ($\theta_2 = \theta_3 = 0$) and equal learning rates for correctly predicted flashes and nonflashes ($\theta_1 = \theta_4 = \theta$). Thus

$$\Delta X_n = \begin{cases} \theta(1 - X_n) & \text{with probability} \quad X_n \pi, \\ -\theta X_n & \text{with probability} \quad (1 - X_n)(1 - \pi), \\ 0 & \text{otherwise}. \end{cases}$$

When the linear model was discussed in the last section, it was tacitly assumed that π did not vary with θ. Here we assume that the departure of π from 1/2 is proportional to θ, i.e., $\pi = (1 + c\theta)/2$.

Then

$$E(\Delta X_n | X_n = x) = \theta^2 cx(1 - x),$$
$$E((\Delta X_n)^2 | X_n = x) = \theta^2 x(1 - x)/2 + O(\theta^3),$$
$$E(|\Delta X_n|^3 | X_n = x) = O(\theta^3).$$

Clearly $X_n = X_n^\theta$ satisfies the assumptions of Theorem 3.1 with $w(x) \equiv 0$ and $s(x) = a(x) = x(1 - x)/2$. Consequently $f(t) = x$, $g(t) = ts(x)$, and

$$(X_n - x)/\sqrt{\theta} \sim \mathcal{N}(0, ts(x))$$

as $\theta \to 0$ and $n\theta \to t < \infty$.

This result suggests that

$$X_n \sim \mathcal{N}(x, us(x))$$

as $\theta \to 0$ and $n\theta \to \infty$ in such a way that $n\theta^2 \to u < \infty$. Our point of departure in this section is the observation that this conjecture is definitely incorrect if $0 < x < 1$ and $u > 0$. For under these conditions, the normal distribution $\mathcal{N}(x, us(x))$ admits arbitrarily large observations, but $0 \leqq X_n \leqq 1$. Theorem 3.3 gives the correct asymptotic distribution of X_n as $\theta \to 0$ and $n\theta^2 \to u < \infty$. The parameter τ in Theorem 3.3 corresponds to θ^2.

The assumptions of Theorem 3.3 are as follows. For every $\tau \in (0, 1]$, $X_0^\tau, X_1^\tau, X_2^\tau, \cdots$ is a Markov process with stationary transition probabilities in a closed bounded interval I. For notational convenience we take $I = [0, 1]$. The increments ΔX_n^τ satisfy

$$E(\Delta X_n^\tau | X_n^\tau = x) = \tau b(x) + o(\tau), \tag{3.21}$$

$$E((\Delta X_n^\tau)^2 | X_n^\tau = x) = \tau a(x) + o(\tau), \tag{3.22}$$

and, for any $\delta > 0$,

$$P(|\Delta X_n^\tau| > \delta | X_n^\tau = x) = o(\tau), \tag{3.23}$$

where the orders of magnitude are uniform over x. The functions a and b are continuous throughout I (we write $a, b \in C(I)$), and $a(x) > 0$ if $0 < x < 1$. For

$i = 0$ and $i = 1$, $a(i) = b(i) = 0$,

$$\int_{1/2}^{i} (i - x)a(x)^{-1}\,dx < \infty, \tag{3.24}$$

and the function $r(x) = b(x)/a(x)$ on $(0, 1)$ has a finite limit, denoted $r(i)$, as $x \to i$. Thus $r \in C(I)$.

Let $C^2(0, 1)$ be the set of functions that have two continuous (but not necessarily bounded) derivatives on $(0, 1)$, and, for $f \in C^2(0, 1)$, let

$$Af(x) = \tfrac{1}{2}a(x)(\partial^2 f(x)/\partial x^2) + b(x)(\partial f(x)/\partial x).$$

Let $\mathscr{D}(A)$ be the set of functions in $C(I) \cap C^2(0, 1)$ with $Af(0^+) = Af(1^-) = 0$. If $f \in \mathscr{D}(A)$, let $Af(i) = 0$, so that $Af \in C(I)$.

THEOREM 3.3. *The linear operator A on $\mathscr{D}(A)$ generates a strongly continuous contractive semigroup T_t on $C(I)$. Let P be the associated transition function, i.e.,*

$$T_t f(x) = \int P(t; x, dy) f(y)$$

for $f \in C(I)$. If $X_0^\tau = x$ a.s. for all τ, then $X_n^\tau \sim P(t; x, \cdot)$ as $\tau \to 0$ and $n\tau \to t < \infty$.

For the concepts and terminology of semigroup theory, see [12].

The transition function P is conservative in the sense that $P(t; x, I) = 1$. It corresponds to a Markov process X_t with continuous sample paths, that is, a diffusion, via the equation

$$P(X_{t+s} \in A \mid X_s = x) = P(t; x, A).$$

The states 0 and 1 are absorbing.

The proof of Theorem 3.3 is based on Trotter's semigroup approximation theory [17].[2] A rather different approach to theorems of this type is given in [14, Chap. 9].

Proof. We begin by introducing some further notation. Let

$$B(x) = 2\int_{1/2}^{x} r(y)\,dy,$$

$$p(x) = \int_{1/2}^{x} e^{-B(y)}\,dy,$$

$$m(x) = 2\int_{1/2}^{x} a(y)^{-1} e^{B(y)}\,dy,$$

and

$$h(x) = \int_{1/2}^{x} m(y)\,dp(y).$$

The significance of p and m stems from the fact that

$$\frac{d}{dm}\frac{d}{dp} f = Af$$

for $f \in \mathscr{D}(A)$.

[2] I am indebted to Dr. Walter A. Rosenkrantz for reminding me of this theory.

Since r is bounded over I, so are p' and $m'(x)a(x)$. Therefore

$$h(1) = \int_{1/2}^{1} \left(\int_{s}^{1} p'(y)\,dy \right) m'(s)\,ds \tag{3.25}$$
$$\leqq k \int_{1/2}^{1} (1 - s)a(s)^{-1}\,ds < \infty$$

by (3.24). Therefore 1 is an accessible boundary [12, p. 24]. Similarly $h(0) < \infty$, so 0 is accessible. Granted this, the fact that A generates a strongly continuous contractive semigroup on $C(I)$ follows directly from Feller's semigroup theory [12, Chap. II]. Briefly, the equation $\lambda F - AF = f$, for $\lambda > 0$ and $f \in C(I)$, has a unique solution

$$F = F_0 + \lambda^{-1} f(0) u_0 + \lambda^{-1} f(1) u_1$$

in $\mathscr{D}(A)$ [12, p. 34], $F \geqq 0$ if $f \geqq 0$, and $F = \lambda^{-1}$ if $f = 1$. Since $a(i) = b(i) = 0$, $C^2(I) \subset \mathscr{D}(A)$, so $\mathscr{D}(A)$ is dense in $C(I)$. Thus the Hille–Yosida theorem [12, p. 2] applies.

We shall establish below that $C^2(I)$ is a core of A, i.e., $\{(f, Af): f \in C^2(I)\}$ is dense in the graph of A. It follows that $(\lambda - A)C^2(I)$ is dense in $C(I)$ if $\lambda > 0$.

Let $|\cdot|$ denote the supremum norm on $B(I)$ (the bounded measurable real-valued functions on I). If $g_n, g \in B(I)$, $g_n \to g$ means $|g_n - g| \to 0$. Let

$$A_\tau f = \tau^{-1}(U_\tau f - f),$$

where U_τ is the transition operator of X_n^τ. If $f \in C(I)$ and $A_\tau f$ converges to an element of $C(I)$, let

$$\Omega f = \lim_{\tau \to 0} A_\tau f.$$

A routine argument using (3.21)–(3.23) shows that $A_\tau f \to Af = \Omega f$ for $f \in C^2(I)$. Thus Ω is densely defined and $\lambda - \Omega$ has a dense range if $\lambda > 0$. By Trotter's [17] Theorem 5.3, $U_\tau^m f \to S_t f$ as $\tau \to 0$ for all $f \in C(I)$ and $t \geqq 0$, where $m = [t/\tau]$ and S_t is the semigroup on $C(I)$ generated by the closure, $\bar{\Omega}$, of Ω. Moreover, it can be seen from the proof of Trotter's theorem that convergence is uniform over bounded t intervals. It follows that $U_\tau^n f \to S_t f$ as $\tau \to 0$ and $n\tau \to t$.

Since $(\lambda - \bar{\Omega})^{-1}$ and $(\lambda - A)^{-1}$ agree on the dense set $(\lambda - A)C^2(I)$, they agree throughout $C(I)$. Therefore $\bar{\Omega} = A$, $S_t = T_t$, and $U_\tau^n f \to T_t f$. This, in turn, implies $X_n^\tau \sim P(t; x, \cdot)$.

To complete the proof, we must show that $C^2(I)$ is a core of A. If $f \in \mathscr{D}(A)$, then

$$f(x) = \int_{1/2}^{x} \int_{1/2}^{y} g(s)\,dm(s)\,dp(y) + f(1/2) + p(x)\frac{d}{dp}f(1/2), \tag{3.26}$$

where $g = Af \in C(I)$. For $0 < \delta < 1/2$, define g_δ as follows: $g_\delta(x) = g(x)$ if $\delta \leqq x \leqq 1 - \delta$, $g_\delta(x) = 0$ if $x \leqq \delta/2$ or $x \geqq 1 - \delta/2$, and $g_\delta(x)$ is defined by linear interpolation on $[\delta/2, \delta]$ and $[1 - \delta, 1 - \delta/2]$. Let

$$f_\delta(x) = \int_{1/2}^{x} \int_{1/2}^{y} g_\delta(s)\,dm(s)\,dp(y) + f(1/2) + p(x)\frac{d}{dp}f(1/2). \tag{3.27}$$

Now $r \in C(I)$, hence $p \in C^2(I)$ and $m \in C^1(0, 1)$. Since $g_\delta \in C(I)$, $f_\delta \in C^2(0, 1)$, and, since g_δ vanishes on $[0, \delta/2]$ and $[1 - \delta/2, 1]$, $f_\delta(x)$ is a linear function of $p(x)$ on these intervals. Thus $f_\delta \in C^2(I)$. Clearly $Af_\delta = g_\delta \to g = Af$ uniformly as $\delta \to 0$. Subtracting (3.26) from (3.27) we see that

$$|f_\delta - f| \leqq |g_\delta - g| \max \{h(1), h(0)\}.$$

However $h(i) < \infty$, so $f_\delta \to f$ and the proof is complete.

4. Multidimensional processes. A substantial proportion of the conceptions of contemporary learning theory can be expressed in terms of models with one-dimensional state variables X_n. However some of the most exciting recent work involves "multiprocess" models with multidimensional states. These models are intrinsically more complex than those considered in preceding sections, and the opportunities for mathematical contributions are correspondingly greater. It is fitting, then, to conclude this survey with a brief description of a particularly successful two-process model.

The experimental paradigm is called *discrimination learning*. A child has before him two small shallow wells with distinctive covers, one to his left and one to his right. On any trial, one of these wells contains a piece of candy. The position of the candy is correlated with some feature of the cover, and the child must learn this in order to find the candy consistently. Suppose, for example, that the covers differ in shape (triangle or circle) and color (red or blue), and that the candy is always under the blue cover. There are four possible stimulus displays,

left	**right**
blue triangle	red circle
blue circle	red triangle
red triangle	blue circle
red circle	blue triangle,

and each appears on a quarter of the trials. We say that color is the *relevant dimension* and form and position are *irrelevant*.

According to the model, the subject's state of learning at the beginning of trial n is characterized by two random variables, V_n and Y_n. The variable V_n represents his probability of attending to color on that trial, and the variable Y_n is his probability of choosing blue given attention to color. If he does not attend to color, he chooses blue with probability 1/2. Let B_n and R_n be the events "blue is chosen on trial n" and "red is chosen on trial n," and let $X_n = (V_n, Y_n)$. Then

$$P(B_n|X_n) = Y_n V_n + 2^{-1}(1 - V_n).$$

It remains to specify trial-to-trial changes in V_n and Y_n. The probability V_n of attending to color increases when the subject attends to color and finds candy or does not attend to color and does not find candy. Otherwise, it decreases. The probability Y_n of choosing blue given attention increases when the subject attends to color, and does not change when he fails to attend to color. These changes are effected by linear operators of the type considered in previous sections. Thus, if the subject attends to color,

$$\Delta V_n = \begin{cases} \varphi_1(1 - V_n) & \text{if } B_n, \\ -\varphi_2 V_n & \text{if } R_n, \end{cases}$$

$$\Delta Y_n = \begin{cases} \theta_1(1 - Y_n) & \text{if } B_n, \\ \theta_2(1 - Y_n) & \text{if } R_n. \end{cases}$$

If he does not attend to color,

$$\Delta V_n = \begin{cases} -\varphi_3 V_n & \text{if } B_n, \\ \varphi_4(1 - V_n) & \text{if } R_n, \end{cases}$$

$$\Delta Y_n = 0.$$

We assume that $\varphi_i, \theta_j > 0$ for all i and j.

This model was formulated by Zeaman and House [18] and Lovejoy [11], and some of its properties were described in [14, Chap. 16]. The process X_n is Markovian, and $X_n \to (1, 1)$ a.s. as $n \to \infty$. There are analogues of Theorems 2.1 and 3.1. If $\varphi_i = \varphi$ and $\theta_j = \theta$ for all i and j, as Zeaman and House assumed, there is a simple expression

$$E_x(T) = \varphi^{-1}(1 - v) + 2\theta^{-1}(1 - y)$$

for the expected total number T of errors, when the initial state is $x = (v, y)$.

This formula has served as the basis for comparison of performance after various kinds of problem shifts. In a reversal shift, for example, a subject would be given pretraining with color relevant but the other hue, red, correct. In an extra-dimensional shift, shape would be relevant in pretraining. The duration of pre-training is an important variable, and considerable attention has been focused on the effect of extending pretraining well past the point at which the child makes the correct choice reliably. This is called overtraining. Such manipulations affect x in different ways. It has been shown that the model can predict many interesting experimental results at least qualitatively [15]. For example, if $\varphi < \theta$, the model predicts that there will be fewer errors after reversal than after extradimensional shift. In addition, overtraining will reduce the number of errors after reversal. Both effects have been observed in experiments on children [3], [6].

The conditions $\varphi_i = \varphi$ and $\theta_j = \theta$ are not always psychologically realistic, and no useful formula for $E_x(T)$ is known when these conditions fail. Investigation of the dependence of this quantity on φ_i, θ_j, and x is a challenging mathematical problem.

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