Leverage Dynamics and Credit Quality*

Guillermo Ordoñez† David Perez-Reyna† Motohiro Yogo§

September 28, 2017

Abstract

We study a dynamic model of credit markets with asymmetric information, which allows for a rich set of signaling strategies based on the path of debt and repayment. Whether credit history reveals private information about credit quality in equilibrium depends on the degree of uncertainty in collateral value. When uncertainty is low, good borrowers fully separate by deleveraging, that is borrowing a sufficiently high amount such that subsequent repayment reveals the presence of unobservable income. When uncertainty is higher, an amount of debt that is necessary for separation forces bad borrowers to default in bad states. Therefore, good borrowers must trade off an ex-post benefit of separation against an ex-ante cost of adverse selection through a higher interest rate. The cost outweighs the benefit when uncertainty is sufficiently high, so credit quality is not revealed in equilibrium.

Keywords: Adverse Selection, Collateral, Credit Score, Reputation, Signaling

JEL codes: D82, G32

*An earlier version was circulated under the title “Debt: Deleveraging or Default”. For comments and discussions, we thank seminar participants at the Federal Reserve Board, McGill University, Universidad de los Andes, Wharton, the 2013 SED Annual Meeting, the 2013 Midwest Macroeconomics Conference, and the 2014 North American Summer Meeting of the Econometric Society.

†University of Pennsylvania and NBER (e-mail: ordonez@econ.upenn.edu)
‡Universidad de los Andes (e-mail: d.perez-reyna@uniandes.edu.co)
§Princeton University and NBER (e-mail: myogo@princeton.edu)
1. Introduction

A lender can potentially infer a borrower’s credit quality, as in private information about income or assets, based on the repayment history. Repayment history is indeed the most important of five factors that determine the FICO score, which is a leading measure of consumer credit quality (FICO 2015). Similarly, repayment history is the only factor that determines the PAYDEX score, which is a leading measure of credit quality for small businesses (Dun & Bradstreet 2017). We study how the lender’s perception of credit quality evolves through borrowing, repayment, and default in a dynamic model of credit markets with asymmetric information. Under what conditions can repayment history resolve asymmetric information at no cost? Under what conditions does information revelation involve costly default? Under what conditions does information revelation not even happen? We show that the answers to these questions depend on the degree of uncertainty in collateral value.

We develop a three-period model of credit markets in which borrowers roll over one-period debt in each period. There are two types of borrowers, good and bad. Both types of borrowers have a pledgable asset that can be used as collateral, whose value is subject to uncertainty. Only good borrowers have a non-pledgable asset that cannot be used as collateral but generates unobservable income. Borrowers maximize net worth, as perceived by the lender or outside investors, which is increasing in reputation (i.e., the perceived probability that the borrower is good).\(^1\) The lender is risk neutral and prices debt to break even, conditional on reputation. Reputation is updated through Bayes’ rule, based on repayment versus default and the amount of new debt conditional on repayment.

Good borrowers have an incentive to signal through a strategic path of debt that reveals the presence of unobservable income. Bad borrowers have an incentive to mimic the path of debt, if possible, to delay or prevent information revelation. When uncertainty in collateral value is low, good borrowers fully separate by borrowing a sufficiently high amount and subsequently repaying with unobservable income. Bad borrowers, who do not have unobservable income, must roll over a higher amount of debt to repay. Therefore, the ability to deleverage signals that the borrower is good. Such information revelation through deleveraging is costless because it does not involve default in equilibrium.

When uncertainty in collateral value is higher, full separation is no longer possible through deleveraging alone. An amount of debt that is necessary for separation through deleveraging when the collateral value rises forces bad borrowers to default when the collateral value falls. Although good borrowers do not default, they bear an ex-ante cost of adverse selection.

\(^1\)Reputation is publicly observable in our model so that we are only concerned with “directly placed debt” in the language of Diamond (1991).
through a higher interest rate that reflects the possibility that bad borrowers default. Since there are no deadweight costs of default in our model, the higher interest rate arises from adverse selection only and is a redistributive cost from bad to good borrowers. In choosing the optimal amount of debt, good borrowers must trade off the ex-post benefit of separation against the ex-ante cost of a higher interest rate. This tradeoff depends on uncertainty in collateral value because the benefit of separation is constant, while the interest rate for a given amount of debt increases with uncertainty.

For intermediate uncertainty in collateral value, the benefit of separation outweighs the higher interest cost, so there is full separation in equilibrium. Good borrowers borrow a sufficiently high amount to fully separate by deleveraging if the collateral value subsequently rises. However, if the collateral value falls instead, bad borrowers do not have sufficient collateral and are forced to default. Thus, there is full information revelation in equilibrium through bad borrowers defaulting in bad states.

For high uncertainty in collateral value, the higher interest cost outweighs the benefit of separation, so full separation is no longer optimal. Good borrowers borrow a relatively low amount so that, if the collateral value subsequently rises, even bad borrowers repay by rolling over a low amount of debt that does not reveal credit quality. If the collateral value falls instead, good borrowers fully separate by deleveraging. Thus, there is partial information revelation in equilibrium, only in bad states.

In summary, the cost of asymmetric information rises with uncertainty in collateral value. When uncertainty in collateral value is low, deleveraging is a costless way for good borrowers to signal, and asymmetric information is fully resolved. When uncertainty in collateral value is intermediate, information revelation entails default by bad borrowers in bad states. Although asymmetric information is fully resolved, good borrowers must bear an adverse selection cost through a higher interest rate. When uncertainty in collateral value is high, asymmetric information is not resolved in all states.

An important implication of our results is that credit history is a less precise signal of credit quality in environments with high uncertainty. For example, uncertainty could be high in recessions or for collateral that are hard to value. In such environments, asymmetric information becomes a more important friction that credit history cannot fully resolve, which leads to higher interest rates and higher probability of default.

Static models of credit markets with asymmetric information, pioneered by Leland and Pyle (1977) and Ross (1977), predict that good borrowers signal through higher leverage.\(^2\)

---

\(^2\)Some extensions of the baseline model include managers with different objective functions (Heinkel 1982), projects with different mean returns (Blazenko 1987, John 1987), and projects with different variance of returns (Brick, Frierman and Kim 1998). See Harris and Raviv (1991) for a review of this literature.
Moreover, and somewhat counterintuitively, good borrowers default with higher probability because the higher deadweight cost of default is precisely what signals their type. In contrast to these static models, we show that a dynamic model predicts that the change in debt (rather than the level) could resolve asymmetric information. Furthermore, the feasibility and the cost of information revelation depend on uncertainty in collateral value. Our model provides new insights into the joint dynamics of collateral value and credit market outcomes such as leverage, interest rates, and default.

Our focus on debt is motivated by a long tradition of studying asymmetric information and reputation in the context of credit markets (Ross 1977, Diamond 1989). This paper complements more recent effort to extend models of credit markets with asymmetric information to a dynamic setting (Hennessy, Livdan and Miranda 2010, Morellec and Schürhoff 2011, Streubalaev, Zhu and Zryumov 2016). The previous papers essentially reduce the optimal choice of debt to a static problem by assuming that private information is short-lived or that debt is a one-time choice in a real options framework. In contrast, this paper allows for a richer set of signaling strategies through the path of debt and repayment, which are ruled out by assumption in the previous papers. However, the previous papers model optimal investment choice, which we abstract from in order to isolate how leverage dynamics could resolve asymmetric information. Thus, the only role of debt in our model is to signal credit quality.

Our work is related to a recent literature that studies signaling in dynamic models with persistent asymmetric information. Sannikov (2007) finds that an increasing credit line is an optimal contract in a dynamic principal-agent model with asymmetric information and moral hazard. Bond and Zhong (2016) study a dynamic model of equity issuance and repurchase under asymmetric information. Guerrieri and Shimer (2014a) study the frequency of trade as a signal of asset quality in an exchange economy. Guerrieri and Shimer (2014b) and Chang (2017) extend this analysis to the case of multi-dimensional private information and show the limitations of signaling in dynamic settings.

The remainder of the paper is organized as follows. Section 2 presents a dynamic model of credit markets with asymmetric information and uncertainty in collateral value. Section 3 discusses some important properties of the equilibrium that will be used to prove our main results. Section 4 presents our main results on how uncertainty in collateral value determines information revelation through deleveraging or default. Section 5 concludes.
2. A Dynamic Model of Credit Markets

We present a dynamic model of credit markets with asymmetric information and uncertainty in collateral value. The dynamic model allows for a rich set of signaling strategies through the path of debt and repayment, which could resolve asymmetric information. Uncertainty in collateral value affects the precision of the signaling strategies and whether separation entails default in equilibrium.

2.1. Pledgable and non-pledgable assets

There are two types of assets, pledgable and non-pledgable, which generate stochastic income streams. The pledgable asset can be used as collateral in credit transactions, whereas the non-pledgable asset cannot be used as collateral. A pledgable asset can be thought of as a tangible and observable asset such as land, structure, or equipment. A non-pledgable asset can be thought of as an intangible and unobservable asset such as innovative ability, managerial skill, or organizational structure.

The pledgable asset generates observable income $X_t$ in each period $t$, which follows a martingale (i.e., $E_t[X_{t+s}] = X_t$). Let $R > 1$ denote the gross riskless interest rate, which satisfies $R^2(R - 1) < 1$. The value of the pledgable asset is the present value of its income:

$$V_t = \frac{E_t[X_{t+1} + V_{t+1}]}{R} = \sum_{s=1}^{\infty} \frac{E_t[X_{t+s}]}{R^s} = \frac{X_t}{R - 1}. \quad (1)$$

The non-pledgable asset generates unobservable income $Y_t$ in each period $t$, which also follows a martingale (i.e., $E_t[Y_{t+s}] = Y_t$). All income is perishable and must be immediately consumed or used to repay debt.

2.2. Borrowers with private information

There are two types of risk-neutral borrowers, “good” and “bad”. Both types of borrowers are endowed with a unit of the pledgable asset. Only good borrowers are also endowed with a unit of the non-pledgable asset. Whether a given borrower is good or bad is private information to the borrower, which arises from the fact that income from the non-pledgable asset is unobservable.

Each borrower is in the credit market for at most three periods, which we denote as $t \in \{1, 2, 3\}$. Let $F_0$ be the face value of existing debt that matures in period 1. Let $\pi_0 \in (0, 1)$ be the lender’s perceived probability that the borrower is good, which we refer
to as reputation, at the beginning of period 1. More generally, the borrower enters each period $t$ with maturing debt $F_{t-1}$ and reputation $\pi_{t-1}$.

The borrower receives income $X_t + Y_t$ if good or $X_t$ if bad. The borrower can either repay the face value of maturing debt or default. Let $D_{i,t}$ denote an endogenous default boundary such that it is feasible and optimal for a borrower of type $i \in \{g, b\}$ (i.e., good or bad) to repay if $F_{t-1} \leq D_{i,t}$ and to default otherwise. The borrower can repay using his income as well as the proceeds from rolling over one-period debt with face value $F_{i,t}$ at the equilibrium price $P_t$. Conditional on repayment, the lender updates reputation to $\pi_t$. Note that not only repayment, but also the face value of new debt, can serve as signals for the updating of reputation. Conditional on default, the lender takes possession of the collateral (i.e., the pledgable asset and its income in period $t$) and updates reputation to $\hat{\pi}_t$.

The borrower essentially faces the same problem in period 3, which is the terminal period. The only difference is that instead of rolling over debt, he can sell the pledgable asset at market value to repay. Therefore, only repayment can serve as a signal for the updating of reputation in the terminal period.

Following Ross (1977), we assume that the borrower maximizes net worth, as perceived by the lender (or outside investors with knowledge of only reputation). The value of the non-pledgable asset in period 3 is $W_3 = \frac{\pi_3 Y_3}{R-1}$ in case of repayment and $\hat{W}_3 = \frac{\hat{\pi}_3 Y_3}{R-1}$ in case of default. That is, the value of the non-pledgable asset is equal to the probability that the borrower owns the non-pledgable asset times its value conditional on ownership. Let $\mathbb{1}_g(i)$ be an indicator function that is equal to one if the borrower is good and zero otherwise. The net worth for a type $i$ borrower in period 3 is

$$J_{i,3} = \begin{cases} X_3 + \mathbb{1}_g(i)Y_3 + V_3 + W_3 - F_2 & \text{if } D_{i,3} \geq F_2 \\ \mathbb{1}_g(i)Y_3 + \hat{W}_3 & \text{if } D_{i,3} < F_2 \end{cases}.$$  \hfill (2)

In case of repayment, net worth is income plus the terminal value of both types of assets minus the face value of maturing debt. In case of default, net worth is the terminal value of the non-pledgable asset and (for a good borrower) its income.

We define the borrower’s net worth in period $t \in \{1, 2\}$ recursively as

$$J_{i,t} = \begin{cases} X_t + \mathbb{1}_g(i)Y_t + P_tF_t - F_{t-1} + \frac{\mathbb{E}_t[J_{i,t+1}]}{R} & \text{if } D_{i,t} \geq F_{t-1} \\ \mathbb{1}_g(i)Y_t + \hat{W}_{i,t} & \text{if } D_{i,t} < F_{t-1} \end{cases},$$  \hfill (3)

\footnotetext{3}{We assume that $\pi_0$ is exogenous for simplicity, but it can be endogenized as the mass of borrowers who choose to invest in the non-pledgable asset (Atkeson, Hellwig and Ordoñez 2015).}
where
\[
\hat{W}_{i,t} = 3 - t \sum_{s=1}^{3-t} \frac{\mathbb{I}_g(i) \mathbb{E}_t[Y_{i+t+s}]}{R^t} + \frac{\mathbb{E}_t[\hat{\pi}_3 Y_3]}{R^{3-t}(R-1)} = \frac{((R^{3-t} - 1)\mathbb{I}_g(i) + \hat{\pi}_t)Y_t}{R^{3-t}(R-1)}. \tag{4}
\]

In case of repayment, net worth is income plus the net proceeds from rolling over debt plus the borrower's continuation value. In case of default, net worth is the terminal value of the non-pledgable asset and (for a good borrower) its income through period 3. Note that once the borrower defaults in period 2, there is no further updating of reputation so that $\hat{\pi}_3 = \hat{\pi}_2$.

2.3. Lender

The representative lender is risk neutral and earns an expected gross return $R$ on each loan. The lender does not know whether a given borrower is good or bad. However, the lender updates reputation based on repayment versus default and the amount of new debt conditional on repayment.

We assume throughout that $F_0 \leq X_1$ to rule out a trivial outcome of immediate default in period 1. As we discussed above, there is no refinancing in the terminal period. This means that the lender updates reputation based on the amount of new debt in period 1, the amount of new debt and the default decision in period 2, and the default decision in period 3. We have three periods in our model because this is the minimum necessary to capture strategies in which the borrower increases the amount of debt in period 1 and subsequently decreases it in period 2.

Conditional on repayment in period $t \in \{1, 2\}$, the lender updates reputation through Bayes rule:
\[
\pi_t = \left(1 + \frac{(1 - \pi_{t-1}) \Pr(D_{b,t} \geq F_{t-1}) \cap \{F_{b,t} = F_t\}}{\pi_{t-1} \Pr(D_{g,t} \geq F_{t-1}) \cap \{F_{g,t} = F_t\}}\right)^{-1}. \tag{5}
\]

This formula accounts for the fact that not only repayment, but also the face value of new debt $F_t$, potentially reveals borrower type.\footnote{The fact that reputation depends on repayment history and debt outstanding is consistent with the determinants of FICO score. According to FICO (2015), repayment history determines 35% and debt outstanding determines 30% of the FICO score.} Conditional on default in period $t$, reputation is
\[
\hat{\pi}_t = \left(1 + \frac{(1 - \pi_{t-1}) \Pr(D_{b,t} < F_{t-1})}{\pi_{t-1} \Pr(D_{g,t} < F_{t-1})}\right)^{-1}. \tag{6}
\]

In period 3, the lender updates reputation based on repayment alone because there is no refinancing in the terminal period. Conditional on repayment in period 3, the terminal
reputation is

$$\pi_3 = \left(1 + \frac{(1 - \pi_2) \Pr(D_{b,3} \geq F_2)}{\pi_2 \Pr(D_{g,3} \geq F_2)}\right)^{-1}.$$  \hspace{1cm} (7)

Since a borrower’s type is time invariant, his actions are either fully revealing or not at all, conditional on the realized collateral value. Therefore, reputation either remains the same or updates fully to one or zero for good and bad borrowers, respectively. Shocks to the borrower type could prevent the full updating of reputation, but such an extension would complicate both the analysis and the exposition.

To complete the model, we must make auxiliary assumptions about beliefs off the equilibrium path. We make a natural assumption that repayment reveals that the borrower is good if all borrowers are expected to default in equilibrium. Similarly, default reveals that the borrower is bad if all borrowers are expected to repay in equilibrium. These restrictions on off-equilibrium beliefs arise naturally from the intuitive criterion (Cho and Kreps 1987). We state our assumptions more formally as follows.

**Assumption 1.** The lender’s off-equilibrium beliefs are given by

$$\pi_t = 1 \text{ if } \Pr(D_{g,t} \geq F_{t-1}) = \Pr(D_{b,t} \geq F_{t-1}) = 0,$$ \hspace{1cm} (8)

$$\tilde{\pi}_t = 0 \text{ if } \Pr(D_{g,t} < F_{t-1}) = \Pr(D_{b,t} < F_{t-1}) = 0.$$ \hspace{1cm} (9)

In period $t \in \{1, 2\}$, the lender’s break-even condition determines the equilibrium price of debt $P_t$, given face value $F_t$ and reputation $\pi_t$:

$$P_tF_t = \pi_tC_{g,t} + (1 - \pi_t)C_{b,t},$$ \hspace{1cm} (10)

where

$$C_{i,t} = \frac{\Pr(D_{i,t+1} \geq F_t)F_t + \Pr(D_{i,t+1} < F_t)\mathbb{E}_t[X_{t+1} + V_{t+1}|D_{i,t+1} < F_t]}{R}.$$ \hspace{1cm} (11)

That is, the lender breaks even if the value of debt is equal to the expected repayment discounted at $R$. The expected repayment is equal to the probability that the borrower is good multiplied by good borrowers’ expected repayment plus the probability that the borrower is bad multiplied by bad borrowers’ expected repayment.
2.4. Summary of the model

The borrower can signal through the amount of new debt in periods 1 and 2 and through repayment in periods 2 and 3. We summarize the model as follows.

Period 1. The borrower starts with face value of debt $F_0 \leq X_1$ and reputation $\pi_0$.

(a) The borrower receives income $X_1 + Y_1$ if good and $X_1$ if bad.

(b) The borrower takes out a new loan with face value $F_1$ at the equilibrium price $P_1$. The lender updates reputation to $\pi_1$.

Period 2. The borrower enters with face value of debt $F_1$ and reputation $\pi_1$.

(a) The borrower receives income $X_2 + Y_2$ if good and $X_2$ if bad.

(b) The borrower decides whether or not to repay $F_1$.

- In case of repayment, the borrower takes out a new loan with face value $F_2$ at the equilibrium price $P_2$. The lender updates reputation to $\pi_2$.
- In case of default, the lender takes possession of the pledgable asset (i.e., $X_2 + V_2$) and updates reputation to $\widehat{\pi}_2$. The borrower’s terminal value is the non-pledgable asset and its income (i.e., $\mathbb{I}_g(i)Y_2 + \widehat{W}_{i,2}$).

Period 3. In case of repayment in period 2, the borrower enters with face value of debt $F_2$ and reputation $\pi_2$.

(a) The borrower receives income $X_3 + Y_3$ if good and $X_3$ if bad.

(b) The borrower decides whether or not to repay $F_2$.

- In case of repayment, the lender updates reputation to $\pi_3$.
- In case of default, the lender takes possession of the pledgable asset (i.e., $X_3 + V_3$) and updates reputation to $\widehat{\pi}_3$. The borrower’s terminal value is the non-pledgable asset and its income (i.e., $\mathbb{I}_g(i)Y_3 + \widehat{W}_3$).

3. Properties of the Equilibrium

We first characterize some important properties of the equilibrium that do not depend on additional parametric assumptions. We will use the lemmas in this section to prove our main results in Section 4.
3.1. Borrowers’ maximization problem

In period 3, a type \(i\) borrower can repay if \(X_3 + \mathbb{I}_g(i)Y_3 + V_3 \geq F_2\). That is, he can repay if his income plus the value of the pledgable asset exceeds the face value of maturing debt. Moreover, equation (2) implies that it is optimal for the borrower to repay if \(X_3 + V_3 + W_3 - \hat{W}_3 \geq F_2\). Combining feasibility and optimality, the default boundary in period 3 is

\[
D_{i,3} = X_3 + V_3 + \min \left\{ \mathbb{I}_g(i)Y_3, W_3 - \hat{W}_3 \right\} .
\] (12)

In period \(t \in \{1, 2\}\), a type \(i\) borrower can repay if

\[
X_t + \mathbb{I}_g(i)Y_t + \max_{F_t} P_t F_t \geq F_{t-1} .
\] (13)

That is, the borrower can repay if his income plus the maximum amount that he can borrow exceeds the face value of maturing debt. The following lemma establishes the condition under which repayment is optimal, which implies the default boundary when combined with feasibility.

**Lemma 1.** In period \(t \in \{1, 2\}\), the borrower’s net worth is

\[
J_{i,t} = \begin{cases} 
X_t + \mathbb{I}_g(i)Y_t + V_t + W_{i,t} - F_{t-1} & \text{if } D_{i,t} \geq F_{t-1} \\
\mathbb{I}_g(i)Y_t + \hat{W}_{i,t} & \text{if } D_{i,t} < F_{t-1} 
\end{cases},
\] (14)

where the value of the non-pledgable asset conditional on repayment is

\[
W_{i,t} = - (\mathbb{I}_g(i) - \pi_t) (C_{g,t} - C_{b,t}) + \mathbb{I}_g(i)Y_t + \Pr(D_{i,t+1} \geq F_t) \mathbb{E}_t[W_{i,t+1} | D_{i,t+1} \geq F_t] \\
+ \frac{\Pr(D_{i,t+1} < F_t) \mathbb{E}_t[\hat{W}_{i,t+1} | D_{i,t+1} < F_t]}{R} .
\] (15)

The default boundary is

\[
D_{i,t} = X_t + V_t + \min \left\{ \mathbb{I}_g(i)Y_t + \max_{F_t} P_t F_t - V_t, W_{i,t} - \hat{W}_{i,t} \right\} ,
\] (16)

where

\[
W_{i,t} - \hat{W}_{i,t} = - (\mathbb{I}_g(i) - \pi_t) (C_{g,t} - C_{b,t}) + \frac{\mathbb{E}_t[\hat{\pi}_{t+1} Y_{t+1}] - \hat{\pi}_i Y_t}{R^{3-t} (R - 1)} \\
+ \frac{\Pr(D_{i,t+1} \geq F_t) \mathbb{E}_t[W_{i,t+1} - \hat{W}_{i,t+1} | D_{i,t+1} \geq F_t]}{R} .
\] (17)
Proof. See Appendix A.

In case of repayment in period $t \in \{1, 2\}$, the borrower chooses $F_t$ to maximize his net worth (14). However, all components of net worth are predetermined, except for the value of the non-pledgable asset. Therefore, the borrower’s maximization problem simplifies to

$$\max_{F_t} W_{i,t} \text{ subject to } X_t + \mathbb{1}_g(i)Y_t + P_t F_t \geq F_{t-1}. \quad (18)$$

3.2. Benchmark with perfect information

Private information about whether or not the borrower owns the non-pledgable asset is the only friction in our model. The benchmark with perfect information is a special case of our model where reputation is $\pi_{t-1} \in \{0, 1\}$. In this special case, we recover the standard result that debt (or leverage) is indeterminate.

Lemma 2 (Modigliani and Miller (1958)). If $F_{t-1} \leq X_t$ and $\pi_{t-1} \in \{0, 1\}$, borrowers are indifferent between any amount of debt such that $P_t F_t \leq V_t$. The equilibrium interest rate is $P_t^{1-1} = R$.

Proof. See Appendix A.

3.3. Signaling through deleveraging or default

In the presence of asymmetric information, good borrowers have an incentive to signal through repayment and the amount of new debt conditional on repayment. Bad borrowers have an incentive to mimic good borrowers’ actions in order to delay (or if possible avoid) information revelation. The incentive of bad borrowers to mimic good borrowers comes from two sources. First, bad borrowers pay interest that is lower than under perfect information, given that they are more likely to default in the future. This source is captured by the first term, $\pi_t(C_{g,t} - C_{b,t}) \geq 0$, in equation (15). Second, there is a higher terminal value of the non-pledgable asset if bad borrowers can altogether avoid information revelation. This source is captured by the last two terms in equation (15).

The following lemma formally establishes that bad borrowers are more likely to default than good borrowers.

Lemma 3. The default boundary for good borrowers is higher than that for bad borrowers:

$$X_t + V_t \leq D_{b,t} \leq D_{g,t} \leq X_t + Y_t + V_t. \quad (19)$$

To simplify the statement of our results, we follow the convention that bad borrowers mimic good borrowers in the knife-edge case of indifference.
In the event of full separation in period $t$, the first and third inequalities are equalities, and the second inequality is strict.

Proof. See Appendix A.

Based on Lemma 3, we define four regions for the face value of maturing debt relative to the realized collateral value and the default boundaries as illustrated in Figure 1. Lemmas 4 to 7 that follow correspond to the four regions. For each region, we state the optimal strategy of good borrowers and whether there is separation in equilibrium.

---

**Figure 1: Signaling regions**

**Lemma 4 (No separation).** Suppose that $F_{t-1} \leq X_t$ in period $t \in \{1, 2\}$ or $F_2 \leq D_{b,3}$ in period 3. All borrowers repay, so borrower type is not revealed.

Proof. If $F_{t-1} \leq X_t$, both types of borrowers can repay without rolling over debt.

**Lemma 5 (Separation by deleveraging).** Suppose that $F_{t-1} \in (X_t, D_{b,t}]$ in period $t \in \{1, 2\}$. Good borrowers repay by rolling over $F_t \in [R \max\{0, F_{t-1} - X_t - Y_t\}, R(F_{t-1} - X_t)]$. Bad borrowers repay by rolling over $F_t \in [R(F_{t-1} - X_t), RV_t]$. Thus, borrower type is fully revealed.

Proof. In this region, it is optimal for all borrowers to repay. Good borrowers can repay by rolling over at least $P_tF_t \geq \max\{0, F_{t-1} - X_t - Y_t\}$. Bad borrowers can repay by rolling over at least $P_tF_t \geq F_{t-1} - X_t$. Therefore, good borrowers separate by rolling over at most $P_tF_t < F_{t-1} - X_t$. Lemma 2 implies that the equilibrium interest rate is $P_t^{-1} = R$. 
Lemma 6 (Separation by default). Suppose that $F_{t-1} \in (D_{b,t}, D_{g,t}]$ in any period $t$. Only bad borrowers default, so borrower type is fully revealed. In period $t \in \{1, 2\}$, good borrowers repay by rolling over $F_t \in [R \max\{0, F_{t-1} - X_t - Y_t\}, RV_t]$.

Proof. In this region, bad borrowers are forced to default. In period $t \in \{1, 2\}$, good borrowers can repay by rolling over at least $P_t F_t \geq \max\{0, F_{t-1} - X_t - Y_t\}$. Lemma 2 implies that the equilibrium interest rate is $P_t^{-1} = R$.

Lemma 7 (No separation). Suppose that $F_{t-1} > D_{g,t}$ in any period $t$. All borrowers default, so borrower type is not revealed.

Lemmas 5 and 6 establish that there are two ways in which the borrower type is fully revealed in period 2. First consider a low level of maturing debt $F_1 \in (X_2, D_{b,2}]$, shown as the red shaded region in Figure 1. In this region, good borrowers can roll over a lower amount of debt by repaying with unobservable income. Bad borrowers, who do not have unobservable income, must roll over a higher amount of debt in order to repay. Therefore, rolling over an amount of debt that is lower than the maturing debt minus observable income signals that the borrower is good because only borrowers with unobservable income can follow such a strategy.

Next consider a higher level of maturing debt $F_1 > D_{b,2}$ in Figure 1. In this region, good borrowers can repay with unobservable income, while bad borrowers are forced to default. Therefore, repayment signals that the borrower is good.

Lemmas 5 and 6 describe the optimal strategy conditional on the face value of maturing debt and the realized collateral value in period 2. In Section 4, we will work backwards to solve for the optimal choice of debt in period 1. Before we go into the formal analysis, we discuss the intuition for the tradeoff that good borrowers face in choosing the optimal amount of debt in period 1. Good borrowers have a choice of borrowing a lower amount to prepare for signaling by deleveraging or a higher amount to prepare for signaling by forcing default in period 2. If possible, good borrowers prefer deleveraging because equation (15) implies that the value of the non-pledgable asset is $W_{g,1} = \frac{Y_1}{R-1}$ under deleveraging and

$$W_{g,1} = -(1 - \pi_t)(C_{g,1} - C_{b,1}) + \frac{Y_1}{R - 1}$$

under forcing default. Forcing default is costly because good borrowers have to pay higher interest in period 1 due to adverse selection, captured by the first term in equation (20).

Recall that asymmetric information is the only friction in our model. We do not have deadweight costs of default, which is the key friction that allows good borrowers to signal
through higher debt in the static model of Ross (1977). In a dynamic setting, deleveraging is a superior way of signaling, which is ruled out by construction in the static model.

In the absence of uncertainty in collateral value, $X_2$ is known when borrowers choose $F_1$ in period 1. In that case, good borrowers can choose $F_1 \in (X_2, D_{b,2}]$ to always separate by deleveraging in period 2. When there is uncertainty in collateral value, however, good borrowers may not be able to ensure that $F_1 \in (X_2, D_{b,2}]$ in all states. A sufficiently high level of debt that ensures full separation through deleveraging when collateral value rises could cause bad borrowers to default when collateral value falls. This introduces a tradeoff between the benefit of separation and the higher interest cost. We analyze how this tradeoff depends on uncertainty in collateral value in Section 4.

4. CHARACTERIZATION OF THE EQUILIBRIUM

The tradeoff between the benefit of separation and the higher interest cost depends on parametric assumptions about uncertainty in collateral value. We make such assumptions and fully characterize how the equilibrium depends on uncertainty in collateral value.

4.1. Parametric assumptions

Our first assumption is that unobservable income is a constant proportion of observable income. Moreover, unobservable income is less than the collateral value so that signaling plays a non-trivial role in the model.

**Assumption 2.** Unobservable income is a constant proportion $y$ of observable income. Moreover, unobservable income is less than the collateral value:

$$y = \frac{Y_t}{X_t} < \frac{X_t + V_t}{X_t} = \frac{R}{R - 1}. \quad (21)$$

Our second assumption is that observable income follows a binomial version of the geometric random walk. This assumption reduces the analysis to checking a manageable number of cases.

**Assumption 3.** The growth rate of observable income is distributed as

$$x_t = \frac{X_t}{X_{t-1}} = \begin{cases} \bar{x} & \text{with probability } 1 - p \\ x & \text{with probability } p \end{cases}, \quad (22)$$

where $\bar{x} \geq x$ and $(1 - p)\bar{x} + px = 1.$
There are only two free parameters between $\overline{\tau}$, $\overline{x}$, and $p$ because of the normalization that the mean growth rate of observable income is one. In characterizing the equilibrium, it is convenient to divide the parameter space into regions along $\overline{\tau}$ and $(1 - p)\overline{\tau}$. $\overline{\tau}$ captures uncertainty in collateral value, and $(1 - p)\overline{\tau}$ captures asymmetry in the distribution of collateral value. In this section, we present the results for the case $(1 - p)\overline{\tau} \geq 0.5$, where collateral value is unlikely to fall and thus has negative skew. We present the results for the complementary case $(1 - p)\overline{\tau} < 0.5$ in Appendix C.

4.2. Low uncertainty in collateral value

As we discussed in Section 3, deleveraging is a costless and optimal strategy for separation in the absence of uncertainty in collateral value. By continuity, the equilibrium should remain full separation through deleveraging as long as uncertainty in collateral value is low.

**Proposition 1 (Costless full separation).** Suppose that uncertainty in collateral value is low. That is, $\frac{\overline{\tau}}{\overline{x}} < \frac{R}{R-1}$. In period 1, all borrowers borrow $F_1 \in (X_1\overline{x}, RV_1\overline{x}]$ at the interest rate $P_1^{-1} = R$. In period 2, borrower type is fully revealed. Good borrowers repay by rolling over $F_2 \in [\max\{0, F_1 - X_2 - Y_2\}, R(F_1 - X_2)]$. Bad borrowers repay by rolling over $F_2 \in [R(F_1 - X_2), RV_2]$.

**Proof.** Lemma 9 in Appendix B implies the equilibrium in period 1. Lemma 5 implies the equilibrium in period 2.

Collateral value rises
$F_1 \quad X_2 = X_1\overline{x} \quad D_{b,2} = RV_1\overline{x}$

Collateral value falls
$F_1 \quad X_2 = X_1\overline{x} \quad D_{b,2} = RV_1\overline{x}$

Figure 2: Signaling regions for low uncertainty in collateral value
In period 2, good borrowers can fully separate by deleveraging if the face value of maturing debt $F_1$ is greater than observable income $X_2$. In that case, good borrowers repay from their observable income, part of their unobservable income, and the remainder from rolling over debt. Bad borrowers, who do not have unobservable income, must roll over a higher amount of debt in order to repay. This implies that in period 1, good borrowers must borrow at least $F_1 > X_1 \pi \geq X_2$ to ensure separation in period 2, even if the realized collateral value is high. Thus, $X_1 \pi$ is the lower bound on $F_1$ for costless full separation. This is illustrated as the red shaded region in the upper line in Figure 2.

Bad borrowers can repay in period 2 as long as the face value of maturing debt $F_1$ is less than the collateral value $X_2 + V_2$. Therefore, good borrowers do not want to borrow more than $F_1 \leq RV_1 \pi \leq X_2 + V_2$ to ensure that bad borrowers do not default even if the collateral value falls. As we discussed in Section 3, good borrowers prefer not to force default because they bear a higher interest cost due to adverse selection in period 1. Thus, $RV_1 \pi$ is the upper bound on $F_1$ for costless full separation. This is illustrated as the red shaded region in the lower line in Figure 2.

The range of equilibrium debt in period 1, which is the gray shaded region of overlap in Figure 2, shrinks as uncertainty in collateral value rises. This is because the lower bound on debt must rise so that bad borrowers cannot repay with observable income alone, even if the collateral value rises in period 2. At the same time, the upper bound on debt must fall to prevent bad borrowers from defaulting and surrendering collateral, even if the collateral value falls in period 2. The range of equilibrium debt shrinks until it becomes a point at which $F_1 = X_1 \pi = RV_1 \pi$, which is equivalent to $\pi = \frac{R}{R-1}$. At this point, good borrowers face a tradeoff. On the one hand, a higher amount of debt in period 1 would force bad borrowers to default if collateral value falls in period 2. On the other hand, a lower amount of debt in period 1 would allow even bad borrowers to repay without rolling over debt if collateral value rises in period 2.

4.3. Higher uncertainty in collateral value

When uncertainty in collateral value is higher such that $\pi \geq \frac{R}{R-1}$, good borrowers cannot costlessly separate by deleveraging in all states. A sufficiently high level of debt that allows good borrowers to separate by deleveraging when collateral value rises causes bad borrowers to default when collateral value falls. A lower level of debt allows good borrowers to separate by deleveraging when collateral value falls, but it does not allow good borrowers to separate when collateral value rises.

When uncertainty in collateral value is intermediate, the equilibrium turns out to be full separation through deleveraging in good states and default in bad states. The optimal
amount of debt in period 1 is $F_1 > X_1\overline{x}$, illustrated as the red shaded region in the upper line in Figure 3. If the collateral value rises in period 2, good borrowers fully separate by rolling over a lower amount of debt than bad borrowers. If the collateral value falls in period 2, good borrowers fully separate by repaying, while bad borrowers default. Thus, full separation is costly in the sense that bad borrowers default in bad states.

Collateral value rises

$X_2 = X_1\overline{x}$

$D_{b,2} = RV_1\overline{x}$

Collateral value falls

$X_2 = X_1\underline{x}$

$D_{b,2} = RV_1\overline{x}$

Figure 3: Signaling regions for intermediate uncertainty in collateral value

**Proposition 2 (Costly full separation).** Suppose that uncertainty in collateral value is intermediate. That is, $\frac{R}{R-1} \leq \overline{x} < \min\{d_{g,2}, z\}$, where

$$d_{g,2} = \frac{R}{R-1} + y,$$

$$z = \frac{R}{R-1} + \frac{(1-p)\overline{x}y}{R(R-1)}.$$  \hspace{1cm} (23) \hspace{1cm} (24)

In period 1, all borrowers borrow $F_1 > X_1\overline{x}$ at an interest rate $P_1^{-1} > R$ that satisfies

$$P_1 F_1 = \frac{(1-(1-\pi_0)p)F_1}{R} + \frac{(1-\pi_0)pX_1\overline{x}}{R-1}. $$

(25)

If the collateral value rises in period 2 (i.e., $x_2 = \overline{x}$), borrower type is fully revealed. Good borrowers repay by rolling over $F_2 = 0$, and bad borrowers repay by rolling over $F_2 \in (0, RV_2]$. If the collateral value falls in period 2 (i.e., $x_2 = \underline{x}$), only bad borrowers default, so borrower type is fully revealed. Good borrowers repay by rolling over $F_2 \in [R \max\{0, F_1-X_2-Y_2\}, RV_2]$. 

17
Proof. Lemma 9 in Appendix B implies the equilibrium in period 1. Lemma 5 implies the equilibrium if the collateral value rises in period 2. Lemma 6 implies the equilibrium if the collateral value falls in period 2.

When uncertainty in collateral value is high, the equilibrium turns out to be partial separation. The optimal amount of debt in period 1 is $F_1 \in (X_1 \bar{x}, RV_1 \bar{x}]$, which is illustrated as the red shaded region in the lower line in Figure 4. If the collateral value rises in period 2, all borrowers roll over the same amount of debt, so borrower type is not revealed. If the collateral value falls in period 2, good borrowers fully separate by rolling over a lower amount of debt than bad borrowers.

![Figure 4: Signaling regions for high uncertainty in collateral value](image)

**Proposition 3 (Partial separation).** Suppose that uncertainty in collateral value is high. That is, $\bar{x} \geq \min\{d_2, z\}$. In period 1, all borrowers borrow $F_1 \in (X_1 \bar{x}, RV_1 \bar{x}]$ at the interest rate $P_1^{-1} = R$.

If the collateral value rises in period 2 (i.e., $x_2 = \bar{x}$), borrower type is not revealed. All borrowers repay by rolling over $F_2 > RV_2 \bar{x}$ at an interest rate $P_2^{-1} > R$ that satisfies

$$P_2 F_2 = V_2 + \pi_0 (1 - p) \left( \frac{F_2}{R} - V_2 \bar{x} \right).$$

(26)

Subsequently, if the collateral value rises in period 3 (i.e., $x_3 = \bar{x}$), only bad borrowers default, so borrower type is fully revealed. If the collateral value falls instead (i.e., $x_3 = \bar{x}$), all borrowers default, so borrower type is not revealed.
If the collateral value falls in period 2 (i.e., \(x_2 = \bar{x}\)), borrower type is fully revealed. Good borrowers repay by rolling over \(F_2 \in [R \max \{0, F_1 - X_2 - Y_2\}, R(F_1 - X_2)]\). Bad borrowers repay by rolling over \(F_2 \in [R(F_1 - X_2), RV_2]\).

**Proof.** Lemma 9 in Appendix B implies the equilibrium in period 1. Lemma 8 implies the equilibrium if the collateral value rises in period 2. Lemma 6 implies the equilibrium if the collateral value subsequently rises in period 3, and Lemma 7 implies the equilibrium if the collateral value falls instead. Lemma 5 implies the equilibrium if the collateral value falls in period 2.

We now discuss the intuition for Propositions 2 and 3 by sketching out the essential elements of the formal proofs in Appendix B. As we discussed above, full separation through deleveraging alone is no longer possible when \(\frac{\bar{x}}{x} \geq \frac{R}{R-1}\). Good borrowers then face a tradeoff between **costly full separation** and **partial separation**.

Costly full separation occurs if good borrowers borrow \(F_1 > X_1 \bar{x}\) as in Proposition 2. The value of the non-pledgable asset under this strategy is

\[
W_{g,1} = -(1 - \pi_0)p \left(\frac{F_1}{R} - \frac{X_1 \bar{x}}{R - 1}\right) + \frac{Y_1}{R - 1}.
\]  

(27)

The first term is the interest cost in period 1, which arises from bad borrowers defaulting if the collateral value falls in period 2 with probability \(p\). The second term is the benefit of full separation. To minimize the interest cost, good borrowers optimally choose \(F_1\) that is arbitrarily close to \(X_1 \bar{x}\) so that

\[
W_{g,1} = -(1 - \pi_0)X_1 p \bar{x} \left(\frac{\bar{x}}{x} - \frac{R}{R - 1}\right) + \frac{Y_1}{R - 1}.
\]  

(28)

Fixing \((1 - p)\bar{x}\) (or equivalently fixing \(p \bar{x} = 1 - (1 - p)\bar{x}\)), the interest cost increases with uncertainty \(\frac{\bar{x}}{x}\).

Partial separation occurs if good borrowers borrow \(F_1 \leq RV_1 \bar{x}\) as in Proposition 3. Under this strategy, good borrowers avoid the higher interest cost at the sacrifice of not being able to fully separate in all future states. The value of the non-pledgable asset under this strategy is

\[
W_{g,1} = \frac{Y_1}{R - 1} - \frac{(1 - \pi_0)(1 - p) \bar{x} Y_1}{R^2 (R - 1)}.
\]  

(29)

The first term is the benefit of full separation. The second term accounts for the fact that good borrowers cannot separate if the collateral value rises in period 2 with probability
1 - p, then falls in period 3 with probability p. Fixing \((1 - p)\bar{x}\), equation (29) is constant in uncertainty \(\frac{\bar{x}}{x}\).

Comparing equations (28) and (29), good borrowers prefer costly full separation to partial separation when

\[
\frac{(1 - \pi_0)X_1 p \bar{x}}{R} \left( \frac{\bar{x}}{x} - \frac{R}{R - 1} \right) > -\frac{(1 - \pi_0)(1 - p)\bar{x}Y_1}{R^2(R - 1)},
\]

which is equivalent to

\[
\frac{\bar{x}}{x} < \frac{R}{R - 1} + \frac{(1 - p)\bar{y}}{R(R - 1)} = z.
\]

That is, good borrowers prefer costly full separation when uncertainty in collateral value is sufficiently low. As the interest cost increases with uncertainty, the preferred strategy switches to partial separation at the point \(z\). Even if costly full separation is preferred, it may not be feasible if the face value of debt must be so high that even good borrowers would want to default. Therefore, the boundary between Propositions 2 and 3 is the minimum of \(z\) (i.e., the point at which partial separation becomes preferred) and \(d_{2,2}\) (i.e., the point at which full separation becomes infeasible).

In this section, we have presented the results for the case \((1 - p)\bar{x} > 0.5\). In Appendix C, we present the results for the complementary case \((1 - p)\bar{x} < 0.5\) as summarized by Figure 5. The condition for costless full separation remains the same as Proposition 1. The results for Proposition 2 and 3 also remain essentially the same, except for two small differences. First, the threshold \(z\) for uncertainty in collateral value at which partial separation becomes preferred to full separation takes a different expression. Second, the optimal amount of debt when the collateral value falls in period 2 is lower than in Proposition 3. Therefore, if the collateral value rises in period 3, all borrowers repay, so borrower type is not revealed. If the collateral value falls instead, only bad borrowers default, so borrower type is fully revealed.

5. Conclusion

We have developed a dynamic model of credit markets with asymmetric information to allow for a richer set of signaling strategies through the path of debt and repayment, which were ruled out by assumption in a previous literature dominated by static models. In particular, we have shown the importance of deleveraging strategies in which good borrowers borrow a sufficiently high amount such that subsequent repayment reveals the presence of unobservable income. The precision of deleveraging as a signal depends on the degree of uncertainty in
collateral value. When uncertainty is high, information revelation could entail default of bad borrowers or no information revelation at all in equilibrium.

If we interpret uncertainty in collateral value as a property of an asset class, our model suggests that assets with more certain value are more useful as collateral for borrowers to signal in a dynamic setting. Interestingly, Dang, Gorton and Holmström (2015) reach a similar conclusion that optimal collateral in credit markets is debt instead of equity because of its low information sensitivity.

If we interpret uncertainty in collateral value as systematic uncertainty that is common across borrowers, our model provides an alternative view of leverage cycles in the macroeconomy. The usual view of leveraging followed by a wave of deleveraging or default is that these are negative outcomes caused by financial frictions or credit constraints (Kiyotaki and Moore 1997, Geanakoplos 2009). In our model, leverage is not an outcome of constraints but rather determined by the optimal choice of borrowers trying to resolve asymmetric information. Deleveraging is a signaling mechanism that reveals credit quality, sometimes through...
default of bad borrowers in equilibrium. Such information revelation is more likely when collateral value falls, such as in periods of falling housing or stock prices. Our model thus highlights a potentially positive side of deleveraging that complements existing theories of leverage cycles.
References


APPENDIX A. PROOFS OF LEMMAS 1 TO 3

Proof of Lemma 1. We show that equations (3) and (14) are equivalent by induction. Suppose that equation (14) holds for period \( t + 1 \). Then the continuation value is

\[
\frac{\mathbb{E}_t[J_{i,t+1}]}{R} = \frac{\Pr(D_{i,t+1} \geq F_t)\mathbb{E}_t[X_{t+1} + Y_{t+1} + W_{i,t+1} - F_t | D_{i,t+1} \geq F_t]}{R} + \frac{\Pr(D_{i,t+1} < F_t)\mathbb{E}_t[Y_{t+1} + W_{i,t+1} | D_{i,t+1} < F_t]}{R} = V_t - C_{i,t} + \frac{\Pr(D_{i,t+1} < F_t)\mathbb{E}_t[Y_{t+1} + W_{i,t+1} | D_{i,t+1} \geq F_t]}{R} + \frac{\Pr(D_{i,t+1} < F_t)\mathbb{E}_t[Y_{t+1} + W_{i,t+1} | D_{i,t+1} < F_t]}{R}.
\]

Substituting equations (10) and (A1) into equation (3), equation (14) holds for period \( t + 1 \). Then the default boundary (12) simplifies to

\[
\pi_3 = 1, \quad \pi_3 = 0, \quad \text{and riskless.}
\]

Proof of Lemma 2. When there is no further updating of reputation, the value of the non-pledgable asset is \( W_{i,t} = \hat{W}_{i,t} \). That is, the borrower’s objective function is independent of \( F_t \). Therefore, the borrower is indifferent between any amount of debt such that repayment is feasible (i.e., \( P_tF_t \geq F_{t-1} - X_t - \mathbb{1}_g(i)Y_t \)). Moreover, the maximum amount of new debt is

\[
P_tF_t = \frac{\Pr(X_{t+1} + V_{t+1} \geq F_t)F_t}{R} = \frac{\Pr(X_{t+1} + V_{t+1} \geq F_t)}{R} \mathbb{E}_t[X_{t+1} + V_{t+1} | X_{t+1} + V_{t+1} \geq F_t]
\]

with equality when \( F_t = \mathbb{E}_t[X_{t+1} + V_{t+1} | X_{t+1} + V_{t+1} \geq F_t] \). That is, debt is fully collateralized and riskless.

Proof of Lemma 3. We first show that inequality (19) holds in period 3. By equation (12), it suffices to show that \( 0 \leq \pi_3 - \hat{\pi}_3 \leq 1 \), which would imply that \( 0 \leq W_3 - \hat{W}_3 \leq \frac{\gamma_3}{R-1} \). If \( F_2 \leq \min\{D_{b,3}, D_{g,3}\} \), all borrowers repay. Therefore, equation (7) and Assumption 1 imply that \( \pi_3 = \pi_2 \) and \( \hat{\pi}_3 = 0 \). If \( F_2 > \max\{D_{b,3}, D_{g,3}\} \), all borrowers default. Therefore, equation (6) and Assumption 1 imply that \( \pi_3 = 1 \) and \( \hat{\pi}_3 = \pi_2 \).

If \( F_2 \in (\min\{D_{b,3}, D_{g,3}\}, \max\{D_{b,3}, D_{g,3}\}) \), we show that \( D_{b,3} \leq D_{g,3} \) by contradiction. Suppose that \( D_{b,3} > D_{g,3} \). Equations (6) and (7) imply that \( \pi_3 = 0 \) and \( \hat{\pi}_3 = 1 \). Equation (12) then implies that \( D_{b,3} = D_{g,3} = X_3 + V_3 - \frac{\gamma_3}{R-1} \), which contradicts \( D_{b,3} > D_{g,3} \). Therefore, \( D_{b,3} \leq D_{g,3} \). In the event of full separation in period 3, \( \pi_3 = 1, \hat{\pi}_3 = 0, \) and \( W_3 - \hat{W}_3 = \frac{\gamma_3}{R-1} \). Therefore, the default boundary (12) simplifies to \( D_{i,3} = X_3 + V_3 + \mathbb{1}_g(i)Y_3 \).

We now show that inequality (19) holds in period \( t \in \{1, 2\} \). By equation (16), it suffices
to show that
\[ \max_{F_t} P_t F_t \geq V_t \quad (A3) \]

and
\[ 0 \leq W_{b,t} - \hat{W}_{b,t} \leq W_{g,t} - \hat{W}_{g,t} \leq \frac{Y_t}{R^{3-t} (R-1)}. \quad (A4) \]

Suppose that inequalities (19) and (A4) hold in period \( t + 1 \). The proof is by induction.

Equation (11) implies that
\[ C_{g,t} - C_{b,t} = \Pr(F_t \in (D_{b,t+1}, D_{g,t+1}]) \mathbb{E}_t \left[ \frac{F_t}{R} - \frac{X_{t+1}}{R-1} \big| F_t \in (D_{b,t+1}, D_{g,t+1}] \right] \geq 0, \quad (A5) \]

where the inequality follows from
\[ \frac{F_t}{R} - \frac{X_{t+1}}{R-1} \geq 0 \iff F_t \geq \frac{RX_{t+1}}{R-1} = X_{t+1} + V_{t+1} \quad (A6) \]

and the induction hypothesis. Inequality (A3) then follows from \( \max_{F_t} C_{b,t} = V_t \).

We now prove the first part of inequality (A4). Inequality (A5) implies that the first term in equation (17) is weakly positive for bad borrowers. The third term in equation (17) is weakly positive by the induction hypothesis. The numerator in the second term of equation (17) can be rewritten as
\[ \mathbb{E}_t [\tilde{\pi}_{t+1} Y_{t+1}] - \tilde{\pi}_t Y_t = \begin{cases} \pi_t \Pr(D_{g,t+1} < F_t) \mathbb{E}_t [Y_{t+1} | D_{g,t+1} < F_t] & \text{if } F_{t-1} \leq \min \{D_{b,t}, D_{g,t}\} \\ \Pr(D_{g,t+1} < F_t) \mathbb{E}_t [Y_{t+1} | D_{g,t+1} < F_t] & \text{if } F_{t-1} \in (D_{b,t}, D_{g,t}] \\ -Y_t & \text{if } F_{t-1} \in (D_{g,t}, D_{b,t}] \\ 0 & \text{if } F_{t-1} > \max \{D_{b,t}, D_{t,t}\} \end{cases} \quad (A7) \]

which is weakly positive unless \( F_{t-1} \in (D_{g,t}, D_{b,t}] \). In this case, we show that \( D_{b,t} \leq D_{g,t} \) by contradiction. Suppose that \( D_{b,t} > D_{g,t} \). Equations (5) and (6) imply that \( \pi_t = 0 \) and \( \tilde{\pi}_t = 1 \). Moreover, Lemma 2 implies that \( D_{b,t+1} = D_{g,t+1} \) and \( W_{t,t+1} = \hat{W}_{t,t+1} \). Equations (16) and (A3) then imply that \( D_{b,t} = D_{g,t} = X_t + V_t - \frac{Y_t}{R^{3-t} (R-1)} \), which contradicts \( D_{b,t} > D_{g,t} \).

Therefore, \( D_{b,t} \leq D_{g,t} \).

We now prove the third part of inequality (A4). Inequality (A5) implies that the first term in equation (17) is weakly negative for good borrowers. If \( F_{t-1} \leq D_{b,t} \), the sum of the second and third terms of equation (17) is less than or equal to
\[ \frac{\pi_t \Pr(D_{g,t+1} < F_t) \mathbb{E}_t [Y_{t+1} | D_{g,t+1} < F_t]}{R^{3-t} (R-1)} + \frac{\Pr(D_{g,t+1} \geq F_t) \mathbb{E}_t [Y_{t+1} | D_{g,t+1} \geq F_t]}{R^{3-t} (R-1)} \leq \frac{Y_t}{R^{3-t} (R-1)} \quad (A8) \]
by equation (A7) and the induction hypothesis. If \( F_{t-1} \in (D_{b,t}, D_{g,t}] \), the second term of equation (17) is less than or equal to \( \frac{Y_{i+1}}{R^{3-t}(R-1)} \) by equation (A7), and the third term is zero by Lemma 2.

We now prove the second part of inequality (A4). Equations (17) and (A5) imply that

\[
W_{g,t} - \tilde{W}_{g,t} - \left( W_{b,t} - \tilde{W}_{b,t} \right) = \frac{Y_{i+1}}{R^{3-t}(R-1)} - \frac{F_t}{R} + \frac{X_{t+1}}{R - 1} \leq 0 \iff F_t \leq X_{t+1} + V_{i+1} + \frac{Y_{i+1}}{R^{3-t}(R-1)},
\]

which holds by the induction hypothesis. The second term is also positive by the induction hypothesis.

In the event of full separation in period \( t \), \( \max_{F_t} P_t F_t = \max_{F_t} C_{i,t} = V_t \) and \( W_{i,t} - \tilde{W}_{i,t} = \frac{1}{R^{3-t}(R-1)} \). Therefore, the default boundary (16) simplifies to \( D_{i,t} = X_t + V_t + \frac{1}{R^{3-t}(R-1)} \).

**Appendix B. Lemmas Used to Prove Propositions 1 to 3**

To simplify notation, we now let lowercase letters denote the corresponding variables divided by \( X_t \). That is, \( f_t = \frac{F_t}{X_t}, d_{i,t} = \frac{D_{i,t}}{X_t}, w_{i,t} = \frac{W_{i,t}}{X_t}, \) and \( c_{i,t} = \frac{C_{i,t}}{X_t} \).

**Lemma 8.** If \( f_1 \leq x_2 \) in period 2, all borrowers borrow \( f_2 = d_{b,3} \bar{x} + \varepsilon_2 \) for an arbitrarily small \( \varepsilon_2 > 0 \) at an interest rate \( P_2^{-1} < R \) that satisfies

\[
P_2 f_2 = \begin{cases} \frac{1}{R-1} + \frac{\pi_1}{R-1} + \frac{\varepsilon_2}{R} & \text{if } \frac{\varepsilon_2}{R} < \frac{d_{b,3}}{d_{b,3}}, \\ \frac{1}{R-1} + \frac{\pi_1(1-p)\varepsilon_2}{R} & \text{if } \frac{\varepsilon_2}{R} \geq \frac{d_{b,3}}{d_{b,3}}. \end{cases}
\]

The value of non-pledgable assets for good borrowers is

\[
w_{g,2} = \begin{cases} -\frac{(1 - \pi_1)}{R} \left( \frac{x_2}{R-1} + \frac{\varepsilon_2}{R} \right) + \frac{y}{R-1} & \text{if } \frac{\varepsilon_2}{R} < \frac{d_{b,3}}{d_{b,3}}, \\ -\frac{(1 - \pi_1)(1-p)\varepsilon_2}{R} + \frac{y}{R-1} - \frac{(1 - \pi_1)p\varepsilon_2}{R(R-1)} & \text{if } \frac{\varepsilon_2}{R} \geq \frac{d_{b,3}}{d_{b,3}}. \end{cases}
\]

**Proof.** If \( f_1 \leq x_2 \), Lemma 4 implies no updating of reputation in period 2 so that \( \pi_2 = \pi_1 \). Good borrowers choose \( f_2 \) that maximizes \( w_{g,2} \), and bad borrowers mimic good borrowers.

In period \( t \in \{1, 2\} \), expected repayment for a type \( i \) borrower is

\[
c_{i,t} = \begin{cases} \frac{1}{R-1} \left( \frac{x_2}{R-1} + \frac{\varepsilon_2}{R} \right) & \text{if } f_t > d_{i,t+1} \bar{x}, \\ \frac{f_t}{R-1} + \frac{p\varepsilon_2}{R-1} & \text{if } f_t \in (d_{i,t+1} \bar{x}, d_{i,t+1} \bar{x}], \\ \frac{f_t}{R} & \text{if } f_t \leq d_{i,t+1} \bar{x}. \end{cases}
\]
If \( d_{t+1} < d_{g,t+1} \), the difference in expected payment is

\[
c_g - c_b = \begin{cases} 
0 & \text{if } f_t > d_{g,t+1} \\
(1 - p) \left( \frac{f_t}{R} - \frac{\pi_g}{R-1} \right) & \text{if } f_t \in (d_{g,t+1}, d_{b,t+1}] \\
\frac{f_t}{R} - \frac{1}{R-1} & \text{if } f_t \in (d_{g,t+1}, d_{b,t+1}] \\
p \left( \frac{f_t}{R} - \frac{\pi_b}{R-1} \right) & \text{if } f_t \in (d_{b,t+1}, d_{g,t+1}] \\
0 & \text{if } f_t \leq d_{b,t+1}. 
\end{cases} 
\]  
(B4)

If \( d_{t+1} \geq d_{g,t+1} \), the difference in expected payment is

\[
c_g - c_b = \begin{cases} 
0 & \text{if } f_t > d_{g,t+1} \\
(1 - p) \left( \frac{f_t}{R} - \frac{\pi_g}{R-1} \right) & \text{if } f_t \in (d_{g,t+1}, d_{b,t+1}] \\
0 & \text{if } f_t \in (d_{g,t+1}, d_{b,t+1}] \\
p \left( \frac{f_t}{R} - \frac{\pi_b}{R-1} \right) & \text{if } f_t \in (d_{b,t+1}, d_{g,t+1}] \\
0 & \text{if } f_t \leq d_{b,t+1}. 
\end{cases} 
\]  
(B5)

We obtain equation (B1) by substituting \( \pi_2 = \pi_1 \) as well as equations (B4) and (B5) into equation (10).

If \( d_{b,3} < d_{g,3} \), equations (4), (15) and (B4) imply that

\[
w_{g,2} = \begin{cases} 
\frac{1 - \pi_2}{R-1} - \frac{(1 - \pi_2) y}{R(R-1)} & \text{if } (1) \ f_2 > d_{g,3} \\
-(1 - \pi_2)(1 - p) \left( \frac{f_2}{R} - \frac{\pi}{R-1} \right) + \frac{y}{R-1} - \frac{(1 - \pi_2) y (1 - p)}{R(R-1)} & \text{if } (2) \ f_2 \in (d_{g,3}, d_{g,3}] \\
-(1 - \pi_2) \left( \frac{f_2}{R} - \frac{\pi}{R-1} \right) + \frac{y}{R-1} - \frac{(1 - \pi_2)(1 - p) y}{R(R-1)} & \text{if } (3) \ f_2 \in (d_{g,3}, d_{g,3}] \\
\frac{1 - \pi_2}{R-1} - \frac{(1 - \pi_2) y}{R(R-1)} & \text{if } (4) \ f_2 \in (d_{g,3}, d_{g,3}] \\
\frac{1 - \pi_2}{R-1} - \frac{y}{R(R-1)} & \text{if } (5) \ f_2 \leq d_{b,3}. 
\end{cases} 
\]  
(B6)

Note that \( w_{g,2} \) is decreasing in \( f_2 \) in regions (2), (3) and (4). In the other regions, \( w_{g,2} \) is independent of \( f_2 \). Let \( w_{g,2}(n) \) denote the maximized value of \( w_{g,2} \) in region \( n \). \( w_{g,2}(3) \) is greater than \( w_{g,2}(2) \). \( w_{g,2}(4) \) is greater than \( w_{g,2}(1) \) and \( w_{g,2}(5) \). \( w_{g,2}(3) \) is greater than \( w_{g,2}(4) \) when \( (1 - p) \pi \geq 0.5 \). Therefore, \( w_{g,2} \) is maximized in region (3) when \( f_2 = d_{g,3} + \varepsilon_2 \).

If \( d_{b,3} \geq d_{g,3} \), equations (4), (15) and (B5) imply that

\[
w_{g,2} = \begin{cases} 
\frac{1 - \pi_2}{R-1} - \frac{(1 - \pi_2) y}{R(R-1)} & \text{if } (1) \ f_2 > d_{g,3} \\
\frac{1 - \pi_2}{R-1} - \frac{(1 - \pi_2) y}{R(R-1)} & \text{if } (2) \ f_2 \in (d_{g,3}, d_{g,3}] \\
\frac{1 - \pi_2}{R-1} - \frac{(1 - \pi_2) y}{R(R-1)} & \text{if } (3) \ f_2 \in (d_{g,3}, d_{g,3}] \\
\frac{1 - \pi_2}{R-1} - \frac{(1 - \pi_2) y}{R(R-1)} & \text{if } (4) \ f_2 \in (d_{g,3}, d_{g,3}] \\
\frac{1 - \pi_2}{R-1} - \frac{(1 - \pi_2) y}{R(R-1)} & \text{if } (5) \ f_2 \leq d_{b,3}. 
\end{cases} 
\]  
(B7)

Note that \( w_{g,2} \) is decreasing in \( f_2 \) in regions (2) and (4). In the other regions, \( w_{g,2} \) is independent of \( f_2 \). \( w_{g,2}(4) \) is greater than \( w_{g,2}(1) \), \( w_{g,2}(3) \), and \( w_{g,2}(5) \). \( w_{g,2}(2) \) is greater than \( w_{g,2}(4) \) when \( (1 - p) \pi \geq 0.5 \). Therefore, \( w_{g,2} \) is maximized in region (2) when \( f_2 = d_{b,3} + \varepsilon_2 \).
Lemma 9. If \( f_0 \leq x_1 \) in period 1, all borrowers borrow

\[
f_1 \in \begin{cases} 
(\bar{x}, d_{b,2}x) & \text{if } \frac{\bar{x}}{x} \in [1, d_{b,2}] \\
\bar{x} + \varepsilon_1 & \text{if } \frac{\bar{x}}{x} \in [d_{b,2}, \min\{d_{g,2}, z\}] \\
(\bar{x}, d_{b,2}x) & \text{if } \frac{\bar{x}}{x} \geq \min\{d_{g,2}, z\}
\end{cases}
\quad (B8)
\]

for an arbitrarily small \( \varepsilon_1 > 0 \) at an interest rate \( P_1^{-1} < R \) that satisfies

\[
P_1 f_1 = \begin{cases} 
\frac{f_1}{R} & \text{if } \frac{\bar{x}}{x} \in [1, d_{b,2}] \\
\frac{f_1}{R} \left(1 - (1 - \pi_0) p_f_1\right) + \frac{(1 - \pi_0) p y}{R - 1} & \text{if } \frac{\bar{x}}{x} \in [d_{b,2}, \min\{d_{g,2}, z\}] \\
\frac{f_1}{R} & \text{if } \frac{\bar{x}}{x} \geq \min\{d_{g,2}, z\},
\end{cases}
\quad (B9)
\]

Proof. If \( f_0 \leq x_1 \), Lemma 4 implies no updating of reputation in period 1 so that \( \pi_1 = \pi_0 \). Good borrowers choose \( f_1 \) that maximizes \( w_{g,1} \), and bad borrowers mimic good borrowers. We obtain equation (B9) by substituting \( \pi_1 = \pi_0 \) as well as equations (B4) and (B5) into equation (10).

In period \( t \in \{1, 2\} \), Lemma 3 and Assumption 2 imply that the default boundaries satisfy the inequality

\[
d_{g,t} \leq \frac{R}{R-1} - \frac{y}{R-1} < \frac{R + y}{R} < \frac{R}{R - 1} \leq d_{b,t}.
\quad (B10)
\]

In addition, Lemmas 5, 6, and 8 imply that

\[
w_{g,2} = \begin{cases} 
\frac{y}{R-1} & \text{if } f_1 \in (x_2, d_{g,2}x_2] \\
w_{g,2}(2,3) & \text{if } f_1 \leq x_2
\end{cases}
\quad (B11)
\]

where \( w_{g,2}(2,3) \) denotes equation (B2).

If \( \frac{\bar{x}}{x} < \frac{d_{g,2}}{d_{b,2}} \), equations (4), (15), (B4), and (B11) imply that

\[
w_{g,1} = \begin{cases} 
\frac{y}{R-1} - \frac{(1 - \pi_0) y}{R^2(R-1)} & \text{if } (1) f_1 > d_{g,2}\bar{x} \\
-(1 - \pi_0)(1 - p) \left( \frac{f_1}{R} - \frac{\bar{x}}{R-1} \right) + \frac{y}{R-1} - \frac{(1 - \pi_0) p y}{R(R-1)} & \text{if } (2) f_1 \in (d_{g,2}\bar{x}, d_{g,2}x_1] \\
-(1 - \pi_0) \left( \frac{f_1}{R} - \frac{1}{R-1} \right) + \frac{y}{R-1} & \text{if } (3) f_1 \in (d_{b,2}\bar{x}, d_{g,2}x_1] \\
-(1 - \pi_0) p \left( \frac{f_1}{R} - \frac{\bar{x}}{R-1} \right) + \frac{y}{R-1} & \text{if } (4) f_1 \in (d_{b,2}\bar{x}, d_{b,2}\bar{x}) \\
\frac{y}{R-1} & \text{if } (5) f_1 \in (\bar{x}, d_{b,2}\bar{x}) \\
\frac{y}{R-1} - \frac{(1 - p) y}{R(R-1)} (\frac{y}{R-1} - w_{g,2}(2,3)) & \text{if } (6) f_1 \in (\bar{x}, \bar{x}) \\
\frac{y}{R-1} - \frac{1}{R} (\frac{y}{R-1} - w_{g,2}(2,3)) & \text{if } (7) f_1 \leq \frac{\bar{x}}{x}.
\end{cases}
\quad (B12)
\]

Note that \( w_{g,1} \) is maximized in region (5) for any \( f_1 \in (\bar{x}, d_{b,2}\bar{x}] \).
If $\frac{x}{z} \in \left[\frac{d_{b,2}}{d_{b,2}}, d_{b,2}\right)$, equations (4), (15), (B5), and (B11) imply that

$$w_{g,1} = \begin{cases} 
\frac{y}{R - 1} - \frac{(1 - \pi_0) y}{R^2 (R - 1)} & \text{if } (1) f_1 > d_{g,2}x \\
- (1 - \pi_0) (1 - p) \left( \frac{f_1}{R} - \frac{x}{R - 1} \right) + \frac{y}{R - 1} - \frac{(1 - \pi_0) p y}{R^2 (R - 1)} & \text{if } (2) f_1 \in (d_{b,2}, d_{g,2}x] \\
- (1 - \pi_0) p \left( \frac{f_1}{R} - \frac{x}{R - 1} \right) + \frac{y}{R - 1} & \text{if } (3) f_1 \in (d_{g,2}x, d_{b,2}x] \\
\frac{y}{R - 1} - \frac{(1 - \pi_0) p y}{R^2 (R - 1)} & \text{if } (4) f_1 \in (d_{b,2}x, d_{g,2}x] \\
\frac{y}{R - 1} - \frac{(1 - \pi_0) p y}{R^2 (R - 1)} & \text{if } (5) f_1 \in (\frac{x}{z}, d_{b,2}x] \\
\frac{y}{R - 1} - \frac{(1 - \pi_0) p y}{R^2 (R - 1)} & \text{if } (6) f_1 \in (x, d_{b,2}x] \\
\frac{y}{R - 1} - \frac{(1 - \pi_0) p y}{R^2 (R - 1)} & \text{if } (7) f_1 \leq \frac{x}{z}.
\end{cases} \tag{B13}$$

Note that $w_{g,1}$ is maximized in region (5) for any $f_1 \in (\frac{x}{z}, d_{b,2}x]$. If $\frac{x}{z} \in [d_{b,2}, d_{g,2})$, equations (4), (15), (B5), and (B11) imply that

$$w_{g,1} = \begin{cases} 
\frac{y}{R - 1} - \frac{(1 - \pi_0) y}{R^2 (R - 1)} & \text{if } (1) f_1 > d_{g,2}x \\
- (1 - \pi_0) (1 - p) \left( \frac{f_1}{R} - \frac{x}{R - 1} \right) + \frac{y}{R - 1} - \frac{(1 - \pi_0) p y}{R^2 (R - 1)} & \text{if } (2) f_1 \in (d_{b,2}, d_{g,2}x] \\
- (1 - \pi_0) p \left( \frac{f_1}{R} - \frac{x}{R - 1} \right) + \frac{y}{R - 1} & \text{if } (3) f_1 \in (d_{g,2}x, d_{b,2}x] \\
\frac{y}{R - 1} - \frac{(1 - \pi_0) p y}{R^2 (R - 1)} & \text{if } (4) f_1 \in (\frac{x}{z}, d_{b,2}x] \\
\frac{y}{R - 1} - \frac{(1 - \pi_0) p y}{R^2 (R - 1)} & \text{if } (5) f_1 \in (d_{g,2}x, \frac{x}{z}] \\
\frac{y}{R - 1} - \frac{(1 - \pi_0) p y}{R^2 (R - 1)} & \text{if } (6) f_1 \in (x, d_{b,2}x] \\
\frac{y}{R - 1} - \frac{(1 - \pi_0) p y}{R^2 (R - 1)} & \text{if } (7) f_1 \leq \frac{x}{z}.
\end{cases} \tag{B14}$$

Note that $w_{g,1}$ is decreasing in $f_1$ in regions (2), (4) and (5). In the other regions, $w_{g,1}$ is independent of $f_1$. Let $w_{g,1}(n)$ denote the maximized value of $w_{g,1}$ in region (n). $w_{g,1}(3)$ is greater than $w_{g,1}(1)$ and $w_{g,1}(2)$. $w_{g,1}(6)$ is greater than $w_{g,1}(3)$, $w_{g,1}(5)$, and $w_{g,1}(7)$. Moreover, $w_{g,1}(4)$ is greater than $w_{g,1}(6)$ if and only if $\frac{x}{z} < z$. Therefore, $w_{g,1}$ is maximized in region (4) for $f_1 = \frac{x}{z} + \varepsilon_1$ if $\frac{x}{z} < z$. Otherwise, $w_{g,1}$ is maximized in region (6) for any $f_1 \in (x, d_{b,2}x]$. If $\frac{x}{z} \geq d_{g,2}$, equations (4), (15), (B5), and (B11) imply that

$$w_{g,1} = \begin{cases} 
\frac{y}{R - 1} - \frac{(1 - \pi_0) y}{R^2 (R - 1)} & \text{if } (1) f_1 > d_{g,2}x \\
- (1 - \pi_0) (1 - p) \left( \frac{f_1}{R} - \frac{x}{R - 1} \right) + \frac{y}{R - 1} - \frac{(1 - \pi_0) p y}{R^2 (R - 1)} & \text{if } (2) f_1 \in (d_{b,2}x, d_{g,2}x] \\
- (1 - \pi_0) p \left( \frac{f_1}{R} - \frac{x}{R - 1} \right) + \frac{y}{R - 1} & \text{if } (3) f_1 \in (d_{g,2}x, d_{b,2}x] \\
\frac{y}{R - 1} - \frac{(1 - \pi_0) p y}{R^2 (R - 1)} & \text{if } (4) f_1 \in (d_{g,2}x, \frac{x}{z}] \\
\frac{y}{R - 1} - \frac{(1 - \pi_0) p y}{R^2 (R - 1)} & \text{if } (5) f_1 \in (d_{g,2}x, d_{b,2}x] \\
\frac{y}{R - 1} - \frac{(1 - \pi_0) p y}{R^2 (R - 1)} & \text{if } (6) f_1 \in (x, d_{b,2}x] \\
\frac{y}{R - 1} - \frac{(1 - \pi_0) p y}{R^2 (R - 1)} & \text{if } (7) f_1 \leq \frac{x}{z}.
\end{cases} \tag{B15}$$

30
Note that \( w_{g,1} \) is decreasing in \( f_1 \) in regions (2) and (5). In the other regions, \( w_{g,1} \) is independent of \( f_1 \). \( w_{g,1}(3) \) is greater than \( w_{g,1}(1) \) and \( w_{g,1}(2) \). \( w_{g,1}(6) \) is greater than \( w_{g,1}(3), w_{g,1}(4), w_{g,1}(5), \) and \( w_{g,1}(7) \). Therefore, \( w_{g,1} \) is maximized in region (6) for any \( f_1 \in (x, d_{b,2}\bar{x}) \).

**APPENDIX C. RESULTS FOR THE CASE \((1 - p)\bar{\pi} < 0.5\)**

We first present Lemmas 8' and 9' for the case \((1 - p)\bar{\pi} < 0.5\), which correspond to Lemmas 8 and 9 for the case \((1 - p)\bar{\pi} > 0.5\). We then present Propositions 2' and 3' for the case \((1 - p)\bar{\pi} < 0.5\), which correspond to Propositions 2 and 3 for the case \((1 - p)\bar{\pi} \geq 0.5\).

**Lemma 8'.** Suppose that \((1 - p)\bar{\pi} < 0.5\) and \( \frac{\bar{\pi}}{\bar{x}} \geq \frac{d_{g,3}}{d_{b,3}} \). If \( f_1 \leq x_2 \) in period 2, all borrowers borrow \( f_2 = d_{b,3}\bar{x} + \varepsilon_2 \) for an arbitrarily small \( \varepsilon_2 > 0 \) at an interest rate \( P_2^{-1} < R \) that satisfies

\[
P_2 f_2 = \frac{(1 - (1 - \pi_1)p)f_2}{R} + \frac{(1 - \pi_1)px}{R - 1}. \tag{C1}
\]

The value of non-pledgable assets for good borrowers is

\[
w_{g,2} = -\frac{(1 - \pi_1)p\varepsilon_2}{R} + \frac{y}{R - 1} - \frac{(1 - \pi_1)(1 - p)\bar{\pi}y}{R(R - 1)}. \tag{C2}
\]

**Proof.** The proof essentially follows that for Lemma 8. The only difference is that \( w_{g,2}(4) \) is greater than \( w_{g,2}(2) \) when \((1 - p)\bar{\pi} < 0.5\). Therefore, \( w_{g,2} \) is maximized in region (4) when \( f_2 = d_{b,3}\bar{x} + \varepsilon_2 \).

**Lemma 9'.** Suppose that \((1 - p)\bar{\pi} < 0.5\). If \( f_0 \leq x_1 \) in period 1, all borrowers borrow

\[
f_1 \in \begin{cases} 
(x, d_{b,2}\bar{x}) & \text{if } \frac{\bar{\pi}}{\bar{x}} \in [1, d_{b,2}) \\
\bar{x} + \varepsilon_1 & \text{if } \frac{\bar{\pi}}{\bar{x}} \in [d_{b,2}, \min\{d_{g,2}, z'\}) \\
(x, d_{b,2}\bar{x}) & \text{if } \frac{\bar{\pi}}{\bar{x}} \geq \min\{d_{g,2}, z'\}
\end{cases} \tag{C3}
\]

for an arbitrarily small \( \varepsilon_1 > 0 \) at an interest rate \( P_1^{-1} < R \) that satisfies

\[
P_1 f_1 = \begin{cases} 
\frac{f_1}{R} & \text{if } \frac{\bar{\pi}}{\bar{x}} \in [1, d_{b,2}) \\
\frac{f_1}{R} + \frac{(1 - (1 - \pi_0)p)f_1}{R - 1} & \text{if } \frac{\bar{\pi}}{\bar{x}} \in [d_{b,2}, \min\{d_{g,2}, z'\}) \\
\frac{f_1}{R} & \text{if } \frac{\bar{\pi}}{\bar{x}} \geq \min\{d_{g,2}, z'\}
\end{cases}. \tag{C4}
\]

**Proof.** The proof essentially follows that for Lemma 9. Lemmas 5, 6, and 8' imply that

\[
w_{g,2} = \begin{cases} 
\frac{y}{R - 1} & \text{if } f_1 \in (x_2, d_{g,2}x_2) \\
w_{g,2}(4) < \frac{y}{\bar{x} - 1} & \text{if } f_1 \leq x_2
\end{cases} \tag{C5}
\]

where \( w_{g,2}(4) \) denotes equation (C2).
If $\frac{g}{2} < \frac{d_{g,2}}{d_{g,2}}$, $w_{g,1}$ is given by equation (B12) with $w_{g,2}(2,3)$ replaced by $w_{g,2}(4)$. Similarly, if $\frac{g}{2} \in [\frac{d_{g,2}}{d_{g,2}}, d_{g,2})$, $w_{g,1}$ is given by equation (B13) with $w_{g,2}(2,3)$ replaced by $w_{g,2}(4)$. In both cases, $w_{g,1}$ is maximized in region (5) for any $f_1 \in (\pi, d_{b,2} \pi)$.

If $\frac{g}{2} \in [d_{b,2}, d_{g,2})$, equations (4), (15), (B5), and (C5) imply that

$$w_{g,1} = \left\{ \begin{array}{ll}
\frac{y}{R - 1} - \frac{(1-\pi_0)y}{R^2(1-R)} \\
-\frac{(1-\pi_0)(1-p)}{R(1-R)} + \frac{y}{R} - \frac{(1-\pi_0)p}{R^2(1-R)} \\
\frac{y}{R - 1} - \frac{(1-\pi_0)p}{R^2(1-R)} \\
\frac{-(1-\pi_0)p}{R(1-p)\pi x} + \frac{y}{R - 1} - \frac{(1-\pi_0)p}{R^2(1-R)} \\
\frac{-(1-\pi_0)p}{R(1-p)\pi x} + \frac{y}{R - 1} - \frac{(1-\pi_0)p}{R^2(1-R)} \\
\frac{-(1-\pi_0)p}{R(1-p)\pi x} + \frac{y}{R - 1} - \frac{(1-\pi_0)p}{R^2(1-R)} \\
\frac{-(1-\pi_0)p}{R(1-p)\pi x} + \frac{y}{R - 1} - \frac{(1-\pi_0)p}{R^2(1-R)} \\
\end{array} \right. \quad (C6)$$

Note that $w_{g,1}$ is decreasing in $f_1$ in regions (2), (4) and (5). In the other regions, $w_{g,1}$ is independent of $f_1$. Let $w_{g,1}(n)$ denote the maximized value of $w_{g,1}$ in region (n). $w_{g,1}(3)$ is greater than $w_{g,1}(1)$ and $w_{g,1}(2)$, $w_{g,1}(6)$ is greater than $w_{g,1}(3)$, $w_{g,1}(5)$, and $w_{g,1}(7)$. $w_{g,1}(4)$ is greater than $w_{g,1}(6)$ if and only if $\frac{g}{2} < z'$, where $z'$ is given by equation (C9). Therefore, $w_{g,1}$ is maximized in region (4) for $f_1 = \pi + \varepsilon_1$ if $\frac{g}{2} < z'$. Otherwise, $w_{g,1}$ is maximized in region (6) for any $f_1 \in (\pi, d_{b,2} \pi)$.

If $\frac{g}{2} \geq d_{g,2}$, equations (4), (15), (B5), and (C5) imply that

$$w_{g,1} = \left\{ \begin{array}{ll}
\frac{y}{R - 1} - \frac{(1-\pi_0)y}{R^2(1-R)} \\
-\frac{(1-\pi_0)(1-p)}{R(1-R)} + \frac{y}{R} - \frac{(1-\pi_0)p}{R^2(1-R)} \\
\frac{y}{R - 1} - \frac{(1-\pi_0)p}{R^2(1-R)} \\
\frac{-(1-\pi_0)p}{R(1-p)\pi x} + \frac{y}{R - 1} - \frac{(1-\pi_0)p}{R^2(1-R)} \\
\frac{-(1-\pi_0)p}{R(1-p)\pi x} + \frac{y}{R - 1} - \frac{(1-\pi_0)p}{R^2(1-R)} \\
\frac{-(1-\pi_0)p}{R(1-p)\pi x} + \frac{y}{R - 1} - \frac{(1-\pi_0)p}{R^2(1-R)} \\
\frac{-(1-\pi_0)p}{R(1-p)\pi x} + \frac{y}{R - 1} - \frac{(1-\pi_0)p}{R^2(1-R)} \\
\frac{-(1-\pi_0)p}{R(1-p)\pi x} + \frac{y}{R - 1} - \frac{(1-\pi_0)p}{R^2(1-R)} \\
\frac{-(1-\pi_0)p}{R(1-p)\pi x} + \frac{y}{R - 1} - \frac{(1-\pi_0)p}{R^2(1-R)} \\
\end{array} \right. \quad (C7)$$

Note that $w_{g,1}$ is decreasing in $f_1$ in regions (2) and (5). In the other regions, $w_{g,1}$ is independent of $f_1$. $w_{g,1}(3)$ is greater than $w_{g,1}(1)$ and $w_{g,1}(2)$. $w_{g,1}(6)$ is greater than $w_{g,1}(3)$, $w_{g,1}(4)$, $w_{g,1}(5)$, and $w_{g,1}(7)$. Therefore, $w_{g,1}$ is maximized in region (6) for any $f_1 \in (\pi, d_{b,2} \pi)$.

**Proposition 2'.** Suppose that $(1-p)\pi < 0.5$ and uncertainty in collateral value is interme-
Proof. That is, \[ \frac{R}{P-1} \leq \frac{\pi}{x} < \min\{d_{g,2}, z'\}, \] where

\[ d_{g,2} = \frac{R}{R-1} + y, \tag{C8} \]

\[ z' = \frac{R}{R-1} + \frac{((1-p)\bar{x})y}{R(R-1)(1-(1-p)\bar{x})}. \tag{C9} \]

In period 1, all borrowers borrow \( F_1 > X_1\bar{x} \) at an interest rate \( P_1^{-1} > R \) that satisfies

\[ P_1F_1 = \frac{(1-(1-\pi_0)p)F_1}{R} + \frac{(1-\pi_0)pX_1\bar{x}}{R-1}. \tag{C10} \]

If the collateral value rises in period 2 (i.e., \( x_2 = \bar{x} \)), borrower type is fully revealed. Good borrowers repay by rolling over \( F_2 = 0 \). Bad borrowers repay by rolling over \( F_2 \in (0, RV_2] \). If the collateral value falls in period 2 (i.e., \( x_2 = \underline{x} \)), only bad borrowers default, so borrower type is fully revealed. Good borrowers repay by rolling over \( F_2 \in [R \max\{0, F_1 - X_2 - Y_2\}, RV_2]. \)

Proof. Lemma 9' implies the equilibrium in period 1. Lemma 5 implies the equilibrium if the collateral value rises in period 2. Lemma 6 implies the equilibrium if the collateral value falls in period 2.

**Proposition 3'**. Suppose that \((1-p)\bar{x} < 0.5 \) and uncertainty in collateral value is high. That is, \( \frac{\pi}{x} \geq \min\{d_{g,2}, z'\} \). In period 1, all borrowers borrow \( F_1 \in (X_1\bar{x}, RV_1\bar{x}] \) at the interest rate \( P_1^{-1} = R \).

If the collateral value rises in period 2 (i.e., \( x_2 = \bar{x} \)), borrower type is not revealed. All borrowers repay by rolling over \( F_2 > RV_2\bar{x} \) at an interest rate \( P_2^{-1} > R \) that satisfies

\[ P_2F_2 = \frac{(1-(1-\pi_0)p)F_2}{R} + \frac{(1-\pi_0)pX_2\bar{x}}{R-1}. \tag{C11} \]

Subsequently, if the collateral value rises in period 3 (i.e., \( x_3 = \bar{x} \)), all borrowers repay, so borrower type is not revealed. If the collateral value falls instead (i.e., \( x_3 = \underline{x} \)), only bad borrowers default, so borrower type is fully revealed.

If the collateral value falls in period 2 (i.e., \( x_2 = \underline{x} \)), borrower type is fully revealed. Good borrowers repay by rolling over \( F_2 \in [R \max\{0, F_1 - X_2 - Y_2\}, R(F_1 - X_2)] \). Bad borrowers repay by rolling over \( F_2 \in [R(F_1 - X_2), RV_2] \).

Proof. Lemma 9' implies the equilibrium in period 1. Lemma 8' below implies the equilibrium if the collateral value rises in period 2. Lemma 4 implies the equilibrium if the collateral value subsequently rises in period 3, and Lemma 6 implies the equilibrium if the collateral value falls instead. Lemma 5 implies the equilibrium if the collateral value falls in period 2.