Higher Interpolation and Extension for Persistence Modules

Peter Bubenik, Vin de Silva, and Vidit Nanda

Abstract. The use of topological persistence in contemporary data analysis has provided considerable impetus for investigations into the geometric and functional-analytic structure of the space of persistence modules. In this paper, we isolate a coherence criterion which guarantees the extensibility of non-expansive maps into this space across embeddings of the domain to larger ambient metric spaces. Our coherence criterion is category-theoretic, allowing Kan extensions to provide the desired extensions. As a consequence of such “higher interpolation”, it becomes possible to compare Vietoris-Rips and Čech complexes built within the space of persistence modules.

Introduction

The combination of rigorously-developed foundations [6, 10], efficient computability [25, 21] and stability properties [11, 8] have resulted in the widespread adoption of topological persistence [15, 25] as a technique for the analysis of large and complex datasets [7, 16, 22]. The output of this process is a collection of persistent homology groups, which are typically represented via a barcode or a persistence diagram. Recent applications of persistence often confront dynamically evolving data [17, 2], and in these cases one requires the ability to make inferences about the dynamics from collections of persistence diagrams. Substantial efforts have been devoted to this end; among the best-known outcomes are vineyards [12], Fréchet means [24] and persistence landscapes [4].

In this work, we provide a new geometric lens with which to view the space of persistence diagrams. Our main result is in fact a statement about the space of (sufficiently tame) persistence modules — these consist of vector spaces and linear maps indexed by the real line $\mathbb{R}$, and their representation theory produces persistence diagrams [23]. The class $\text{Mod}$ of persistence modules admits an interleaving metric, and the interpolation lemma from [8] establishes that $\text{Mod}$ is a path metric space — two modules which are $e$-interleaved for $0 \leq e < \infty$ can always be connected by a path in $\text{Mod}$ of length $e$. This lemma plays a key role in the proof of the stability theorem, which confirms that if two point-clouds are within Hausdorff distance $e$ of each other then (their persistence modules are $e$-interleaved, and hence) their persistence diagrams are also within bottleneck distance $e$ of each other [11, 8].

The interpolation lemma provides an affirmative answer to the Lipschitz extension problem [3, Ch 1] encoded in the following commutative diagram of metric spaces (and
1-Lipschitz maps):

\[
\begin{array}{c}
\{0, e\} \\
\downarrow \quad f' \\
[0, e]
\end{array} \quad \xrightarrow{f} \quad \text{Mod}
\]

Here \(\{0, e\}\) and \([0, e]\) are given the traditional Euclidean metric inherited from \(\mathbb{R}\), and the fact that \(f\) is 1-Lipschitz follows immediately from our assumption that \(f(0)\) and \(f(e)\) are \(e\)-interleaved. The existence of an extension \(f'\) allows us to assign intermediate persistence modules \(f'(x)\) to all \(x\) in \([0, e]\) so that \(f'\) agrees with \(f\) on the endpoints \(\{0, e\}\) and the interleaving distance between \(f'(x)\) and \(f'(y)\) does not exceed \(|x - y|\).

Similarly, one seeks 1-Lipschitz extensions across more general choices of metric inclusions. Our objective here is to prescribe sufficient categorical conditions on \(f\) which guarantee the existence of such extensions. Here is a consequence of our main result:

**Theorem (Higher interpolation and extension).** Let \(M\) be any metric space and \(A\) a subspace. If a map \(f: A \to \text{Mod}\) is coherent (in the sense of Definition 3.2 below), then it admits three 1-Lipschitz extensions \(M \to \text{Mod}\).

In order to precisely describe what it takes for \(f: A \to \text{Mod}\) to be coherent, we examine a pair of functors relating \(\text{Cat}\), the usual category of small categories, and \(\text{Met}\), the category of metric spaces with 1-Lipschitz maps as morphisms. Although our functors fail to form an adjoint pair in general, there is a distinguished natural transformation \(\eta\) from the identity functor on \(\text{Met}\) to their composite. Coherent maps are precisely those \(f: A \to \text{Mod}\) which factor through this natural transformation. The rest of this paper is organized as follows: in Section 1 we use known facts about the metric space of persistence modules to describe a functor \(\text{Cat} \to \text{Met}\), and in Section 2 we describe a functor \(\text{Met} \to \text{Cat}\). The proof of the higher interpolation theorem occupies Section 3 and some of its consequences are explored in Section 4.

1. The Geometry of Persistence Modules

We assume that the reader has prior familiarity with the basics of category theory [20, 1]. We also adopt the following conventions throughout: given a category \(C\) we write \(C_0\) for its class of objects and \(C(x, y)\) for its set of morphisms from an object \(x\) to an object \(y\). For a small category \(C\), we will denote the category of functors from \(C\) to \(D\) by \(D^C\). Although we will survey some relevant definitions and results here, the reader is invited to consult [6, 5, 8, 10, 23] for detailed background material on the categorical and metric aspects of persistence modules.

1.1. Persistence Modules as Functors. Let \(\mathbb{R}\) denote the category whose objects are the real numbers \(\mathbb{R}\), and which admits a unique morphism \(a \to b\) whenever \(a \leq b\). Persistent homology associates algebraic invariants to filtered topological spaces, which are naturally regarded as members of \(\text{Top}^\mathbb{R}\) – these are functors from \(\mathbb{R}\) to the category \(\text{Top}\) of topological spaces and continuous maps. In practice, one also encounters filtered spaces indexed by proper subcategories of \(\mathbb{R}\) (typically finite sets \(n = \{0, 1, \ldots, n\}\), natural numbers \(\mathbb{N}\), or integers \(\mathbb{Z}\)). In such cases, a standard dictionary may be used to modulate
between indexing subcategories: in the diagram

\[
\begin{array}{cccccc}
\mathbb{N} & \rightarrow & \mathbb{Z} & \rightarrow & \mathbb{R} & \rightarrow \\
\downarrow & & & & & \downarrow \\
\text{Top} & & & & & \\
\end{array}
\]

horizontal arrows are inclusions and pullbacks are given by restriction. Conversely, one may extend \(U : \mathbb{N} \rightarrow \text{Top}\) to \(U' : \mathbb{N} \rightarrow \text{Top}\) by assigning \(U'(k) = U(n)\) for \(k > n\); a functor \(U' : \mathbb{N} \rightarrow \text{Top}\) extends to \(U'' : \mathbb{Z} \rightarrow \text{Top}\) by assigning \(U''(k) = \emptyset\) for \(k < 0\); and a functor \(U'' : \mathbb{Z} \rightarrow \text{Top}\) extends to \(U''' : \mathbb{R} \rightarrow \text{Top}\) by assigning \(U'''(a) = U''(\lfloor a \rfloor)\) for all \(a \in \mathbb{R}\) (where \(\lfloor \cdot \rfloor\) indicates the floor function). That such constructions are possible in each case depicted above is a pleasant consequence of the fact that \(\text{Top}\) admits left Kan extensions.

Since we may pass from one of these functors to another, we will henceforth treat all filtered spaces as functors \(\mathbb{R} \rightarrow \text{Top}\). Letting \(\textbf{Vect}\) denote the category of vector spaces and linear maps over a fixed underlying field, we note that any functor \(H : \text{Top} \rightarrow \textbf{Vect}\) (such as singular homology) induces a push-forward from \(\text{Top}^R\) to \(\text{Vect}^R\) via post-composition. The resulting structure is a persistence module.

**Definition 1.2.** The category \(\text{Mod}\) of persistence modules is \(\text{Vect}^R\) – its objects are functors \(U : \mathbb{R} \rightarrow \text{Vect}\) and morphisms in \(\text{Mod}(U, V)\) are natural transformations \(U \Rightarrow V\).

The morphisms from \(U\) to \(V\) in \(\text{Mod}\) admit a convenient pointwise description as collections of linear maps \(\{\Phi(a) : U(a) \rightarrow V(a) \mid a \in \mathbb{R}\}\) which satisfy the following property. Across all choices of \(a \leq b\) in \(\mathbb{R}\), the following diagram commutes:

\[
\begin{array}{ccc}
U(a) & \xrightarrow{U(a \leq b)} & U(b) \\
\Phi(a) & \downarrow & \Phi(b) \\
V(a) & \xrightarrow{V(a \leq b)} & V(b)
\end{array}
\]

It is often convenient to pass to a more general setting by considering different choices of target categories. We therefore follow [6] and work with \(\textbf{C}^R\) for an arbitrary category \(\textbf{C}\), keeping in mind that this functor category specializes to \(\text{Mod}\) whenever \(\textbf{C} = \text{Vect}\).

### 1.2. The Interpolation Lemma

For each \(e \geq 0\), one has a translation functor \(T_e : \mathbb{R} \rightarrow \mathbb{R}\) (sending \(a\) to \(a + e\)) as well as a unique natural transformation \(\sigma_e\) from the identity to this functor. It is readily confirmed that every such translation induces an endofunctor on \(\textbf{C}^R\) which

1. sends each \(U \in \textbf{C}^R_0 := (\textbf{C}^R)_0\) to \(UT_e\) satisfying \(UT_e(a) = U(a + e)\) for \(a \in \mathbb{R}\), and
2. admits a distinguished natural transformation from the identity.

**Definition 1.3.** Given \(e \geq 0\), two functors \(U, V \in \textbf{C}^R\) are said to be \(e\)-interleaved if there are morphisms \(\Phi : U \rightarrow VT_e\) and \(\Psi : V \rightarrow UT_e\) in \(\textbf{C}^R\) satisfying \((VT_e)\Phi = U\sigma_{2e}\) and
\((\Phi T_e)\Psi = V\sigma_{2e}\), as encoded in commutativity of the following diagrams:

The \textit{interleaving distance} \cite{6, 8, 10, 19} on \(C^R_0\) is defined as follows:
\[
d_{\text{Int}}(U, V) = \inf\{e \geq 0 \mid U, V \text{ are e-interleaved}\},
\]
with the understanding that \(d_{\text{Int}}(U, V) = \infty\) if no interleaving exists.

We want to say that \(C^R_0\) together with the interleaving distance is a metric space. To make this possible, throughout this article, we relax the usual requirements for a metric space \((M, d)\) in three ways:

1. we allow \(d(x, y)\) to attain the value \(+\infty\),
2. we allow \(d(x, y) = 0\) for \(x \neq y\) in \(M\), and
3. we allow \(M\) to be a class rather than a set.

In other words, we work with \textit{symmetric Lawvere metric spaces} as defined in \cite{18}.

At times, we will not allow the third generalization. That is, we will need the collection of elements in a metric space to be a set. Let \(\text{Met}\) be the category whose objects are metric spaces with a set of elements, and whose morphisms are \textit{non-expansive} or \(1\)-\textit{Lipschitz maps}.\(^{1}\) Similarly, let \(\text{Cat}\) denote the category of small categories and functors.

**Theorem 1.4** (\cite{6}).

1. Given a category \(C\), \((C^R_0, d_{\text{Int}})\) is a metric space.
2. For a functor \(H : C \to D\), the map \(H^R : C^R_0 \to D^R_0\) which maps \(U\) to \(HU\) is \(1\)-\textit{Lipschitz} with respect to \(d_{\text{Int}}\).
3. Specializing to small categories, these combine to define a functor \(\bullet^R : \text{Cat} \to \text{Met}\).

Recall that a metric space \((M, d)\) is a \textit{path metric space} if for each \(x, y\) in \(M\) the infimum of the lengths of all paths between them equals \(d(x, y)\). The following \textit{interpolation lemma} \cite{8} establishes that \(\text{Mod}_0\) is a path metric space when endowed with the interleaving distance as a metric.

**Lemma 1.5.** Given \(e \geq 0\) and two \(e\)-interleaved persistence modules \(U_0\) and \(U_e\) in \(\text{Mod}_0\), there exists a one-parameter family \(\{U_a \mid a \in (0, e)\}\) in \(\text{Mod}_0\) so that \(U_a\) and \(U_b\) are \(|a - b|\)-interleaved for all \(a, b \in [0, e]\).

Note in general that the interpolating family \(U_t\) is not unique, and that the lemma need not hold for general categories \(C^R\).

As mentioned in the Introduction, our main result is a higher interpolation lemma. A simple example illustrating that higher interpolations are not always possible may be found in \cite{14}, and we will reproduce it here in Section 1.3. On the other hand, a pathway towards the desired generalization is provided by the following \textit{sharp interpolation lemma} from \cite{10}: not only can one find an interpolating family of modules, but one can also find a compatible family of interleaving maps between them.

**Lemma 1.6.** Given persistence modules \(U_0\) and \(U_e\) along with morphisms \(\Phi : U_0 \to U_eT_e\) and \(\Psi : U_e \to U_0T_e\) which realize an \(e\)-interleaving, there exist

\(^{1}\)That is, a map \(f : (M, d_M) \to (N, d_N)\) satisfying \(d_N(f(x), f(y)) \leq d_M(x, y)\) for all \(x, y \in M\).
Figure 1. Three overlaid persistence diagrams $\Delta$, $\times$ and $\bullet$. The corresponding persistence modules have pairwise interleaving distance 1, but there is no persistence module within distance $e$ of all three for any $e < 1$.

(1) persistence modules $\{ U_a \mid a \in (0, e) \}$, and
(2) module morphisms $\Phi^b_a : U_a \to U_b T_{b-a}$ and $\Psi^a_b : U_b \to U_a T_{b-a}$ for all $a \leq b$ in $[0, e]$, so that

(3) $\Phi^e_0 = \Phi$ and $\Psi^0_e = \Psi$,
(4) $\Phi^b_a$ and $\Psi^a_b$ realize a $(b-a)$-interleaving between $U_a$ and $U_b$, and
(5) $(\Phi^c_b T_{b-a}) \Phi^b_a = \Phi^c_a$ and $(\Psi^c_b T_{b-a}) \Psi^a_b = \Psi^c_a$ hold for all $a \leq b \leq c$ in $[0, e]$.

In general, the intermediate modules $U_a$ and maps $\Phi^b_a$ and $\Psi^a_b$ are not uniquely defined.

1.3. Failure of Higher Interpolation. The main result of [13] asserts that (isomorphism classes of) tame\(^2\) persistence modules are faithfully represented by their persistence diagrams [25, 11, 8], which are multi-sets of points in the upper half-plane\(^3\). Moreover, these diagrams admit a bottleneck distance $d_{\text{Bot}}$ and the following isometry theorem [10, 6] establishes that the assignment $\text{dgm}$ which sends a (tame) persistence module to its corresponding diagram preserves distances.

**Theorem 1.7.** The equality

$$d_{\text{Int}}(U, V) = d_{\text{Bot}}(\text{dgm}U, \text{dgm}V)$$

holds across all pairs $U, V$ of tame persistence modules\(^4\).

It was shown in [14] that higher interpolations may fail to exist even for simple choices of $A \hookrightarrow M$.

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2These are modules $U : \mathbb{R} \to \text{Vect}$ for which $U(t)$ is finite-dimensional for each $t$.
3To be precise, one needs to take decorated persistence diagrams [10]. This result extends to the $q$-tame modules described in [10] in a somewhat weaker form [9].
4This result also holds for q-tame persistence modules [10].
Example 1.8. Let \((A, d)\) be the three-point metric space \(\{x_1, x_2, x_3\}\) with \(d(x_i, x_j) = 1\) for \(i \neq j\), and let \(M\) be \(A\) together with \(x_0\) where \(d(x_0, x_i) = \frac{1}{2}\) for \(i \geq 1\). Let \(f: A \rightarrow \text{Mod}_0\) be the function whose images \(f(x_i)\) prescribe the three persistence diagrams shown in Figure 1. It is easily confirmed that no persistence diagram is within distance \(< 1\) of each of the three persistence diagrams in Figure 1. So there is no tame persistence module which may be assigned to \(x_0\) to provide an extension of \(f\) from \(A\) to \(M\) without strictly increasing the Lipschitz constant.

This sharp interpolation lemma suggests the reason for the failure of the higher-order interpolation in Example 1.8: the 1-interleavings do not satisfy the compatibility condition (5).

2. Categories from Metric Spaces

In Theorem 1.4 we defined a functor \(\bullet^R : \text{Cat} \rightarrow \text{Met}\). The central goal of this section is to describe the construction of a functor \(\text{Met} \rightarrow \text{Cat}\), whose interaction with \(\bullet^R\) will be of crucial importance in the proof of our main result. To this end, consider \((M, d) \in \text{Met}_0\) and let \(M \times \mathbb{R}\) denote the product \(M \times \mathbb{R}\) equipped with the binary relation

\[
(x, s) \leq_M (y, t) \text{ if and only if } d(x, y) \leq t - s.
\]

Henceforth, we will often drop the subscript and simply write \((x, s) \leq (y, t)\), relying on context for clarity. The following result is straightforward.

Proposition 2.1. The relation \(\leq\) induces a pre-order on \(M \times \mathbb{R}\). Moreover, if \(d\) is a genuine metric in the sense that \(d(x, y) = 0\) holds only for \(x = y\), then \(\leq\) induces a partial order on \(M \times \mathbb{R}\).

Proof. Since \(d(x, x) = 0\), we have reflexivity: \((x, s) \leq (x, s)\). Turning to transitivity, assume \((x, s) \leq (y, t)\) and \((y, t) \leq (z, u)\). By the triangle inequality, we have

\[
d(x, z) \leq d(x, y) + d(y, z) \leq (t - s) + (u - t) = u - s.
\]

Hence, \((x, s) \leq (z, u)\) so \(\leq\) is transitive. Finally, if \((x, s) \leq (y, t) \leq (x, s)\), then \(0 \leq d(x, y) \leq \min(s - t, t - s)\). So \(s = t\). Thus, we have anti-symmetry only if \(d(x, y) = 0\) forces \(x = y\) in \(M\).

Since \(M \times \mathbb{R}\) is a preordered set, it may be treated as a thin category\(^5\). Given a 1-Lipschitz map \(f \in \text{Met}(M, N)\), define \(f^R : M \times \mathbb{R} \rightarrow N \times \mathbb{R}\) via the mapping \((x, s) \mapsto (f(x), s)\).

Theorem 2.2. The assignment \(\bullet^R\) prescribes a functor \(\text{Met} \rightarrow \text{Cat}\).

Proof. We first confirm that \(f^R : M \times \mathbb{R} \rightarrow N \times \mathbb{R}\) is a morphism in \(\text{Cat}\) whenever \(f : (M, d_M) \rightarrow (N, d_N)\) is 1-Lipschitz. If \((x, s) \leq_M (y, t)\), we have \(d_M(x, y) \leq t - s\). Since \(f\) is 1-Lipschitz, we obtain \(d_N(f(x), f(y)) \leq d_M(x, y) \leq t - s\). So by definition,

\[
f^R(x, s) = (f(x), s) \leq_N (f(y), t) = f^R(y, t).
\]

Thus, \(f^R\) is order-preserving. It is easy to confirm that \(1_M^R = 1_{M \times \mathbb{R}}\), so we turn to the task of establishing functoriality. Consider \(M \xrightarrow{f} N \xrightarrow{g} P\) in \(\text{Met}\), and note that

\[
g^R \circ f^R(x, s) = g^R(f(x), s) = (gf(x), s) = [(gf)^R](x, s),
\]

which concludes the proof. \(\square\)

---

\(^5\)A category is thin if it admits at most one morphism between any pair of objects.
Figure 2. An illustration of the order \( \leq \) on \( MR \) in the special case where \( M = \mathbb{R} \) with the usual metric. Given \((x, s) \in MR\), the darker region consists of the up-set \((y, t) \geq (x, s)\) while the lighter region consists of the down-set.

With the existence of \( \bullet R : \text{Met} \to \text{Cat} \) established, one might hope for an adjunction with the functor \( \bullet R : \text{Cat} \to \text{Met} \) from Theorem 1.4. If such an adjunction existed, then for each metric space \( M \in \text{Met}_0 \) and category \( C \in \text{Cat}_0 \), we would expect a natural bijection of sets between \( \text{Cat}(M^R, C) \) and \( \text{Met}(M, C^R) \), i.e., the set of functors from \( M^R \) to \( C \) would correspond with \( 1 \)-Lipschitz maps from \( M \) to \( C^R \). Example 1.8 confirms that there is no such bijection in general, whence our functors \( \bullet R \) and \( \bullet \mathbb{R} \) do not constitute an adjoint pair. Instead, we seek solace in the existence of a unit, as described below.

Let \( F \) denote the endofunctor on \( \text{Met} \) arising from the following composition:

\[
F : \text{Met} \longrightarrow \text{Cat} \longrightarrow \text{Met}.
\]

(2.3)

Chasing definitions, one can explicitly describe the effect of \( F \) on the objects and morphisms of \( \text{Met} \): each metric space \( M \) is mapped to \( M^R \) (with the interleaving distance), and every \( 1 \)-Lipschitz \( f : M \to N \) is sent to the map \( M^R \to N^R \) which takes \( U : R \to M^R \) to \( f \circ U : R \to N^R \).

**Theorem 2.4.** The functor \( F \) admits a natural transformation \( \eta : 1_{\text{Met}} \Rightarrow F \) from the identity endofunctor on \( \text{Met} \).

**Proof.** For each \( (M, d_M) \in \text{Met}_0 \), we require a \( 1 \)-Lipschitz map \( \eta_M \) in \( \text{Met}(M, FM) \) which sends points of \( M \) to functors \( R \to M^R \). We provisionally define this map as follows: for each \( x \in M \), let \( \eta_M(x) \) be the functor which sends \( s \in R \) to \( (x, s) \) and \( s \leq t \) to \( (x, s) \leq_M (x, t) \). To check that the latter inequality holds in \( M^R \), we verify that \( d_M(x, x) \leq t - s \). This definition sends identities to identities and respects composition since \( M^R \) is a thin category. Thus, \( \eta_M(x) \) is indeed a functor.

Next, we confirm that \( \eta_M \) is \( 1 \)-Lipschitz by exhibiting a \( d_M(x, y) \)-interleaving between \( \eta_M(x) \) and \( \eta_M(y) \) for every pair of points \( x, y \) in \( M \). By definition, \( (x, s) \leq_M (y, t) \) if and only if \( d_M(x, y) \leq t - s \). Thus, \( (x, s) \leq_M (y, s + d_M(x, y)) \) and similarly, \( (y, t) \leq
\( (x, t + d_M(x, y)) \). Since \( M\mathcal{R} \) is a thin category, everything commutes and so \( \eta_M(x) \) and \( \eta_M(y) \) are \( d_M(x, y) \)-interleaved. Hence \( \eta_M : M \to FM \) is 1-Lipschitz, as desired.

Finally, we check that the assignment \( x \mapsto \eta_M(x) \) prescribes a natural transformation. Given \( f : (M, d_M) \to (N, d_N) \) in \( \textbf{Met} \), we must verify that the following diagram commutes:

\[
\begin{array}{ccc}
M & \xrightarrow{\eta_M} & FM \\
\downarrow f & & \downarrow Ff \\
N & \xrightarrow{\eta_N} & FN
\end{array}
\]

Pick any \( x \in M \). For all \( s \in R \), we have

\[
[Ff \circ \eta_M(x)](s) = Ff(x, s) = (f(x), s) = \eta_N(f(x))(s) = [\eta_N \circ f(x)](s),
\]

so our diagram commutes and \( \eta \) is a natural transformation. \( \square \)

### 3. Coherence and Higher Interpolation

Throughout this section, we fix a choice of category \( \mathcal{C} \) and metric space \( A \in \textbf{Met} \). We also let \( \eta : I_{\textbf{Met}} \Rightarrow F \) be the natural transformation from the proof of Theorem 2.4. For a functor \( G : A\mathcal{R} \to \mathcal{C} \) define \( \theta(G) = G^R \circ \eta_A \):

\[
A \xrightarrow{\eta_A} FA = A\mathcal{R} \xrightarrow{G^R} \mathcal{C}^R. \tag{3.1}
\]

**Definition 3.2.** The 1-Lipschitz functions \( g : A \to \mathcal{C}^R \) which lie in the image of \( \theta \) are called **coherent**. In the special case where \( \mathcal{C} = \text{Vect} \), such functions are called **coherent persistence modules**.

By definition, for every coherent 1-Lipschitz map \( g : A \to \mathcal{C}^R \) there is some functor \( G : A\mathcal{R} \to \mathcal{C} \) satisfying \( g = \theta(G) \). The map \( G^R \) now serves as an extension of \( g \) across \( \eta_A : A \to A\mathcal{R} \) because the following diagram commutes by the definition of \( \theta \):

\[
\begin{array}{ccc}
A & \xrightarrow{g} & \mathcal{C}^R \\
\downarrow \eta_A & & \downarrow G^R \\
A\mathcal{R} & \xrightarrow{G^R} & \mathcal{C}^R
\end{array} \tag{3.3}
\]

If \( A \) includes into a larger metric space \( M \in \textbf{Met} \), then it is easy to check that \( A\mathcal{R} \) is a full subcategory of \( M\mathcal{R} \). Given any functor \( G : A\mathcal{R} \to \mathcal{C} \), one has the following functor extension problem:

\[
\begin{array}{ccc}
A\mathcal{R} & \xrightarrow{G} & \mathcal{C} \\
\downarrow \mathcal{M\mathcal{R}} & & \downarrow \mathcal{G} \\
M\mathcal{R} & \xrightarrow{G} & \mathcal{C} \tag{3.4}
\end{array}
\]

Recall that the category \( \mathcal{C} \) is (co)complete if it has all (co)limits. The solution to problems such as (3.4) for functors taking values in (co)complete categories is furnished by **Kan extensions** [20, Ch X].

**Proposition 3.5.** An extension \( \hat{G} \) of \( G \) exists under any of the following circumstances:
• if $C$ is cocomplete, we can take $\hat{G}$ to be the left Kan extension $\text{Lan}_G G$ of $G$,
• if $C$ is complete, we can take $\hat{G}$ to be the right Kan extension $\text{Ran}_G G$ of $G$,
• if $C$ is bicomplete and abelian, we can take $\hat{G}$ to be the image of the universal natural transformation $\text{Lan}_G G \Rightarrow \text{Ran}_G G$.

If $C = \text{Vect}$ (as in the case of persistence modules) then we have all three extensions, but if $C = \text{Top}$ (as in the case of filtered topological spaces), then we only have the left and right extensions. The following theorem is the main result of this paper.

**Theorem 3.6.** Let $A$ be the subspace of a metric space $M \in \text{Met}$, and assume that the 1-Lipschitz map $g : A \to C^R$ is coherent. If $C$ is (co)complete, then $g$ admits a coherent 1-Lipschitz extension $\hat{g} : M \to C^R$.

**Proof.** Since $g$ is coherent, it equals $\theta(G)$ for some functor $G : A^{IR} \to C$. By Proposition 3.5 and the (co)completeness hypothesis on $C$, there is an extension $\hat{G} : M^{IR} \to C$ of $G$ as in (3.4). Note that the following diagram of metric spaces and 1-Lipschitz maps commutes:

$$
\begin{array}{ccc}
A & \xrightarrow{\eta_A} & A^{IR} & \xrightarrow{G^R} & C^R \\
| & & \downarrow & & \\
M & \xrightarrow{\eta_M} & M^{IR} & \xrightarrow{\hat{G}^R} & C^R
\end{array}
$$

The square on the left commutes because $\eta$ is a natural transformation. The triangle on the right commutes since $G^R(F) = G \circ F = \hat{G} \circ i \circ F = \hat{G}^R(i(F))$, where $i : A^{IR} \hookrightarrow M^{IR}$ and the middle equality is by (3.4). Since the composite in the top row of our diagram equals $g$, it is immediately seen that the desired extension $\hat{g} : M \to C^R$ is given by $\theta(\hat{G}) = \hat{G}^R \circ \eta_M$. \hfill \square

The $C = \text{Vect}$ specialization of Theorem 3.6 yields the higher interpolation lemma promised in the Introduction. In this case, for a given $G$ satisfying $\theta(G) = g$ we have at least three possible choices\(^6\) of $\hat{G}$ arising from Proposition 3.5. Regardless of which $\hat{G}$ is chosen, the map $\theta(\hat{G})$ is itself coherent by construction, and hence admits further extensions to larger metric spaces.

4. Consequences

In this section we describe some applications of Theorem 3.6.

4.1. Discrete and Continuous Interpolation. Let $U_1, \ldots, U_n$ be a collection of $n \geq 1$ persistence modules and let $d \geq 0$ be a fixed constant. Assume further that $U_i$ and $U_j$ are $2e$-interleaved for all $i \neq j$. Let $A = \{a_1, \ldots, a_n\}$ be the metric space where all nontrivial distances $d(a_i, a_j)$ equal $2e$, and note that we may describe each $U_i$ as the image $g(a_i)$ of a 1-Lipschitz map $g : A \to \text{Mod}_0$. The following result provides an easily-computable criterion for coherence (compare to Lemma 1.6 as well as [14, Thm 4.2]).

\(^6\)For explicit calculations and a comparison of all three extensions in the context of the sharp interpolation lemma, consult [10, Sec 3.5] (and particularly Prop 3.6 therein). The image extension is optimal among the three $n$ the sense that it satisfies two universal properties instead of one.
Let $M$ be the metric space which consists of $A$ above, along with an additional point $a$ so that $d(a, a_i) = e$. The **discrete interpolation problem** for persistence modules seeks to extend our 1-Lipschitz map $g : A \to \text{Mod}_0$ to a 1-Lipschitz map $\hat{g} : M \to \text{Mod}_0$ across the obvious inclusion $A \hookrightarrow M$. On the other hand, the **continuous interpolation problem** for persistence modules seeks an extension of $g$ across the inclusion of $A$ as vertices of the standard $(n - 1)$-simplex $\Sigma \subset \mathbb{R}^n$, given by:

$$\Sigma = \left\{ (x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1 + \cdots + x_n = \sqrt{2e} \text{ and } x_j \geq 0 \right\}$$

It follows immediately from Theorem 3.6 that the discrete and continuous interpolation problems both admit solutions (in triplicate) whenever the modules $\U_i, \ldots, \U_n$ are connected by morphisms $\Phi_{ij}$ which satisfy the two properties from Proposition 4.1. Note that in the latter case, $g$ extends not only to $\Sigma$ but to $\mathbb{R}^n$.

### 4.2. Čech and Rips Complexes of Persistence Modules

Let $(M, d_M)$ be an ambient metric space with a distinguished subspace $A \subset M$. We recall the **Vietoris-Rips complex** $\mathcal{V}$ and the **Čech complex** $\mathcal{C}$ on $A$. Both are abstract simplicial complexes filtered by a single non-negative real parameter; their vertices are the points of $A$, but the construction of higher-dimensional simplices sets them apart. In particular, a collection $[a_0, \ldots, a_n]$ of points in $A$ forms an n-simplex

1. in $\mathcal{V}(A, e)$ if and only if $d_M(a_i, a_j) \leq e$ for $0 \leq i < j \leq n$, and
2. in $\mathcal{C}(A, e)$ if and only if some $b$ in $M$ satisfies $d_M(a_i, b) \leq e$ for $0 \leq i \leq n$.

Any $b$ which is within distance $e$ of all the points $[a_0, \ldots, a_n]$ is called an $e$-**witness** for those points. It is well-known and immediate from the definitions that for each $e \geq 0$...
one always has the following simplicial sandwich:

\[ C(A, e) \hookrightarrow V(A, 2e) \hookrightarrow C(A, 2e), \]

where the first inclusion follows directly from the triangle inequality, and the second follows from the fact that any \( a_i \) serves as a \( 2e \)-witness for a simplex \([a_0, \ldots, a_n]\) in \( V(A, 2e) \).

We may ask if these inclusions are tight. For the first inclusion, assume that \( M \) is a path metric space (e.g., \( \text{Mod}_0 \)), and that there exist \( a_0, a_1 \in A \) with \( d_M(a_0, a_1) = 2e + \delta \) for some \( \delta > 0 \). Then \([a_0, a_1]\) is not a \( 1 \)-simplex in \( V(A, 2e) \) and there exists a path from \( a_0 \) to \( a_1 \) with length at most \( 2e + 2\delta \). The midpoint of this path is a \((e + \delta)\)-witness for \([a_0, a_1]\), which is therefore a \( 1 \)-simplex in \( C(A, e + \delta) \). Thus \( C(A, e + \delta) \not\hookrightarrow V(A, 2e) \) for \( \delta > 0 \). The second inclusion \( V(A, 2e) \hookrightarrow C(A, 2e) \) might be improved, depending on \( M \). For example, \( V(A, 2e) \) always includes into \( C(A, 2e/\sqrt{3}) \) whenever \( A \) is a subset of \( \mathbb{R}^2 \) with the standard Euclidean metric.

When working within \( \text{Mod}_0 \), one typically strengthens the requirements in the definitions of Rips and Čech complexes slightly since the infimum over interleavings may not actually be attained. In particular, a collection \( \sigma = [U_0, \ldots, U_n] \) of persistence modules forms an \( n \)-simplex

(1) in \( V(\text{Mod}_0, e) \) if and only if \( U_i \) and \( U_j \) are \( e \)-interleaved for all \( 0 \leq i < j \leq n \), and

(2) in \( C(\text{Mod}_0, e) \) if and only if some \( V \) is \( e \)-interleaved with \( U_i \) for all \( 0 \leq i \leq n \).

It is straightforward to check that \( V(\text{Mod}_0, 2e) \) does not include into \( C(\text{Mod}_0, d) \) for any \( d < 2e \) by appealing to Figure 1 and the isometry theorem. On the other hand, the following result from [14] characterizes those simplices of \( V(\text{Mod}_0, 2e) \) which do include into \( C(\text{Mod}_0, e) \).

**Theorem 4.2.** Let \( U_0, \ldots, U_n \) be a collection of persistence modules and let \( e \geq 0 \). Then, \([U_0, \ldots, U_n]\) is an \( n \)-simplex in \( C(\text{Mod}_0, e) \) if and only if there exist morphisms \( \Phi_{ij} \) for \( i \neq j \) which satisfy the conditions of Proposition 4.1

Thus, a simplex in \( V(\text{Mod}_0, 2e) \) forms a simplex in \( C(\text{Mod}_0, e) \) if and only if the module morphisms which realize the pairwise \( 2e \)-interleavings can be chosen to commute (up to factors of \( \sigma \)). From our perspective here, the preceding result is a direct consequence of the discrete interpolation discussed in Section 4.1.

**References**


(Peter Bubenik) DEPARTMENT OF MATHEMATICS, UNIVERSITY OF FLORIDA
E-mail address: peter.bubenik@ufl.edu

(Vin de Silva) DEPARTMENT OF MATHEMATICS, POMONA COLLEGE
E-mail address: Vin.deSilva@pomona.edu

(Vidit Nanda) DEPARTMENT OF MATHEMATICS, UNIVERSITY OF PENNSYLVANIA
E-mail address: vnanda@sas.upenn.edu