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Classical Competitive Analysis of Economies with Islands*

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Arrow–Debreu competitive equilibrium analysis is extended to environments with information sets differing in space as well as in time and with people moving between locations. Equilibrium is shown to exist and to be optimal, and the equilibrium price system is characterized. Such environments include many of those studies in the sectoral reallocation literature. *Journal of Economic Literature* Classification Numbers: 021, 023.

1. INTRODUCTION

The scope of competitive general equilibrium theory has been continually expanding since its first rigorous formulation in the 1940s and 1950s. Initially, the theory was seen as a static, highly stylized concept capable of dealing only with the simplest of environments. This was despite the fact that the idea of indexing goods by date and location, making them different goods, can be traced back to Lindahl in the 1920s and Hicks and Tinter in the 1930s. In the 1950s, Arrow and Debreu introduced uncertainty by considering trades of goods contingent on states of nature. This was a major expansion of the scope of environments that the theory could address. During the 1950s, the major results of the theory were developed,

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including not only the welfare theorems and the standard fixed-point arguments in commodities and prices, but also the fixed-point arguments in utilities and transfers first used by Negishi [21].

Beginning in the late 1960s, important advances in general equilibrium theory expanded its use to the analysis of economies with a large number of products and households. These advances (particularly those of Aumann [1, 2], Bewley [5], and Mas-Colell [18]) were exploited by Prescott and Townsend [25, 26] in an extension of the classical approach to economies with private information. With their approach, the objects that are traded and priced are no longer goods indexed by the state of nature, but are incentive-compatible contracts in the sense of Hurwicz [11]. These contracts take the form of lotteries or probability distributions on the underlying consumption possibility set of the agents. Prescott and Townsend show that the use of lotteries in these typically nonconvex,[1] but otherwise standard, environments restores the standard results in general equilibrium theory.[2] They also show that in their private information environments, equilibria exist and are optimal. The second welfare theorem, however, does not hold. Separating hyperplane arguments can only be applied to sets that involve the incentive-compatibility constraints, and this does not provide the conditions for supporting optimal allocations. A crucial feature of Prescott and Townsend's environments is that agents with characteristics which are distinct and privately observed at the time of initial trading enter the economy-wide resource constraints in a homogeneous way. This condition is satisfied if the private information or hidden action follows contracting, as is true for principal–agent problems. In environments with adverse selection (e.g., the insurance model of Rothschild and Stiglitz [29] or the signaling model of Spence [31]), even with lotteries as the commodities traded, agents do not enter the economy-wide resource constraints in a homogeneous way. As a result, fundamental problems of nonexistence and nonoptimality of equilibria remain.

Mutual gains from lotteries also arise in economies with nonconvex consumption possibility sets or nonconcave utility functions. The type of environments involved with these nonconvexities include those with indivisibility, either in the consumption goods or in certain exclusive attributes that agents may choose, such as physical locations, skills type, and family size. Equilibrium allocations are crucially dependent on the presence or absence of these lottery contracts.[3] Moreover, characterization of equilibria in economies with nonconvexities in the objects of trade

[1] By *nonconvex environments*, we mean nonconvexities in the consumption possibility set and not nonconvexities in the aggregate production possibility set. Yannelis [35] addresses the issue of nonconvexities in the utility function, which we also do not study.

[2] Cole [6] has shown that the use of lotteries is important in some private information environments even without nonconvexities.

becomes very difficult because standard theorems rely heavily on convexity properties.

Only recently has this class of models been used as a tool in economic analysis. Townsend [32], in an illuminating piece, describes how the type of models used in general competitive analysis (*Arrow–Debreu programs*, as he calls them) can be redefined to expand its scope in a variety of directions. Among them are the existence of capital, nonconvexities in the consumption possibility set, private information, limited communication and commitment, and, in a limited sense, spatial separation. Townsend sees these models as an excellent tool for studying actual economies. This line of research has proven very useful in the field of business cycles. (See, for example, Kydland and Prescott [14] and Hansen [10] for environments with lotteries that build on the work of Rogerson [28].)

The class of Phelps [23] island economies has received substantial attention in the macroeconomic literature (by, e.g., Lucas [15] and Barro [3]). In this class of environments, people move between geographic locations or firms or occupations or industries. In these economies, a key feature is the lack of perfect information in each island about the state of nature in the other islands. Analyses of such economies have not been of the classical general equilibrium variety—except for a few. A notable exception is Rogerson's [28] work with movements of people between the household and the market sector when preferences are additively separable over consumption and leisure. Another class of important models in macroeconomics is the equilibrium search economies (e.g., Lucas and Prescott [16], Mortensen [20], and Diamond [9]). These search models share some of the key features of the island economies. This literature does not use the Arrow–Debreu type of equilibrium. Instead, it uses either the sequence of markets approach or a notion of Nash equilibrium.

The purpose of this paper is to show how island economies can be analyzed using classical, competitive, general equilibrium analysis. In particular, we are interested in economies in which people, production, and information are geographically dispersed and in which people and goods can move between locations.[4] We do our analysis in the context of a prototype structure in macroeconomics, the growth model. This implies the use of neoclassical stochastic production functions that have constant

[3] In a recent paper, Shell and Wright [30] have shown that in environments with indivisibilities the equilibrium allocations when lotteries are traded are also equilibrium allocations if elements of the commodity vector are indexed by suitable chosen sunspots.

[4] Rogerson [27] has developed a model for locational decisions within the Arrow–Debreu–McKenzie paradigm. He deals with the nonconvexity problem in pretty much the same way we do. But his approach is very specific and not easily generalized to incorporate sector-specific information. He introduces a moving disutility. This feature is easy to incorporate into our model.

returns to scale. This is not a constraint. It just avoids dealing with issues about the size distribution of firms and about entry conditions. We show that for this class of economies, the standard results in general equilibrium theory—namely, the existence of equilibria and the two welfare theorems—hold.

Our work is also motivated by the absence of general equilibrium tools to assess the consequences of more timely business and employment statistics. For such an analysis, we need general equilibrium tools that can handle environments with informationally decentralized production and consumption.

Key to our analysis is the choice of a commodity space which permits the representation of the environment as a Debreu [7, 8] economy. A point in our commodity space is an infinite sequence of signed measures with elements indexed by date, location, and date–location event. A crucial feature of the environments we consider is that the aggregate production possibility set depends only on the first moments of these signed measures. This feature preserves standard production theory, with its empirically determined production functions, while permitting rich contractual arrangements between firms and households. At the same time, however, standard consumer behavior theory, which requires convexity and no private information, is not preserved. Section 2 formally describes our environment and represents it as an economy. With this commodity space, production and consumption possibility sets are convex, as are preferences.

In Section 3 a topology for the commodity space is introduced for which the utility functions are continuous and the consumption and production possibility sets closed. Under this topology, we show that the set of feasible allocations is compact and nonempty and that these results imply that the set of Pareto optima is also nonempty. Section 4 defines a competitive equilibrium and proves the welfare theorems for our economies. The first welfare theorem is immediate, given local nonsatiation. For the second welfare theorem—namely, that any Pareto optimum can be supported as a quasi-competitive equilibrium with transfers—a stronger topology is required, so that the aggregate production possibility set has a nonempty interior. The constraints defining this set involve only the first moments of the signed measures; that is why we can show that the aggregate production possibility set has a nonempty interior. With this result, the proof is an application of Theorem 2 of Debreu [7].

In Section 5 existence of a quasi-competitive equilibrium without transfers is established. The proof, unlike the welfare theorems, is not an application of an existing theorem. It adopts the proof strategy developed by Negishi [21], and applied by Bewley [4], Magill [17], and Mas-Colell [19], for existence of equilibrium with an infinite-dimensional commodity space. Under additional conditions, a cheaper point for the households is shown to exist. This ensures that a quasi-competitive equilibrium is a competitive equilibrium.

Finally, in Section 6 we address the problem of representing the price system. Theorem 1 of Prescott and Lucas [24] is extended and used to guarantee existence of a quasi-competitive equilibrium with prices being the sum of the values of the signed measures that constitute a commodity point. Theorem 6 is another representation result. It shows that equilibrium prices for our economy can be written as linear functions of the first moments of the measures. Furthermore, the coefficients of these linear functions are derived from marginal rates of substitution and transformation.

2. The Economy

Our economy is an infinite-period economy, with a continuum of agents that is taken to be of measure one. This assumption is used because it simplifies things considerably. With it, even though individual agents have nonconvex consumption possibility sets, in the aggregate the nonconvexity disappears. The economy has a finite number of agent types. Each type $i \in \{1, 2, ..., I\}$ of agents has Lebesgue measure λ^i. The economy includes L islands, and agents can go from island l at date t to island l' at date $t+1$. Agents care about leasure and consumption of the produced good. They have standard preferences over such pairs; their endowment is one unit of time per period. There is a neoclassical production function with capital and labor inputs. This function is subject to date- and location-specific technology shocks. These shocks are observed only in the island they affect; next period, however, their values will be known everywhere. Once installed, capital cannot be moved. Investment can flow from one island to another. The consumption good, however, has to be consumed at the same date–location as it is produced. The date–location information set at island l at date t is the set of all events that have happened up to period $t-1$ plus the current event at (l, t). Based on this information set, the island's allocation actions are taken. This entails not only production and consumption decisions, but also decisions to allocate labor and investment output at date $t+1$. Figure 1 captures the information set and the decisions to allocate resources to other islands.

Consumers choose probabilities over pairs of labor and consumption contingent on available information on each island.

The economy has only one firm or technology, but with a constant returns to scale technology, one and many firms are essentially the same.[5]

[5] Actually, the technology could be strictly convex, and we could use some specialized factor of production to account for the diminishing returns. Then this factor of production could be modeled in the same way as we model capital.

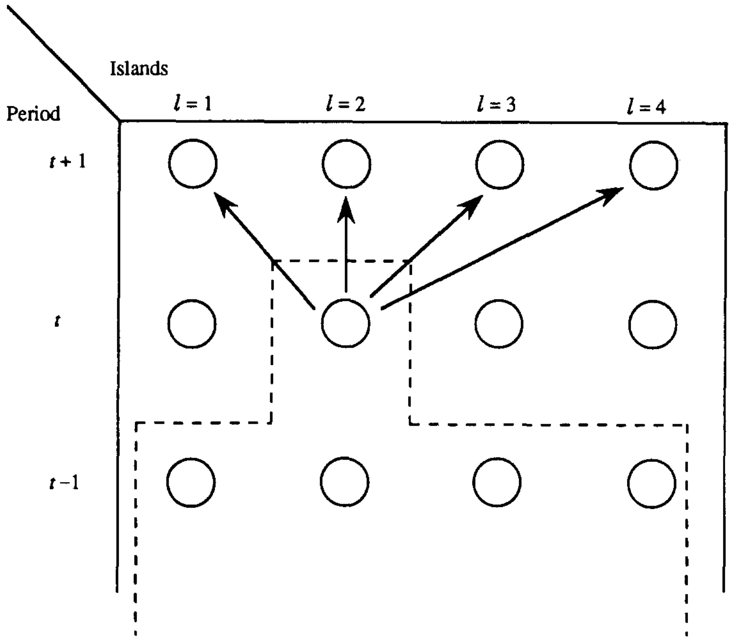

FIG. 1. The information set in island $l=2$ in period t is the set of date–location events z below the dashed line.

This firm, unlike the households, chooses quantities of labor and consumption goods contigent on the history of shocks at each date–location. The way to reconcile objects that interest the firm and the consumers is by letting the firm choose signed measures over labor–consumption pairs. In particular, it can choose an atomic one.

A further argument is needed to be able to say that an allocation is feasible when consumers choose probabilities and the firm chooses measures. In particular, something that guarantees the realization of the lotteries leads to an ex post distribution of the agents that is the same as the ex ante probabilities. In short, we need a law of large numbers. It is well known (see, for example, Judd [13]) that with a continuum of random variables (as in our case, since we have a continuum of consumers), severe measurability problems appear. However, in a recent paper, Uhlig [33], has proved a version of the L_2 law of large numbers for a continuum of random variables that are identically and independently distributed. This fits our problem, so we can equate ex ante probabilities to ex post realizations and, hence, measures of agents. Thus, we can talk about feasible and equilibrium allocations in terms of signed measures on both the consumption and the production sides of the economy.[6]

As of time zero, there is a probability assigned to shocks in each date–location. Let us call these shocks z_{lt}. They have support on a finite zet Z. Let $z_t = \{z_{1t}, ..., z_{Lt}\}$, $z^t = \{z_1, ..., z_t\}$, and $h_{lt} = \{z^{t-1}, z_{lt}\}$. The last element is the information available at (l, t). By Kolmogorov's extension theorem, there exists a probability space $(\Omega, \sigma(\Omega), \pi)$ with the property that for each (l, t) and possible history h_{lt}, $\pi(h_{lt})$ is the probability of history h_{lt} happening. (We write $\pi(h_{lt})$ instead of the cumbersome notation $\pi(\{\omega\colon \text{proj}_{lt}(\omega) = h_{lt}\})$.)

We can identify the set Ω as the set of all possible elements of $\prod_{t=0}^{\infty} Z^L$. Let $\sigma(\Omega)$ be the Borel σ-field generated by Ω.

In the same way, let $\sigma(z^t)$ and $\sigma(h_{lt})$ be the smallest σ-fields on Ω that make z^t and h_{lt} measurable. Clearly, $\{\sigma(z^t)\}_{t=0}^{\infty}$ is an increasing family of σ-fields, a filtration on the probability space $(\Omega, \sigma(\Omega), \pi)$.

Properties of these σ-fields are

$$\sigma(z^{t-1}) \subset \sigma(h_{lt}) \subset \sigma(z^t) \qquad \text{and} \qquad \sigma(h_{lt}) \subset \sigma(h_{l', t+1}),$$

for all (l, l', t). We can think of $\sigma(h_{lt})$ as the information available at (l, t).

Now for each (l, t) the set H_{lt} of possible histories h_{lt} is a finite set, and $H \equiv \bigcup_{lt} H_{lt}$ is a countable set.

There is an underlying consumption set of the agent. It is C, a closed subset of $\mathbf{R}^2$. Moreover, $C \subset \{[-1, 0] \times [0, \bar{c}]\}$, where the first component is the negative of the length of time within each period devoted to work. Hence, $(1 + c_1)$ is an agent's leisure.[7] The second component is the consumption good, where $\bar{c}$ is an upper bound for consumption.

Let $\mathscr{M}$ be the set of finite signed measures defined on the Borel sets of C, the underlying consumption possibility set. Let $S_{lt}(\Omega, \sigma(h_{lt}), \mathscr{M})$ be the space of functions $\Omega \to \mathscr{M}$ such that they are measurable with respect to $\sigma(h_{lt})$.

The commodity space S is

$$S \equiv \{[S_{lt}(\Omega, \sigma(h_{lt}), \mathscr{M}]_{l=1}^{L}\}_{t=0}^{\infty}.$$

[6] There is another way of handling this problem of the relation between ex ante probabilities and ex post measures. It is by generating an artificial stochastic process for all islands, periods, and histories on which each agent's lotteries are contingent. This would not only greatly increase the dimensionality of the problem, but would also oblige us to keep track of the agents' names. We think the use of Uhlig's [33] theorem makes things much easier.

[7] Given the way the commodity space is constructed, we do not require the underlying consumption possibility set to be convex. This is important because indivisibilities and specialization, which often are essential parts of applied general equilibrium analysis, result in this set being nonconvex.

Note that S can be characterized as sequences of signed measures, each element indexed by a specific date–location history.

The consumption possibility set X is

$$X = \{x \in S_+ : x \text{ satisfies conditions (1)–(4)}\},$$

where the $x_{lt}(h_{lt}, A)$ are the probabilities of being at island l at date t and consuming in Borel set $A \subset C$ given h_{lt}. The four conditions are as follows:

$$\text{The } x_{lt}(h_{lt}, C) \text{ are } \sigma(z^{t-1})\text{-measurable.} \tag{1}$$

This condition requires that the probability of being at (l, t) is not contingent on the actual shock z_{lt}, since the decision to be at (l, t) precedes its realization.

$$\sum_l x_{lt}(h_{lt}, C) = 1 \qquad \text{for all } t, \text{ all } z^t. \tag{2}$$

(Note that z^t defines h_{lt} for all l.) This ensures that with probability one a person is always in one and only one of the locations.

There exist functions $b_{l,t+1,l'}$: $\Omega \to \mathbf{R}_+$, measurable with respect to $\sigma(h_{l't})$, for all (l, l', t) such that

$$\sum_l b_{l,t+1,l'}(h_{l't}) = x_{l't}(h_{l't}, C) \qquad \text{for all } (l', t, h_{l't}). \tag{3}$$

The $b_{l,t+1,l'}$ functions are the joint probabilities of being at (l', t) and $(l, t+1)$ given $(h_{l't})$. Thus, Condition (3) is a consistency property for the probabilities of being in each location every period.

Condition (4) is another consistency property for the probabilities of being at the different locations:

$$\sum_{l'} b_{l,t+1,l'}(h_{l't}) = x_{l,t+1}(h_{l,t+1}, C) \qquad \text{for all } (l', t, h_{l,t+1}). \tag{4}$$

Note that the arguments of the functions on the two sides of (4) are different. This reflects the fact that the decisions to move and to choose labor–consumption pairs are based on different information sets. Finally, from Conditions (2) and (3) or (2) and (4),

$$\sum_{l'} \sum_l b_{l,t+1,l'}(h_{l't}) = 1 \qquad \text{for all } t, z^t.$$

Preferences of a type i agent are discounted, expected utility,

$$E_0 \left\{ \sum_{t=0}^{\infty} \beta^t U_i(c_t) \right\},$$

where $0 < \beta < 1$ and $U^i: C \to R^+$ is bounded and strictly increasing.

The aggregate production possibility set Y is

$$Y = \{y \in S\text{: there exist measurable } k_{lt}: Z^{t-1} \to R_+ \text{ and measurable } a_{l,t+1,l'}: H_{lt} \to R_+ \text{ for all } (l, l', t) \text{ satisfying Conditions (5), (6), and (7)}\}.$$

Here, $a_{l,t+1,l'}(h_{lt})$ is the amount of the investment good produced at (l, t) contingent on event h_{lt} and shipped to location l' for use at time $t+1$, while $k_{lt}(z^{t-1})$ is the beginning-of-period capital stock at (l, t) given history z^{t-1}. Condition (5) is

$$k_{lt}(z^{t-1}) = k_{l,t-1}(z^{t-2}) + \sum_{l'} a_{ltl'}(h_{l',t-1}) \tag{5}$$

for all $t \geqslant 1$, l, and all z^{t-1}. This is just the law of motion for the capital stock given the initial capital stock k_{l0}. Condition (6) is

$$\int_C c_2 y_{lt}(h_{lt}, dc) + \sum_{l'} a_{l',t+1,l}(h_{lt}) \leqslant z_{lt} f\left[k_{lt}(z^{t-1}), -\int_C c_1\, y_{lt}(h_{lt}, dc)\right] \tag{6}$$

for all (l, t, h_{lt}). Here, the function f is a constant returns to scale neoclassical production function which has as its first argument the capital input and its second the labor input. Note that c_1 is the first component of C and c_2 the second. Consequently, $-\int c_1\, y_{lt}(h_{lt}, dc)$ is the event-contingent labor input at (l, t) while the left side of (6) is the event-contingent production of the consumption good, also at (l, t). We assume $f(k, 1)$ is bounded. This assumption ensures a uniform bound for output and, therefore, for consumption. Condition (7) is

$$-\int_C c_1\, y_{lt}(h_{lt}, dc) \geqslant 0 \qquad \text{for all } (l, t, h_{lt}). \tag{7}$$

This condition guarantees that labor cannot be an output of the production process.

All agents of the same type choose the same commodity point. This is not a restriction since there are lotteries and the law of large numbers guarantees that the ex post distribution is the same as the ex ante distribution for each of the I classes of consumers.

An allocation $[(x^i), y]$ is *feasible* if $x^i \in X$ for all i, if $y \in Y$, and if for all (l, t, h_{lt}),

$$\sum_i \lambda^i x^i_{lt}(h_{lt}, A) = y_{lt}(h_{lt}, A) \qquad \text{for all Borel sets } A \text{ in } C.$$

This is the standard requirement that objects on the production side must equal those on the consumption side.

3. Some Preliminary Mathematical Results

Our sets X and Y are the projections of sets onto S. Formally, let R be

$$R \equiv \{r = \{r_h\}_{h=1}^{\infty}: r_h \in R^{L(L+1)}, \sup_l |r_{lh}| < \bar{r}_h \text{ for all } h\}.$$

Define T_1, T_2 as

$$T_1 \equiv \{(s, b, 0) \in S_+ \times R: (s, b, 0) \text{ satisfies conditions (1)–(4)}\}$$

$$T_2 \equiv \{(s, a, k) \in S \times R: (s, a, k) \text{ satisfies conditions (5)–(7)}\}.$$

Then

$$X = \text{proj}(T_1 \text{ onto } S)$$

$$Y = \text{proj}(T_2 \text{ onto } S).$$

The set T_i are convex and closed in the product topology over sequences with components that have the weak* topology. The reason is that these sets are defined as the intersection of a countable number of closed and convex constraints.

In this topology, the set R is compact. This ensures that the projections of closed subsets of $S \times R$ onto S are closed. Projections of convex sets are convex. All this can be summarized in the following lemma.

Lemma 1. *The sets X and Y are convex and are closed in the product topology with the weak* topology for components.*

Regarding preferences, the following result holds:

Lemma 2. *Preferences can be represented by $u^i: X \to \mathbf{R}_+$,*

$$u^i(x) = E_0 \left\{ \sum_{t=0}^{\infty} \beta^t \int_C U^i(c) \left[\sum_l x_{lt}(h_{lt}, dc) \right] \right\}.$$

Furthermore, in the product topology generated by the weak topology on $\mathcal{M}(C)$, u^i is a continuous and linear function.*

Proof. See Appendix.

Lemma 3. *The set of feasible allocations is compact.*

Proof. See Appendix.

Consequently, for any $\mu \in \mathbf{R}_+^I$ with $\sum_i \mu^i = 1$, the problem

$$\max_{[(x^i), y]} \sum_i \lambda^i \mu^i u^i(x^i)$$

subject to feasibility has a solution given Lemmas 2 and 3 and the existence of a feasible allocation. Any solution is necessarily a Pareto-optimal allocation if $\mu^i > 0$ for all i.

4. Welfare Theorems

A competitive equilibrium is a feasible allocation $[(\bar{x}^i), \bar{y}]$ together with a price system v (i.e., a nontrivial linear functional on S) for which these conditions hold:

(i) For all i, $x \in X$ and $u^i(x) > u^i(\bar{x}^i)$ imply that $v(x) > v(\bar{x}^i)$.

(ii) $y \in Y$ implies that $v(y) \leqslant v(\bar{y})$.

Condition (i) is utility maximization subject to a budget constraint while Condition (ii) is profit maximization subject to a technology constraint.

Theorem 1 (First Welfare Theorem). *If the allocation* $[(\bar{x}^i), \bar{y}]$, *for* $i = 1, \ldots, I$, *together with the price functional* v, *is a competitive equilibrium, then it is a Pareto-optimal allocation.*

Proof. All we need to show is local nonsatiation for the allocation. The point x which places mass only on the point set $\{(0, \bar{c})\}$ for all (l, t, h_{lt}) is the only satiation point. Given our assumptions, no allocation that places mass one on $\bar{c}$ for any type for any (l, t, h_{lt}) is feasible. Consequently, feasibility rules out satiation for any type. ∎

For technical reasons, the underlying consumption set C was constrained to be bounded. Our theory, however, can also be applied to some environments in which this set is not bounded. Whenever the utility possibility frontier for the n types is not reduced by introducing some sufficiently large bound on consumption, the theory is applicable. For example, if $C = [-1, 0] \times [0, \infty)$ and for fixed c_1 the U^i are strictly concave in c_2, then any

$$\bar{c} > \sup_{z, k, i} \frac{zf(k, 1)}{\lambda^i} < \infty$$

is an appropriate bound.

To see why, note that consumption $\bar{c}$ is larger than the amount that can be feasibly consumed by all type i people. Still, agents of type i could consume an x that at some (l, t, h_{lt}) puts some positive probability on an amount larger than $\bar{c}$ and, hence, necessarily also puts positive probability on an amount smaller than $\bar{c}$. But given the fact that agents are risk averse—i.e., the U^i are strictly concave in c_2—that allocation will not be

Pareto optimal since it is dominated by another that puts probability one on the first moment of the former. This new allocation is feasible; first moments do not change. All economies with a $\bar{c}$ that satisfies the above property will share Pareto-optimal allocations and, as Theorem 2 shows, competitive equilibria.

Debreu ([7], Theorem 2) establishes that if S is a linear topological space, then Pareto-optimal allocations $[(\bar{x}^i), \bar{y}]$ for which no $\bar{x}^i$ is a satiation point can be supported as a quasi-competitive equilibrium if the following five conditions are satisfied:

(I) X is a convex set.

(II) For x', $x'' \in X$, and for all i, $u^i(x') < u^i(x'')$ implies that $u^i(x') < u^i(x^\alpha)$, where $x^\alpha = \alpha x' + (1-\alpha) x''$ for $\alpha \in (0, 1)$.

(III) For all x, x', $x'' \in X$ and for all $\alpha \in [0, 1]$, the set $\{\alpha \in [0, 1] : u^i(x^\alpha) \leqslant u^i(x)\}$ is closed, where x^α is as before.

(IV) Y is a convex set.

(V) Y has an interior point.

THEOREM 2 (SECOND WELFARE THEOREM). *Any Pareto-optimal allocation $[(\bar{x}^i), \bar{y}]$ can be supported as a quasi-competitive equilibrium with transfers (QET); that is, there exists a nontrivial, continuous, linear functional v such that*

(i) *For all i, $x \in X$ and $u^i(x) \geqslant u^i(\bar{x}^i)$ imply that $v(x) \geqslant v(\bar{x}^i)$.*

(ii) *$y \in Y$ implies that $v(y) \leqslant v(\bar{y})$.*

Proof. With the product topology used to establish the existence of optima, Y has an empty interior. Hence, another topology is needed for application of Debreu's Theorem 2. The topology used is the one induced by the following norm:

$$\|s\| = \sup_{h \in H} \{\|s_h\|_M\} = \sup_{h \in H} \left\{ \sup_{g \in C(C)\, \|g\|_\infty = 1} \left| \int_C g \, ds_h \right| \right\}.$$

Note that the term in braces is the usual norm for signed measures, with $C(C)$ being the space of continuous bounded functions on C. The topology induced by this norm is the topology of uniform convergence of sequences of signed measures.

Conditions (I) and (IV) are Lemma 1. Condition (II) is the immediate consequence of the linearity of u^i as established in Lemma 2. Continuity of u^i with respect to a weaker topology, as established in Lemma 2, implies continuity with respect to this stronger topology. Continuity of the u^i is a stronger condition than Condition (III).

To prove condition (V), we pick a point y^0 and show that it is in the interior of Y.[8] Let

$$y_h^0(A) = \begin{cases} 1 & \text{if } (-1/2, 0) \in A \\ 0 & \text{otherwise.} \end{cases}$$

To show that $y^0 \in Y$, let a^0 be identically zero and k^0 be such that $k_{lt}^0(h_{lt}) = k_{l0}$ for all h. Since $(y^0, k^0, a^0) \in T_2$, $y^0 \in Y$.

Let

$$N_1 \equiv \left\{ s_h \in \mathcal{M}(C): \left| \int c_1 \, ds_h - \int c_1 \, dy_h^0 \right| < 1/4 \right\}$$

$$N_2 \equiv \left\{ s_h \in \mathcal{M}(C): \left| \int c_2 \, ds_h - \int c_2 \, dy_h^0 \right| < \varepsilon \right\},$$

where

$$\varepsilon \equiv \inf_{l, z} z f(k_{l0}, 1/4).$$

Sets N_1 and N_2 are weak* neighborhoods of y_h^0. Since the topology induced by $\|\cdot\|_M$ is a stronger one, there exists an $\varepsilon' > 0$ such that

$$O_h(y_h^0) \equiv \{ s_h \in \mathcal{M}(C): \|s_h - y_h^0\|_M < \varepsilon' \} \subset N_1 \cap N_2.$$

This ε' is the same for all h. We will show that the open set

$$O(y^0) \equiv \{ s \in S: \|s - y^0\| < \varepsilon' \}$$

is contained in Y.

If $s \in O(y^0)$, then for all h, $s_h \in O_h(y_h^0) \subset N_1 \cap N_2$. Thus,

$$\int -c_1 \, ds_h \in (1/4, 3/4)$$

and

$$\left| \int c_2 \, ds_h \right| < \varepsilon.$$

This, in turn, implies that (s, k^0, a^0) satisfies (5)–(7) and, hence, $s \in Y$. This completes the proof. ▌

[8] It is in this step of the proof (as well as in a similar argument in the proof of Theorem 5) that the assumption of nondepreciation of capital becomes handy. Its exclusion would make the statement of point y^0 much messier without adding any interesting feature.

Comment. To support the optimum, the transfer to a type i agent must be

$$\psi^i = v(\bar{x}^i) - \theta^i v(\bar{y}),$$

where θ^i is the share of the firm owned by type i. (Note that $\lambda \cdot \theta = 1$.)

5. Existence of Equilibrium

Theorem 2 guarantees the existence of a quasi-competitive equilibrium with transfer payments (again, a QET). In the next theorem, we show that a quasi-competitive equilibrium with zero transfers also exists. The argument proceeds by constructing a correspondence $\chi: M \to M$, where $M \subset R^I$ is compact and convex and χ is convex valued and upper hemicontinuous (uhc), and then applying Kakutani's fixed-point theorem. The argument is in the spirit of the work of Bewley [4], Magill [17], and Mas-Colell [19], who use the Negishi [21] approach.[9]

Theorem 3 (Existence of Quasi-Competitive Equilibrium). *For the class of economies studied, there exists a quasi equilibrium, i.e., a feasible allocation* $[(\bar{x}^i), \bar{y}]$ *and a price system* v *such that*

$$u^i(x) \geqslant u^i(\bar{x}^i) \text{ implies that } v(x) \geqslant \theta^i v(\bar{y}) \qquad \text{for all } i$$

and

$$y \in Y \text{ implies that } v(y) \leqslant v(\bar{y}).$$

Proof. The correspondence is defined by mapping points in the simplex into transfers. This requires, as an intermediate step, the finding of allocations and prices that constitute a QET associated with each point in the simplex. Existence of these allocations and prices and convexity of the set of prices is shown in Step 1. Step 2 defines the transfer correspondence and shows it is convex. Step 3 proves it is also uhc. Since transfers do not typically lie in the simplex, in Step 4 the domain of the correspondence is extended, while preserving its properties, to a set that includes its image. The correspondence is also transformed in order to obtain quasi equilibrium without transfers associated with its fixed points. Here Kakutani's theorem applies.

Step 1. We start by constructing a continuous function mapping the simplex into points on the utility possibility frontier. These points are

[9] There are other strategies for proving this type of result. Jones [12] uses certain properties of asymptotic cones to show existence, and Bewley [4] uses approximation of economies in the right topology. The choice of our approach was influenced by its strong links to the second welfare theorem, which plays a central role in our analysis.

shown to be associated with a QET. Then properties of these QET are characterized.

Let α: $\Delta^I \to \mathbf{R}$, where $\alpha(\mu) = \max\{\alpha \in \mathbf{R}: \alpha\mu \in U\}$. Here U denotes the utility possibility set. Utility functions have been normalized so that $u^i(\underset{\wedge}{x}) = 0$, $u^i(\hat{x}) < 1$, where $\underset{\wedge}{x}$ is the worst point (all measure on $\{-1, 0\}$) and $\hat{x}$ is the best point (all measure on $\{0, \bar{c}\}$).

Note that associated with a point $u \in U$ there is a feasible allocation. Convexity of X, Y, and preferences guarantee that U is convex. The facts that utility functions are continuous and that the set of feasible allocations is weak* compact guarantee that U is closed. All this implies that $\alpha(\mu)$ is well defined and continuous.

Vector $u(\mu) = \alpha(\mu)\,\mu$ is a point on the utility possibility frontier. Therefore, allocations that generate it are Pareto optima. Let $a(\mu)$ be the set of Pareto optima that yield utility $u(\mu)$. By Theorem 2, there is a linear function v that supports every Pareto-optimal allocation as a QET. Let $V(\mu) \equiv \{v \in S^*: v(\hat{x}) = 1, \exists[(x^i), y] \in a(\mu)$ subject to the condition that $\{[(x^i), y], v\}$ is a QET$\}$. First note that if v supports $[(x^i), y] \in a(\mu)$ as a QET and if $[(x'^i), y'] \in a(\mu)$, then v also supports $[(x'^i), y']$. To see this, note that x^i and x'^i yield the same utility and that both y and y' are feasible, so $v(x'^i) \geqslant v(x^i)$ and $v(y') \leqslant v(y)$. Aggregation yields the result that they have the same value, proving that $\{[(x'^i), y'], v\}$ is also a QET. Furthermore, this last fact implies that the transfers for both QETs are the same. The final step is to show that $v(\hat{x}) > 0$, for then the scaled v with value one at $\hat{x}$ belongs to $V(\mu)$.

By the same argument used in proof of Theorem 2, $\underset{\wedge}{x}$ can be shown to be an interior point of Y, guaranteeing that $v(\underset{\wedge}{x}) < 0$. We first show that $v(y) \geqslant 0$. Let y_0 put measure one on the point $\{(0, 0)\}$. Note that $\gamma y_0 \in Y$ for all real γ. Existence of a solution to the firm's problem, therefore, requires that $v(y_0) = 0$. Thus, $v(y) \geqslant 0$. Next we show that if $v(\hat{x}) = 0$, then $v(x_i) < 0$. First, all convex combinations of $\hat{x}$ and $\underset{\wedge}{x}$ have negative value. Pick one—say, $x^{\alpha i}$—for which $u_i(x^{\alpha i}) = u_i(x^i)$. Cost minimization implies that $v(x^i) \leqslant v(x^{\alpha i}) < 0$. This together with feasibility implies that $v(y) < 0$. But we have shown that profit maximization implies that $v(y) \geqslant 0$. This proves $V(\mu)$ is nonempty. It is trivially convex.

Step 2. Here we define a transfers correspondence and show that it is nonempty and convex.

Let ϕ: $\Delta^I \to \mathbf{R}^I$ be

$$\phi(\mu) = \left\{z \in \mathbf{R}^I: \sum_i z^i = 0, \text{ there exists } \{[(x^i), y], v\} \in a(\mu) \times V(\mu) \right.$$

$$\left. \text{such that } z^i = \lambda^i v(x^i) - \lambda^i \theta^i v(y)\right\}.$$

Clearly, z^i/λ^i is an agent i transfer in a QET $\{[(x^i, y], v\}$. Convexity of $V(\mu)$ guarantees convexity of $\phi(\mu)$. Note that a zero of ϕ is a quasi equilibrium without transfers.

Step 3. This step is to show that ϕ is uhc; i.e., if $z_n \to z$, $\mu_n \to \mu$, and $z_n \in \phi(\mu_n)$ for all n, then $z \in \phi(\mu)$.

To prove that ϕ is uhc, we construct a compact set K such that for all μ, $V(\mu) \subset K$. Then we take a weak* limit of v_n along with an allocation in $a(\mu)$, and show that it is a QET and that the transfers implied are precisely z.

Recall from Step 1 that there exists an $\varepsilon > 0$ such that for all μ, $v \in V(\mu)$, $(1-\varepsilon)\, v(\hat{x}) + \varepsilon v(\underset{\wedge}{x}) \geqslant 0$. This implies that $v(\underset{\wedge}{x}) \geqslant 1-(1/\varepsilon)$. An open set $A \subset Y$ with the property that $v(s) > v(\underset{\wedge}{x})$ for all $s \in A$ can be constructed in the same way as in the proof of Theorem 2. Next we define $A(0)$, a neighborhood of zero, as

$$A(0) \equiv \{s \in S\colon s = (s'-y)/(2\gamma),\ \text{all } s' \in A;\ \gamma = \max[1, 1/\varepsilon - 1],\ y \in A\}.$$

It follows that, for any μ, $v \in V(\mu)$, $s \in A(0)$, $|v(s)| \leqslant 1$. Define $K \equiv \{v \in S^*\colon |v(s)| \leqslant 1,\ s \in A(0)\}$. The Banach-Anaoghu theorem says that K is weak* compact. By construction for any μ, $V(\mu) \subset K$.

Consider now the sequence z_n associated with a QET $\{[(x_n^i), y_n], v_n\}$. Since K is compact and S^* complete, there exists a weak* limit v of v_n, but nothing can be said about the convergence of the sequence of allocations. We proceed in an indirect way. Pick $[(\bar{x}_i), \bar{y}] \in a(\mu)$. We will show that $\{[(\bar{x}^i), \bar{y}], v\}$ is a QET with transfers z.

Pick sequences $w_m^i \to \bar{x}^i$, with the property that $u^i(w_m^i) > u^i(\bar{x}^i)$ for all m. Note that $u^i(\mu_n) \to u^i(\mu)$ and, hence, that $u^i(x_n^i) \to u^i(\bar{x}^i)$. This implies that for all m, there exists an $N(m)$ such that for $n > N(m)$, $u^i(w_m^i) \geqslant u^i(x_n^i)$, which in turn implies that $v_n(w_m^i) \geqslant v_n(x_n^i)$. By definition of ϕ, $\lambda^i v_n(x_n^i) = z_n^i + \lambda^i \theta^i v_n(y_n)$, and by profit maximization, for any $y \in Y$, $\lambda^i v_n(x_n^i) \geqslant z_n^i + \lambda^i \theta^i v_n(y)$.

These last three facts imply that for $n > N(m)$, $\lambda^i v_n(w_m^i) \geqslant z_n^i + \lambda^i \theta^i v(y)$. Taking the n limit, we get $v(w_m^i) \geqslant z^i + \lambda^i \theta^i v(y)$. Taking the m limit, we obtain $v(\bar{x}^i) \geqslant z^i + \lambda^i \theta^i v(y)$. Note that by aggregation and the fact that $\bar{y} \in Y$, $v(\bar{x}^i) = z^i + \lambda^i \theta^i v(\bar{y})$. This proves that $z \in \phi(\mu)$, since $\{[(\bar{x}^i), \bar{y}], v\}$ is a QET that generates transfer z.

Step 4. First, we transform ϕ so that a fixed point, rather than a zero, implies a quasi equilibrium without transfers. The following transformation χ has this property: $\chi(\mu) = \mu - \phi(\mu)$. Typically, χ will take values outside the simplex. Consider the set $M = \{s \in \mathbf{R}^I\colon \Sigma_i s^i = 1,\ s^i \geqslant \max_j [1/\lambda_j, (1/\varepsilon - 1)/\lambda_j]\}$. It is straightforward to show that $\chi(\mu) \subset M$ for all μ in the unit simplex. The next step is to extend χ to M is such a way that the extension lies in M and the extended χ is uhc and convex valued. For μ outside the simplex, $\chi(\mu)=\chi(\mu_+)$, where $\mu'^i_+ = \max(0, \mu^i)/\Sigma_j \max(0, \mu^j)$. It is trivial to see that this extension has the required properties for the applications of Kakutani's fixed-point theorem. This completes the proof. ▌

We still have not shown the existence of a competitive equilibrium. We will show its existence by applying the well-known result (of Debreu [7]) that if every agent's budget set includes a cheaper point, then the quasi-competitive equilibrium is a competitive one. In the following theorem, we give sufficient conditions on the Pareto optima to guarantee that every agent has a cheaper point. These conditions are that agents do not consume the least desirable point in the underlying consumption set C for all (l, t, h_{lt}). The argument establishes a contradiction between profit maximization and lack of existence of a cheaper point.

Define X^ε as

$$X^\varepsilon = \left\{ x \in X \colon \int (1 + c_1)\, dx_h < \varepsilon \text{ and } \int c_2\, dx_h < \varepsilon \text{ for some } h \in H \right\}.$$

THEOREM 4 (EXISTENCE OF COMPETITIVE EQUILIBRIUM). *If allocation $[(\bar{x}_i), \bar{y}]$ and nontrivial, continuous, linear functional v is a quasi equilibrium such that there exists $\varepsilon > 0$ with the property that for all i, $\bar{x} \notin X^\varepsilon$, then $\{[(\bar{x}^i), \bar{y}], v\}$ is a competitive equilibrium.*

Proof. Let $x_h^*(A) = 1/L$ if $(-1, 0) \in A$ and zero otherwise for all h. $x^*(A) \in X$. Consider the allocation $[(x^i), y^*]$, where $x^i = x^*$ for some i, $x^j = \bar{x}^j$ for $j \neq i$, and $y^* = \Sigma_{j \neq i} \lambda^j \bar{x}^j + \lambda^i x^*$. Since $\bar{x}^i \notin X^\varepsilon$ for some $\varepsilon > 0$, type i agents are consuming less per capita and supplying more labor per capita for x^* than for $\bar{x}^i$ for all h. Consequently, $y^* \in Y$. Profit maximization implies that

$$v(y^*) \leqslant v(\bar{y}).$$

Linearity of v implies that

$$v(y^*) = \lambda^i v(x^*) + \sum_{j \neq i} \lambda^j v(\bar{x}^j).$$

Hence $v(x^*) \leqslant v(\bar{x}^i)$.

Note that $v(x^*) = v(\bar{x}^i)$ implies that $v(y^*) = v(\bar{y})$, which will be shown to be inconsistent with profit maximization. The nature of the argument is to show that y^* is an interior point of Y. If $v(y^*) = v(\bar{y})$, since v is nontrivial, there would be some point y in every neighborhood of y^* with the property that $v(y) > v(y^*)$.

A neighborhood of y^*, $O(y^*) \subset Y$ is constructed as follows.

Two weak* neighborhoods of y_h^* are those associated with continuous functions c_1 and c_2 and bounds $\varepsilon\lambda^i/2L$. By the same reasoning as in the proof of Condition (V) of Theorem 3, there exists an open set $O(y^*)$ in the topology induced by the norm on S such that $y \in O(y^*)$ implies that for all h,

$$\left| \int c_1 \, dy_h - \int c_1 \, dy_h^* \right| < \frac{\lambda^i \varepsilon}{2L}$$

and

$$\left| \int c_2 \, dy_h - \int c_2 \, dy_h^* \right| < \frac{\lambda^i \varepsilon}{2L}$$

Since $\int c_1 \, d\bar{y}_h - \int c_1 y_h^* > \lambda^i \varepsilon/L$ and $\int c_2 \, d\bar{y}_h - \int c_2 \, dy_h^* > \lambda^i \varepsilon/L$, we know that for all h $\int c_1 \, d\bar{y}_h - \int c_1 \, dy_h > \lambda^i \varepsilon/2L$ and $\int c_2 \, d\bar{y}_h - \int c_2 \, dy_h > \lambda^i \varepsilon/2L$.

Note that $(y, \bar{k}, \bar{a}) \in T_2$, where $\bar{k}$, $\bar{a}$ are the functions associated with the quasi equilibrium allocation $\bar{y}$. Therefore, $O(y^*) \subset Y$. Hence, $v(x^*) < v(\bar{x}^i)$ for all i. This concludes the proof. ∎

6. Price Representation

The equilibrium price v lies in the dual of the commodity space S, a space difficult to characterize. In Theorem 6 we show that there exists a quasi equilibrium price functional p that lies in the predual, i.e., in the space of sequences of continuous bounded functions on C which have summable norms. This price functional p has a dot-product representation which agrees with our intuitions of what prices should be; namely, they are rooted in marginal conditions.

Prescott and Lucas [24] have developed sufficient conditions for existence of price systems with dot-product representations for commodity spaces with elements that are sequences of members of normed linear spaces.

Prescott and Lucas require that if a point is feasible, so is any other point for which each of the first T components are equal and for which the rest are zero. In addition, Prescott and Lucas require that agents discount consumption in distant states. Since an allocation in our economy includes a set of probabilities for consumers, truncating with the zero measure is not feasible, so we cannot apply the Prescott and Lucas result. In the following result, we generalize their Theorem 1 to permit truncation with other than the zero element. For this theorem, the aggregate endowment, ω, is not assumed to be zero, and there are a finite number of technologies indexed by j.

Let S be the linear space $\|s\| \equiv \sup_h \|s_h\|_H$. For any $s \in S$, let $s_{\leqslant T}$ be such that its first T components are those of s and the rest are zero. Let $s_{>T} \equiv s - s_{\leqslant T}$.

THEOREM 5 (DOT-PRODUCT PRICE REPRESENTATION WITH NONZERO TRUNCATION). *Suppose an allocation* $[(\bar{x}^i), (\bar{y}^j)]$ *and a nontrivial, continuous, linear functional* v *are a competitive equilibrium for an economy for which no* $\bar{x}^i$ *is a satiation point. Suppose, in addition to Conditions* (I) *and* (II), *that there exist* $\hat{x}^i \in X^i$ *for all* i *and* $\hat{y}^j \in Y^j$ *for all* j *such that*

(VI) *For all* $x^i \in X^i$, *all* T, $x^i_T \equiv x^i_{\leqslant T} + \hat{x}^i_{>T} \in X^i$; *for all* $y^j \in Y^j$, *all* T, $y^j_T \equiv y^j_{\leqslant T} + \hat{y}^j_{>T} \in Y^j$.

(VII) *If agent* i *strictly prefers* x^i *to* $x^{i'}$, *then there exists* T' *such that for* $T > T'$, x^i_T *is strictly preferred to* $x^{i'}$.

(VIII) $\sum_i \lambda^i \hat{x}^i = \sum_j \hat{y}^j + \omega$.

Then

$$q(s) = \lim_{T \to \infty} v(s_{\leqslant T})$$

along with $[(\bar{x}^i), (\bar{y}^j)]$, *is a quasi equilibrium for this economy.*

Proof. See Appendix.

The final theorem uses this result to establish existence of a specific type of representation of a price system. First, it allows for a price to be represented as a dot product; i.e., the value of a commodity point is the sum of the values of its date–location–event components. Second, within each date–location–event component, the value depends only on the first moments of the signed measures with respect to both the consumption good and time, in a way that their relative values coincide with the marginal rates of substitution and transformation.

THEOREM 6 (PRICE REPRESENTATION). *Suppose an allocation* $[(\bar{x}^i), \bar{y}]$ *and a linear functional* v *are a competitive equilibrium. Then there exists a continuous, linear functional* p *of the form*

$$p(s) = \sum_h \int_C (p_{1h} c_1 + p_{2h} c_2)\, ds_h$$

such that $[(\bar{x}^i), \bar{y}]$ *together with* p *is a quasi-competitive equilibrium.*

Proof. Note that Assumption (VI) holds for our economy for

$\hat{x}^i_h = \{[(-1, 0)]\} = 1/L$ for all i and all h, and $\hat{y}_h\{[(-1, 0)]\} = 1/L$ for all h. Discounting takes care of Assumption (VII), and $\hat{x} = \hat{y}$ suffices for Assumption (VIII). Then Theorem 6 guarantees that a q exists and that it together with $[(\bar{x}^i), \bar{y}]$ is a quasi equilibrium.

A result used in this proof and proven in the Appendix is

LEMMA 4. *Suppose v and p are continuous, nontrivial, linear functionals on topological linear space S. Suppose $\bar{s} \in \operatorname{argmax} v(s)$ subject to $p(s) \leqslant p(\bar{s})$. Then there exists an $\alpha > 0$ such that $\alpha p(s) = v(s)$ for all $s \in S$.*

Define for all h the set

$$Y_h \equiv \{ y \in Y: y_{h'} = \bar{y}_{h'} \text{ for all } h' \neq h \}.$$

Consider the following program:

$$\max_y q(y) \text{ subject to } y \in Y_h.$$

Element $\bar{y}$ solves it.

Let $s^{(h)} \equiv s_{\leqslant h} - s_{\leqslant h-1}$. Then $q(s) = \sum_h v(s^{(h)})$. Define $v_h(s_h) = v(s^{(h)})$. Since for no h do agents put mass on the most preferred point in C, utility maximization guarantees that v_h is nontrivial. Consider also the program

$$\max_{s_h} v_h(s_h)$$

subject to

$$\int_C (c_2 + w_h c_1)\, ds_h \geqslant \int_C (c_2 + w_h c_1)\, d\bar{y}_h,$$

where $w_h = z_h f_2[\bar{k}_h, -\int c_1\, d\bar{y}_h]$. Suppose $\bar{y}_h$ does not solve it. Then there exists an s_h such that $v_h(s_h) > v_h(\bar{y}_h)$ and

$$\int_C (c_2 + w_h c_1)\, ds_h \leqslant \int_C (c_2 + w_h c_1)\, d\bar{y}_h.$$

Then there exists a $\delta \in (0, 1)$ such that $v_h(\delta s_h) > v_h(\bar{y}_h)$ and

$$\int_C (c_2 + w_h c_1)\, d(\delta s_h) < \int_C (c_2 + w_h c_1)\, d\bar{y}_h.$$

Given the value of w_h and the properties of the production function, there exists a $\gamma \in (0, 1)$ such that, for

$$s^\gamma_h = \gamma \delta s_h + (1 - \gamma)\, \bar{y}_h,$$

$v_h(s_h^\gamma) > v_h(\bar{y}_h)$. Furthermore, the set $\{ y \in S: y_h^\gamma = s_h^\gamma \text{ and } y_{h'} = \bar{y}_{h'} \text{ for } h' \neq h \}$ is contained in Y_h. But this contradicts the fact that $\bar{y}$ solves the previous program. Hence, $\bar{y}_h$ solves this one.

Since v_h and the constraint are both nontrivial, continuous, linear functionals on a topological linear space for all h, Lemma 4 implies that there exists $\alpha_h > 0$ such that

$$v_h(s_h) = \alpha_h \int_C (c_2 + w_h c_1)\, ds_h.$$

Since $q(s)$ is well defined,

$$p(s) = \sum_h \int_C (p_{1h} c_1 + p_{2h} c_2)\, ds_h$$

is also well defined, where $p_{1h} = \alpha_h w_h/\alpha_1$ and $p_{2h} = \alpha_h/\alpha_1$. (Note that nontriviality of v_h implies that $\alpha_h \neq 0$ for all h.) Since $q(s)$ was a quasi equilibrium, so is $p(s)$. ∎

Recall that Theorem 4 gives sufficient conditions for a quasi equilibrium to be a competitive equilibrium regardless of whether or not the price function has a dot-product representation.

Appendix

Proof of Lemma 2 (Existence, Continuity, and Linearity of the Utility Function Defined). Let

$$u(x) \equiv E_0 \left\{ \sum_{t=0}^{\infty} \beta^t \int_C U(c) \left[\sum_l x_{lt}(h_{lt}, dc) \right] \right\}.$$

We must show first that the expectation operator is well defined.

First define g_T: $\Omega \times X \to \mathbf{R}$ as

$$g_T(\omega, x) \equiv \sum_{t=0}^{T} \beta^t \sum_l \int_C U(c)\, x_{lt}(h_{lt}(\omega), dc) > 0.$$

Since H_{lt} is finite,

$$u_{\leqslant T}(x) \equiv E_0 \{ g_T(\omega, x) \} = \sum_{t=0}^{T} \beta^t \sum_l \sum_{h_{lt} \in H_{lt}} \pi(h_{lt}) \int_C U(c)\, x_{lt}(h_{lt}(\omega), dc) > 0,$$

where $\pi(h_{lt})$ is the probability of h_{lt}. For each ω, let $g(w, x) \equiv$

$\lim_{T\to\infty} g_T(\omega, x)$. Since there is discounting, C is compact, and $U(c)$ is both nonnegative and bounded, this limit exists.

Now since for all ω and x, $g_T(\omega, x)$ is bounded above by $\psi = \sum_{t=0}^{\infty} \beta^t \sum_l \sup_{c\in C} U(c)$, apply Lebesgue's dominated convergence theorem (see Wheeden and Zygmund ([34], Theorem 5.19) to conclude that

$$u(x) = E_0\{\lim_{T\to\infty} g_T(\omega, x)\} = \lim_{T\to\infty} u_{\leqslant T}(x)$$
$$= E_0\left\{\sum_{t=0}^{\infty} \beta^t \int_C U(c) \sum_l x_{lt}[h_{lt}, dc]\right\}.$$

Hence, $u(x)$ is a well-defined function.

Now turn to continuity. Recall that for each (l, t, h_{lt}) the topology used is the weak* one while over the product—across dates, locations, and histories—the topology is the product one. So $x^n \to x^0$ means that the convergence is pointwise.

The argument of the proof is to show that for all $\varepsilon > 0$ there exists an N such that, for any $n > N$,

$$|u(x^n) - u(x^0)| > \varepsilon.$$

Now let $u_{>T}(x) = u(x) - u_{\leqslant T}(x)$. By the triangle inequality, it follows that

$$|u(x^n) - u(x^0)| \leqslant |u_{\leqslant T}(x^n) - u_{\leqslant T}(x^0)| + |u_{>T}(x^n) - u_{>T}(x^0)|.$$

But notice that, for any $x \in X$,

$$0 \leqslant u_{>T}(x) \leqslant \frac{\beta^{T+1}}{1-\beta} \max_{c\in C} U(c).$$

Consequently, for any $\varepsilon > 0$, there exists a T such that for all $x \in X$ and, hence for x^0 and the x^n, $u_{>T}(x) < \varepsilon/2$, and nonnegativity of u guarantees that for some T it is $< \varepsilon/2$. Note that $u_{\leqslant T}$ is a finite sum of continuous functions, which guarantees that for some N_0, $n > N_0$ makes the first term smaller than $\varepsilon/2$. This establishes continuity of the u functions.

Linearity of $u(x)$ follows trivially. ∎

Proof of Lemma 3. By Theorem 6-4 of Parthasarathy [22], a version of the Banach-Alaoglu theorem, the set

$$T_h \equiv \left\{s_h \in \mathscr{M}_+(C): \int_C ds_h \leqslant 1\right\}$$

is weak* compact. By Tychonov's theorem,

$$T \equiv \{(s, b) \in S \times R \colon s_h \in T_h \text{ for all } h\}$$

is compact in the product topology over components. T_1 is a closed subset of T, a compact set, and so is compact itself. The projection operator is continuous, so X is a compact set.

From our definition of feasibility, allocation $[(x^i), y]$ being feasible implies that $x^i \in X$ for all i, $y \in Y$, and $y = \sum_i \lambda^i x^i$. Convexity of X implies that $y \in X$. So $[(x^i), y]$ being feasible implies that $x^i \in X$ for all i and $y \in X \cap Y$. Y is closed, and X is compact; hence, $X \cap Y$ is compact. By Tychonov's theorem again, the set of feasible allocations, i.e., $\{X^I \times X \cap Y\}$, is compact in the topology being used. ∎

Proof of Theorem 5. Let $q(s) \equiv \lim v(s_{\leqslant T})$. This limit exists (Prescott and Lucas ([24] Lemma 1). Linearity of v and the fact that $x_T = x_{\leqslant T} + \hat{x} - \hat{x}_{\leqslant T}$ imply that

$$\lim_{T \to \infty} v(x_T) = \lim_{T \to \infty} v(x_{\leqslant T} + \hat{x} - \hat{x}_{\leqslant T}) = q(x) + v(\hat{x}) - q(\hat{x}). \tag{A.1}$$

Step 1. The first step of the proof is to show that if x is as good as $\bar{x}^i$ for agent i, then

$$q(x) \geqslant v(\bar{x}^i) + q(\hat{x}^i) - v(\hat{x}^i). \tag{A.2}$$

Start by selecting x^i, $x'^i \in X^i$ such that x'^i is strictly preferred to $\bar{x}^i$ and x^i is as desirable as $\bar{x}^i$. Let $x^{\alpha i} = ax^i + (1 - \alpha) x'^i$. By (I), $x^{\alpha i} \in X^i$, and by (II), $x^{\alpha i}$ is strictly preferred to $\bar{x}^i$. By (VII), there exists a T' such that for all $T > T'$, $x_T^{\alpha i}$ is strictly preferred to $\bar{x}^i$. Since v is an equilibrium price, this implies that

$$v(\bar{x}^i) < v(x_T^{\alpha i}).$$

Taking limits when $T \to \infty$ and $\alpha \to 1$ gives

$$v(\bar{x}^i) \leqslant \lim_{T \to \infty} v(x_T^i)$$

which by (A.1) implies that

$$v(\bar{x}^i) \leqslant q(x) + v(\hat{x}^i) - q(\hat{x}^i).$$

Rearranging gives (A.2).

Step 2. The second step is to show that if $y \in Y^j$, then

$$q(y) \leqslant v(\bar{y}^j) + q(\hat{y}^j) - v(\hat{y}^j). \tag{A.3}$$

To show this step, note that with v being an equilibrium price and by (VI),

$$v(y_T^j) \leqslant v(\bar{y}^j).$$

Taking limits and using (A.1) gives

$$v(\bar{y}^j) \geqslant q(y) + v(\hat{y}^j) - q(\hat{y}^j).$$

Rearranging gives (A.3).

Step 3. The third step is to show that profit maximization implies that for all i

$$q(\bar{x}^i) = v(\bar{x}^i) + q(\hat{x}^i) - v(\hat{x}^i) \tag{A.4}$$

and for all j

$$q(\bar{y}^j) = v(\bar{y}^i) + q(\hat{y}^j) - v(\hat{y}^j). \tag{A.5}$$

Let

$$\bar{x} \equiv \sum_i \lambda^i \bar{x}^i, \qquad \hat{x} \equiv \sum_i \lambda^i \hat{x}^i, \qquad \bar{y} \equiv \sum_j \bar{y}^j, \qquad \hat{y} \equiv \sum_j \hat{y}^j.$$

Now note that (A.2) and (A.3) also apply for $\bar{x}^i$ and $\bar{y}^j$. By linearity of v and aggregating over consumers, we get

$$q(\bar{x}) \geqslant v(\bar{x}) + q(\hat{x}) - v(\hat{x}). \tag{A.6}$$

Aggregating over producers, we get

$$q(\bar{y}) \leqslant v(\hat{y}) + q(\hat{y}) - v(\hat{y}). \tag{A.7}$$

By (VIII), $\hat{y} + \omega = \hat{x}$. By feasibility of the equilibrium allocation, $\omega + \bar{y} = \bar{x}$. Then, from (A.6), we have that

$$q(\bar{y} + \omega) \geqslant v(\bar{y} + \omega) + q(\hat{y} + \omega) - v(\bar{y} + \omega).$$

Linearity of v implies that

$$q(\bar{y}) \geqslant v(\bar{y}) + q(\hat{y}) - v(\hat{y}).$$

This, together with (A.7), implies that

$$q(\bar{y}) = v(\bar{y}) + q(\hat{y}) - v(\hat{y}).$$

Since, for all j

$$q(\bar{y}^j) \geqslant v(\bar{y}^j) + q(\hat{y}^j) - v(\hat{y}^j),$$

it follows that

$$q(\bar{y}^j) = v(\bar{y}^j) + q(\hat{y}^j) - v(\hat{y}^j).$$

By the same reasoning, for all i,

$$q(\bar{x}^i) = v(\bar{x}^i) + q(\hat{x}^i) - v(\hat{x}^i).$$

Step 4. The fourth step is to show that q is a quasi equilibrium. But this is immediate from (A.2), (A.3) (A.4), and (A.5). Just substitute $v(\bar{x}^i) + q(\hat{x}^i) - v(\hat{x}^i)$ for $q(\bar{x}^i)$ in (A.2) and $v(\bar{y}^j) + q(\hat{y}^j) - v(\hat{y}^j)$ for $q(\bar{y}^j)$ in (A.3), and the results are the conditions that define a quasi equilibrium. This completes the proof. ∎

Proof of Lemma 4. Pick any $s_0 \in S$ such that $v(s_0) \neq 0$ and $p(s_0) \neq 0$. Define $\alpha \equiv v(s_0)/p(s_0)$. If $\alpha < 0$, then $\bar{s}$ does not solve the program. Suppose there exists $s_1 \in S$ such that $v(s_1) \neq \alpha p(s_1)$. Pick $\gamma_0 \in \mathbf{R}$ and $\gamma_1 \in \mathbf{R}_+$ such that $\gamma_0 p(s_0) + \gamma_1 p(s_1) = 0$.

Start with $v(s_1) > \alpha p(s_1)$. Let $y = \bar{s} + \gamma_0 s_0 + \gamma s_1$. Then

$$\begin{aligned} p(y) &= p(\bar{s}) + \gamma_0 p(s_0) + \gamma_1 p(s_1) = p(\bar{s}) \\ v(y) &= v(\bar{s}) + \gamma_0 v(s_0) + \gamma_1 v(s_1) \\ &> v(\bar{s}) + \alpha[\gamma_0 p(s_0) + \gamma_1 p(s_1)] = v(\bar{s}). \end{aligned}$$

This contradicts the fact that $\bar{s}$ solves the program. The other alternative is that $v(s_1) < \alpha p(s_1)$. Let $y = \bar{s} - \gamma_0 s_0 - \gamma_1 s_1$. This implies that $v(y) > v(\bar{s})$ and, hence, a contradiction. ∎

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