1. Vector Potential for a Steady Current Distribution
Solution
(a) \[ \vec{B} = \nabla \times \vec{A} = (1/s) \partial(sA_\phi)/\partial s = 4\alpha s^2 \hat{e}_z. \]
(b) \[ \vec{J} = (1/\mu_0) (\nabla \times \vec{B}) = (-1/\mu_0) \partial B_z/\partial s \hat{e}_\phi = -(8\alpha s/\mu_0) \hat{e}_\phi. \]
(c) A simple solution uses Stokes’ theorem to convert the flux integral to a line integral on the boundary: \( \Phi = \oint \vec{A} \cdot d\vec{\ell} \) around the circumference of the cylinder. This gives the enclosed flux \( \Phi = 2\pi \alpha a^4. \)

The flux can also be found by integrating the magnetic field found in part (a) through a series of annuli extending from \( s \) to \( s + ds \)

\[
\Phi = \int_0^a 2\pi s ds 4\alpha s^2 = 2\pi \alpha a^4
\]
as above.

(d) The axial current is zero since

\[
I = \int_0^a \vec{J} \cdot d\vec{A} = \int_0^a J dA (\hat{e}_\phi \cdot \hat{e}_z) = 0
\]
i.e. the current circulates only in the \(-\phi\)-direction, not along the axis of the cylinder.

2. Fields and Currents for a Steady Current Distribution
Solution
(a) \[ \vec{J} = \rho \vec{v} = \rho \vec{\omega} \times \vec{r} = \rho \omega s \hat{e}_\phi. \]
(b) Since the exterior field \( (s > b) \) is zero, we can find the field by using Ampere’s Law (integral form) to a rectangular loop that “closes” outside the cylinder. Thus

\[
\oint \vec{B} \cdot d\vec{\ell} = -B_z^< h = \mu_0 \oint \vec{J} \cdot d\vec{A} = \mu_0 \int_a^b (-\rho \omega s) h ds
\]
\[
= -\frac{\mu_0 h}{2} \rho \omega \left(b^2 - a^2\right) \quad (s < a)
\]
\[
B_z^< = \frac{\mu_0}{2} \rho \omega \left(b^2 - a^2\right) \quad (s < a)
\]

(c) Repeat the calculation of part (b) where the lower limit of integration is \( s \) instead of \( a \). Then

\[
B_z = \frac{\mu_0}{2} \rho \omega \left(b^2 - s^2\right) \quad (a < s < b)
\]

(d) The exterior field is zero by assumption. The results of part (b) demonstrate that the interior field vanishes at \( s = b \). Thus the \textit{values} of the fields match with a discontinuity in slope. This is expected since there is no surface current on the boundary \( s = b \).
3. Dipole-dipole interactions

Solution

(a) This is the conventional dipole field with

\[
B_r = \frac{\mu_0 m_1}{4\pi} \frac{2 \cos \theta}{r^3},
\]

\[
B_\theta = \frac{\mu_0 m_1}{4\pi} \frac{\sin \theta}{r^3},
\]

\[
B_\phi = 0.
\]

(b) Since \( \vec{m}_2 = m_2 \hat{e}_x \) we can express \( \hat{e}_x \) in polar coordinates giving

\[
\vec{m}_2 = m_2 (\sin \theta \cos \phi \hat{e}_r + \cos \theta \cos \phi \hat{e}_\theta - \sin \phi \hat{e}_\phi)
\]

(c) We need \( \vec{F}_2 = \nabla (\vec{m}_2 \cdot \vec{B}) \). Using the results from (a) and (b)

\[
\vec{m}_2 \cdot \vec{B} = \frac{\mu_0 m_1}{4\pi} \left( \frac{3 \cos \theta \sin \theta \cos \phi}{r^3} \right)
\]

\[
= \frac{3\mu_0 m_1}{8\pi} \left( \frac{\sin 2\theta \cos \phi}{r^3} \right)
\]

The force is found by evaluating \( \nabla (m_2 \cdot B) \) and evaluating at \( \theta = \phi = \pi/2 \) (this is along the y axis). Since \( \cos \phi = 0 \) at \( \phi = \pi/2 \) the only nonzero term can come from the \( \phi \) term of the gradient for which

\[
F_{2,\phi} = \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} \left[ \frac{3\mu_0 m_1}{8\pi} \left( \frac{\sin 2\theta \cos \phi}{r^3} \right) \right]
\]

\[
= \frac{3\mu_0 m_1}{4\pi} \left( \frac{\cos \theta \sin \phi}{r^4} \right)
\]

\[
= 0 \quad (1)
\]

So the force is zero in this geometry. This can be understood from the symmetry of the current distribution: the forces between “poles” of the dipoles is exactly compensated in this geometry.

(d) The torque \( \vec{N} = \vec{m}_2 \times \vec{B} \). Then, using the field from (a) and evaluating at \( \theta = \pi/2 \) one has

\[
\vec{N}_2 = \frac{\mu_0 m_1 m_2}{4\pi r^3} \hat{e}_r
\]

which tends to orient \( \vec{m}_2 \) along the field produced by \( \vec{m}_1 \).