1. **Boundary Conditions for** $\vec{B}$ **(20 points)**

**Solution**

(a) Using Ampere’s Law the volume current density $\vec{J} = (\nabla \times \vec{B})/\mu_o$. Then in the positive halfspace ($z > 0$):

$$\vec{J}^> = -\frac{1}{\mu_o} \frac{\partial B_y}{\partial z} \hat{e}_x = -\frac{\alpha}{\mu_o} \hat{e}_x$$

while for the negative halfspace ($z < 0$):

$$\vec{J}^< = \frac{1}{\mu_o} \frac{\partial B_x}{\partial z} \hat{e}_y = \frac{\alpha}{\mu_o} \hat{e}_y$$

(b) The boundary conditions for $\vec{B}$ require that the jump discontinuity $\vec{B}^>(0) - \vec{B}^<(0) = \mu_o(\vec{K} \times \hat{e}_z)$. Therefore we have

$$\vec{B}^>(0) - \vec{B}^<(0) = B_o (\hat{e}_y - \hat{e}_x) = \mu_o (\vec{K} \times \hat{e}_z)$$

Taking the cross product by $\hat{e}_z$ on both sides gives

$$\hat{e}_z \times \Delta \vec{B} = B_o (\hat{e}_y - \hat{e}_x) = \mu_o (\vec{K} \times \hat{e}_z)$$

where the last line uses the fact that $\vec{K}$ lies in the $xy$ plane. Therefore

$$\vec{K} = -\frac{B_o}{\mu_o} (\hat{e}_x + \hat{e}_y)$$

2. **Vector Potential (20 points)**

**Solution**

(a) Use $\nabla \cdot \vec{B} = 0$ to find

$$\frac{1}{s} \frac{\partial}{\partial s} (s B_s) + \frac{\partial B_z}{\partial z} = 0$$

Then integrating twice with respect to $s$ gives

$$B_s = -c_2zs$$

so that the magnetic field in the region of interest is

$$\vec{B} = -c_2zs \hat{e}_s + (c_0 + c_2z^2) \hat{e}_z$$
(b) Since $\vec{A} = A_\phi \hat{e}_\phi$, a simple solution relates the loop integral of $\vec{A}$ to the flux of $\vec{B}$ enclosed by the loop. Using $\nabla \times \vec{A} = \vec{B}$ we have

$$\int d\vec{A} \cdot (\nabla \times \vec{A}) = \int d\vec{A} \cdot \vec{B}$$

$$\oint \vec{A} \cdot d\ell = \int d\vec{A} \cdot \vec{B}$$

$$A_\phi(s, z) 2\pi s = (c_0 + c_2 z^2) \pi s^2$$

$$A_\phi(s, z) = \frac{1}{2} (c_0 + c_2 z^2) s$$

Note that this solution takes the form of the vector potential for a uniform field $\vec{A} = (1/2)(B_z \hat{e}_z \times \vec{r})$ since the flux depends only on the (constant) value of $B_z$ at fixed height $z$. You can check that $\nabla \cdot \vec{A} = 0$ and that the current density $\vec{J} = (\nabla \times \vec{B})/\mu_0$ is purely azimuthal, explaining the azimuthal flow of $\vec{A}$. You could also arrive at this result by integrating the curl equation $\nabla \times \vec{A} = \vec{B}$ directly.

3. Dipole Fields (20 points)

Solution

(a) Using the coordinate-system-independent expression for $\vec{B}$

$$\vec{B} = \left( \frac{\mu_0}{4\pi} \right) \left( \frac{3(\vec{m} \cdot \hat{e}_r) \hat{e}_r - \vec{m}}{r^3} \right)$$

For loop “1” $\vec{m}_1 = I\pi a^2 \hat{e}_z$ and for loop “2” $\vec{m}_2 = I\pi a^2 \hat{e}_y$. Then superposing their fields at observation point $\vec{r} = z \hat{e}_z$ one gets

$$\vec{B}_P = \left( \frac{\mu_0 I a^2}{4} \right) \left( \frac{2\hat{e}_z - \hat{e}_y}{z^3} \right)$$

(b) The dipole moment can be obtained from a line integral over the entire current distribution, so the dipole moments of the two separate loops just add as vectors: $\vec{m} = \vec{m}_1 + \vec{m}_2 = I\pi a^2 (\hat{e}_y + \hat{e}_z)$. Then inserting the total dipole moment into Eqn. 1 it is clear that the field is purely radial only when the line of observation is along the direction of the total $\vec{m}$ (this makes $\vec{m} \parallel \hat{e}_r$). This direction is in the $yz$ plane and tipped at the angle of the total dipole moment: $\theta = \pi/4$ and $\phi = \pi/2$. 