1. **Square Antenna (15 points)**

The magnetic dipole moment is oscillating in time with peak value $I_o a^2$. In the radiation zone this generates a TE wave with

$$E_\phi = \frac{\mu_o I_o \omega^2 a^2}{4\pi c} \frac{e^{i\omega t}}{r} \sin \theta$$

$E_r, E_\theta = 0$

The magnetic field in the radiation zone

$$\vec{B} = \frac{1}{c} (\hat{e}_r \times \vec{E})$$

$$B_\theta = -\frac{\mu_o I_o \omega^2 a^2}{4\pi c^2} \frac{e^{ikr - i\omega t}}{r} \sin \theta$$

$B_r, B_\phi = 0$

Since $\vec{S} = (1/\mu_o)(\vec{E} \times \vec{B})$, we have for the time average

$$\langle \vec{S} \rangle = \frac{\mu_o \omega^4 a^4 I_o^2 \sin^2 \theta}{32\pi^2 c^3} \frac{r^2}{r^2} \hat{e}_r$$

Note that this result follows from carrying out the time average over products of the laboratory measured fields.

2. **Perfect Reflection of an EM Wave (15 points)**

(a) The boundary conditions match $E_\parallel$ at the interface and we already know that $E_\parallel = 0$ just below the interface. Therefore the total field $E_\parallel(0^-, t) = (A + B)e^{-i\omega t}\hat{e}_x = 0$ and $B/A = -1$.

(b) Superposing the incident and reflected fields gives

$$\vec{E}^<(z, t) = Ae^{ikz - i\omega t}\hat{e}_x - Ae^{-ikz - i\omega t}\hat{e}_x = 2\hat{e}_x$$

This is a standing wave with a node pinned at the origin. Note that the laboratory field is the real part of this

$$\vec{E}_L(z, t) = 2A \sin(kz) \sin(\omega t) \hat{e}_x$$

(c) Using Faraday’s Law: $\nabla \times \vec{E} = -\partial \vec{B}/\partial t$, we get that

$$\frac{\partial B_y}{\partial t} = -2kA \cos(kz) \sin(\omega t)$$
and integrating over time we get

\[ \vec{B}(z, t) = \frac{2kA}{\omega} \cos(kz) \cos(\omega t) \hat{e}_y \]

\[ = \frac{2A}{c} \cos(kz) \cos(\omega t) \hat{e}_y \]

Note that this is also a standing wave, but it is not spatially or temporally in phase with the total field \( \vec{E} \) because of the interference between the forward and backward moving parts of the solution.

3. Fields and Currents for a Steady Current Distribution (15 points)

(a) \( \vec{H} \) is a circulating field in this problem (it is divergence-free) and can be calculated from Ampere’s law using the free currents. Note that the free current density is \( \vec{J}_f = I/(\pi(b^2 - a^2)) \hat{e}_z \). Thus \( \vec{H} = I(s^2 - a^2)/((b^2 - a^2) \times 2\pi s) \hat{e}_\phi \), and thus \( \vec{M} = \chi_mI(s^2 - a^2)/((b^2 - a^2) \times 2\pi s) \hat{e}_\phi \).

(b) \( \vec{J}_b = \nabla \times \vec{M} = \chi_mI/(\pi(b^2 - a^2)) \hat{e}_z \), which is just \( \chi_m\vec{J}_f \), as expected in a linear permeable medium.

(c) \( \vec{K}_b = \vec{M} \times \hat{e}_n \) where \( \hat{e}_n \) is the “outward” surface normal, namely \( \hat{e}_n = \hat{e}_r \) at \( s = b \) and \( -\hat{e}_r \) at \( s = a \). This gives zero on the inside surface and \( \vec{K}_b = -I/2\pi b \hat{e}_z \) on the outside surface. The total bound current (volume plus surface) is zero as expected.

(d) The contributions to \( \vec{B} \) from the bound current sum to zero (they compensate and are cylindrically symmetric). Thus the exterior magnetic field is \( \vec{B} = \mu_0I/2\pi s \hat{e}_\phi \) just as in the absence of magnetization.

4. Moving conductor in a magnetic field (15 points)

(a) In the conductor \( \vec{E} + \vec{v} \times \vec{B} = 0 \). To lowest order in \( \omega \) the \( \vec{B} \) is just the applied field. So we have

\[ \vec{E} = -\vec{v} \times \vec{B} \]

\[ = -(\omega \times \vec{r}) \times \vec{B} \]

\[ = -\vec{B} \times (\vec{r} \times \omega) \]

\[ = -\vec{r}(\vec{B} \cdot \omega) + \omega(\vec{B} \cdot \vec{r}) \]

Then \( \Delta V = -\int \vec{E} \cdot d\vec{r} \). Since \( \vec{E} \) is always in the \( xy \) plane the second term in \( \vec{E} \) can’t contribute to any change of potential in the tangent plane. The first term is proportional to \( \cos \theta \) which just gives the \( z \) component of \( \vec{B} \). Thus this represents the “conventional” geometry with \( B_z = B \cos \theta \).

Thus \( \Delta V_{PQ} = 0 \) by symmetry and \( \Delta V_{PR} = (1/2)\omega B \cos \theta((\sqrt{2}a/2)^2 - (a/2)^2) = \omega B \cos \theta a^2/8 \) with the corner at higher potential.
5. Vector Potentials (15 points)

(a) In the far field this is a static magnetic dipole with \( \vec{m} = I_o a^2 \hat{e}_z \). Then the vector potential in Coulomb gauge is

\[
\vec{A} = \frac{\mu_o}{4\pi} \frac{\vec{m} \times \hat{e}_r}{r^2}
= \frac{\mu_o I_o a^2 \sin \theta}{4\pi r^2} \hat{e}_\phi
\]

(b) Taking \( \nabla \times \vec{A} \) one gets the dipole field

\[
\vec{B} = \frac{\mu_o I_o a^2}{4\pi} \left( \frac{2 \cos \theta \hat{e}_r + \sin \theta \hat{e}_\theta}{r^3} \right)
\]

6. Forces Between Magnetic Dipoles (15 points)

(a) This is the conventional dipole field with

\[
B_r = \frac{\mu_o m_1}{4\pi} \frac{2 \cos \theta}{r^3}
\]
\[
B_\theta = \frac{\mu_o m_1}{4\pi} \frac{\sin \theta}{r^3}
\]
\[
B_\phi = 0
\]

(b) Since \( \vec{m}_2 = m_2 \hat{e}_y \) we can express \( \hat{e}_y \) in polar coordinates giving

\[
\vec{m}_2 = m_2 (\sin \theta \sin \phi \hat{e}_r + \cos \theta \sin \phi \hat{e}_\theta + \cos \phi \hat{e}_\phi)
\]

(c) We need \( \vec{F}_2 = \nabla (\vec{m}_2 \cdot \vec{B}) \). Using the results from (a) and (b)

\[
\vec{m}_2 \cdot \vec{B} = \frac{\mu_o m_1 m_2}{4\pi} \left( \frac{3 \cos \theta \sin \theta \sin \phi}{r^3} \right)
= \frac{3\mu_o m_1 m_2}{8\pi} \left( \frac{\sin 2\theta \sin \phi}{r^3} \right)
\]

The force is found by evaluating \( \nabla (m_2 \cdot \vec{B}) \) and evaluating at \( \theta = \phi = \pi/2 \) (this is along the y axis). This gives

\[
F_{2,r} = 0
\]
\[
F_{2,\theta} = -\frac{3\mu_o m_1 m_2}{4\pi r^4}
\]
\[
F_{2,\phi} = 0
\]

This is a sideways force on the dipole.