In class we used an expression for the free space solutions to the Maxwell equations in the form of travelling plane waves:

\[ \vec{E}_k(r, t) = \vec{E}_k e^{i\mathbf{k} \cdot \mathbf{r} - i\omega t} \]  

(1)

where \( \vec{E}_k \) is a (complex) vector amplitude. Equation (1) describes a plane wave propagating along direction \( \hat{e}_k \) so that the surfaces of constant phase are planes perpendicular to \( \hat{e}_k \). In this representation the free space Maxwell’s equations are expressed as the four algebraic equations:

\[ \nabla \cdot \vec{E} = i\mathbf{k} \cdot \vec{E}_k = 0 \]

\[ \nabla \cdot \vec{B} = i\mathbf{k} \cdot \vec{B}_k = 0 \]

\[ \nabla \times \vec{E} = i\mathbf{k} \times \vec{E}_k = i\omega \vec{B}_k \]

\[ \nabla \times \vec{B} = i\mathbf{k} \times \vec{B}_k = -i\omega \varepsilon_0 \mu_0 \vec{E}_k \]

In these notes we use this formulation to study spherical waves. This extension is very important and very nontrivial. Many things that you know from studying plane waves simply do not apply to the (generally more important) case of spherical waves! Here we outline the derivation of spherical wave solutions to the Maxwell equations with some technical details deferred to the assignments in Homework Set 10.

The simplest spherical wave in electrodynamics is not so simple. An exact expression for the electric field in a spherical wave solution is actually given in DJG Problem 9.35, but without derivation. (He asks you to verify that it is correct, which is itself a formidable exercise. It is assigned in Homework Set 10 to give you some appreciation for the subtlety of this result.) Here we exploit representation (1) to develop a compact microscopic derivation of this result.

Because the Maxwell equations are linear, sums of plane wave solutions are also solutions. Therefore let’s consider the superposition

\[ \vec{E}(r, t) = \sum_k \vec{E}_k e^{i\mathbf{k} \cdot \mathbf{r} - i\omega t + i\alpha_k} \]

(2)

We now consider solutions at a single frequency \( \omega \) and convert the sum to a (normalized) integral over the surface of a sphere of radius \( k = \omega / c \).

\[ \vec{E}(r, t) = \frac{1}{4\pi} \int_{k=\omega/c} d\phi_k d\theta_k \sin \theta_k \vec{E}_k e^{i\mathbf{k} \cdot \mathbf{r} - i\omega t + i\alpha_k} \]

(3)

By choosing \( \vec{E}_k \) and \( \alpha_k \) appropriately we can describe any electromagnetic wave at frequency \( \omega \).

We are seeking the simplest spherical wave solutions. Spherical waves are the solutions where the surfaces of constant phase are spheres instead of planes as in Eqn. 1.
Crucially, the construction of such a spherical wave is complicated by the transverse constraint on each plane wave component

$$\mathbf{k} \cdot \mathbf{E}_k = 0$$  \hspace{1cm} (4)\\

This requires that $\mathbf{E}_k$ always lies in the tangent plane of the constant-$\omega$ sphere. These amplitudes clearly can’t be the same and in fact a math theorem further requires that when a vector field in the tangent surface of the sphere is smoothly varying (single-valued and differentiable), it has to vanish at two points. Thus $\mathbf{E}_k$ has to vary in both its magnitude and direction inside the integral in Eqn. 3 leading in general to a very complicated waveform. This is the fundamental difference between a spherical wave in electrodynamics and the simpler situation in sound waves. In the latter case, the relevant field is a scalar (e.g. pressure or density) which has no internal direction so it can be assigned the same value for any propagation direction. This is impossible for the propagating vector field spherical-wave solutions in electrodynamics.

The simplest spherical wave superposes the simplest waveforms satisfying these constraints. We can always satisfy the Eqn. 4 by writing

$$\mathbf{E}_k = \vec{\xi}_k \times \hat{e}_k$$  \hspace{1cm} (5)\\

and the simplest solution is to have a constant “director” $\vec{\xi}_k = \vec{\xi}$ and a constant phase $\alpha_k = \alpha \neq 0$. We will retain this phase to make contact with DJG’s result given in Problem 9.35.

The derivation of the spherical wave now reduces to the evaluation of the integral Eqn. 3 using the amplitudes given in Eqn. 5. Some of the details are deferred to a homework problem, but here’s the setup. The angular integrals can be isolated in the manner

$$\vec{E}(r, t) = \frac{1}{4\pi} \hat{\xi} \times \left[ \int_{\kappa = \omega/c} \int_{\phi_k} d\phi_k d\theta_k \sin \theta_k \hat{e}_k e^{i\kappa r} \right] e^{-i\omega t+i\kappa}$$  \hspace{1cm} (6)\\

and it is useful to rotate the coordinate system so that $\mathbf{r} = r\hat{e}_z$. Then the unit vector in the integrand is resolved

$$\hat{e}_k = \hat{e}_x \sin \theta_k \cos \phi_k + \hat{e}_y \sin \theta_k \sin \phi_k + \hat{e}_z \cos \theta_k.$$  \hspace{1cm} (7)\\

from which we see that the $x$ and $y$ components vanish in the $\phi_k$ integration. Then we have that

$$[...] = 2\pi \int_0^\pi d\theta_k \sin \theta_k \cos \theta_k e^{i\kappa r \cos \theta_k} \hat{e}_z$$  \hspace{1cm} (8)\\

This can be evaluated by a standard substitution $u = \cos \theta_k$ leading to

$$[...] = 2\pi \int_{-1}^1 du \, u e^{iku} \hat{e}_z$$  \hspace{1cm} (9)
The integral is elementary and gives us the result (restoring radial notation $r = r \hat{e}_r$)

$$
\vec{E}(r, t) = \frac{1}{2} \xi \times \hat{e}_r \left[ \left( \frac{1}{(kr)^2} - \frac{i}{kr} \right) e^{i kr - i \omega t - i \alpha} - \left( \frac{1}{(kr)^2} + \frac{i}{kr} \right) e^{-i kr - i \omega t - i \alpha} \right]
$$

(10)

Setting $\alpha = \pi/2$ and taking the real part of the complex field solution is the result quoted in Griffiths Problem 9.35. You might compare the elegance of this solution with the labor involved in a “brute force” verification of 9.35 by plugging it into the Maxwell equations.

**Comments on this result:**

(1) This solution is a superposition of exact plane wave solutions to the Maxwell equations and it is therefore an exact solution.

(2) The propagation factors in the arguments of the exponential ($\pm kr - \omega t$) are in D’Alembert form, but the prefactors $1/kr$ are spatial factors that have no time dependence and therefore are not D’Alembert expressions. Thus the spherical wave changes its shape as it propagates which is quite unlike the one-dimensional solutions which don’t. Also, for sufficiently large $r$ ($kr \gg 1$) the leading order contribution to the field seen by a distant observer decays $\propto 1/kr$. This is much slower than the longest range field for any static spatially localized charge distribution which would have been $\propto 1/r^2$. The radiation field literally escapes from the source.

(3) The solution superposes an outward propagating spherical wave $e^{i kr - i \omega t}$ and an inward propagating wave $e^{-i kr - i \omega t}$. Since the Maxwell equations are linear these two pieces are each solutions. This combination was necessitated by our simplifying choice of a constant $\alpha_k$.

(4) The amplitude of the field is a function of the angle of observation $\theta$

$$
\vec{E}(r, t) \propto \xi \times \hat{e}_r = \xi \sin \theta \, \hat{e}_\phi
$$

(11)

and this has a nodal line along the direction of $\vec{E}$. Elsewhere the electric field is always $\phi$-polarized and is therefore always perpendicular to the line of sight to the observer $\hat{e}_r$. This is called a transverse electric (TE) spherical wave.

(5) The magnetic field associated with this solution can be found from Faraday’s law

$$
\nabla \times \vec{E}(r, t) = i \omega \vec{B}(r, t)
$$

(12)

Note that since this is a spherical wave (and not a plane wave) the curl is most easily evaluated by differentiation of the $\phi$ components of $\vec{E}$. (The algebraic conditions given above apply only to single plane wave solutions, not their superpositions). Evaluating the curl shows that the magnetic field generically contains two terms

$$
\vec{B} = B_r \hat{e}_r + B_\theta \hat{e}_\theta
$$

(13)

(you will find $B_r$ and $B_\theta$ in the homework) which shows that the magnetic field is not always perpendicular to the line of sight! Note that it still satisfies $\nabla \cdot \vec{B}$
but interference between its various plane wave contributions moving in different
directions generally permits a nonzero component of $\vec{B}$ along the net propagation
direction. This is a general property of all three dimensional free space solutions to
Maxwell equations for localized sources.

(6) The propagation factor $kr$ measures distance to the observer in wavelengths of
the wave. In the far field $kr \gg 1$. Then the leading order dependences of $B_r$ and $B_\theta$
are

$$
B_\theta \propto \frac{1}{kr}
$$

$$
B_r \propto \frac{1}{(kr)^2}
$$

(14)

which means that in the far field $\vec{B} \perp \hat{e}_r$. In this limit (called the radiation zone)

$$
\vec{B} = \frac{1}{c} \hat{e}_r \times \vec{E}
$$

(15)

and the spherical wave degenerates (locally) to a simple plane wave solution. Here
the electric and magnetic components are both $\propto 1/kr$, perpendicular to each other
and perpendicular to the propagation direction. The well known plane wave is the
far field limit of a spherical wave. That is, the algebraic conditions given above apply
to the far field solution.

(7) One can generate a dual solution by instead imposing the transverse constraint
to $\vec{B}$, i.e. starting from

$$
\vec{B}_k = \vec{\beta} \times \hat{e}_k
$$

(16)
on the surface of the constant-$\omega$ sphere. Then the roles of $\vec{E}$ and $\vec{B}$ are interchanged:
$\vec{B}(\vec{r}, t) \perp \hat{e}_r$ and $\vec{E}(\vec{r}, t) = E_r\hat{e}_r + E_\theta\hat{e}_\theta$. This is the transverse magnetic (TM) mode
for a localized source. For a localized source in a simply-connected region of space
one cannot have a TEM wave (simultaneously transverse in both the electric and
magnetic degrees of freedom).

(8) A useful homework exercise evaluates the Poynting field for the spherical wave.
One finds $\vec{S} \propto \hat{e}_r/r^2$ in the radiation zone so the energy flux through a large sphere
is independent of the radius of the sphere, no matter how large. The interpretation
is that energy can decouples from the source and can propagate with infinite range.
Note that because of the radial fields occurring in the near field, the total Poynting field
is not always radial. The anomalous near field behavior describes a reactive transfer
of energy between its electric and magnetic components as the radiation fields are
being "launched."

(9) Superposition of solutions as given in Eqn. 2 (with a simple modifications for dy-
namics in linear media given in the second set of notes) provides a general microscopic
framework for all of classical optics.