Abstracts

Computation of Some K-groups

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1. Introduction

By now a well-established result is that the D-brane charges in string theory are precisely the K-theory group of the space-time, see [1]. Hence, computing certain K-groups has immediate physical interest. For example, cancellation of the total D-brane charge for compact directions places additional restrictions on allowed compactifications, which eliminates some torus orientifold constructions.

In this talk, I will review the computation of the twisted K-theory that is relevant for $N = 1$ supersymmetric Wess-Zumino-Witten models. I solved the case for compact, simple, simply connected Lie groups in [2]. As a non-simply connected example, I will present $SO(3)$ in Section 3. The latter is joint work with Sakura Schäfer-Nameki [3]

2. Twisted K-theory for Lie Groups

In the following, let $G$ always be a compact, simple, simply connected Lie group, together with a gerbe on $G$ with characteristic class

$$t \in H^3(G; \mathbb{Z}).$$

The corresponding Grothendieck group of twisted vector bundles on $G$ is the twisted K-theory $^tK(G)$. It is a generalized (twisted) cohomology theory. To compute the K-groups, we relate it to equivariant twisted K-theory by rewriting

$$^tK^*(G) = ^tK^*_G(G^{Tr} \times G^L) = ^tK^*_G(G^{Ad} \times G^L),$$

where the superscripts refer to the Trivial, Left, and Adjoint action of $G$ on itself. The first equality is obvious, the second follows from the $G$-isomorphism $G^{Tr} \times G^L = G^{Ad} \times G^L$ through conjugation. To compute the K-theory of the product, we use a certain equivariant Künneth theorem which follows from [4]:

**Theorem 1** (Equivariant Künneth Theorem). Let $G$ be a compact, simple, simply connected Lie group. Let $X$ be a $G$-space with twist class, let $Y$ be a $G$-space. Then there is a spectral sequence

$$E_2^{p,q} = Tor^G_{RG} \left(^tK^*_G(X), \ K^*_G(Y) \right) \Rightarrow ^tK^{p+q}_G(X).$$

The point of doing so is that we can now apply the theorem of Freed-Hopkins-Teleman [5], which identifies the twisted equivariant K-theory with the Verlinde algebra at level $k = t - \hbar$,

$$^tK^*_G(G^{Ad}) = RG/I_k.$$
Hence, it remains to compute

\[ \text{Tor}_{RG}^p \left( t K^*_G(G^\text{Ad}), K_G^*(G^L) \right) = \text{Tor}_{RG}^p \left( RG/I_k, \mathbb{Z} \right) . \]

A widely believed fact is that the Verlinde algebra is a complete intersection, and hence there exists a Koszul resolution. Although not strictly proven, this was checked for a large number of cases in [6]. Henceforth, I assume that there exists a regular sequence \( y_1, \ldots, y_n, n = rk(G) \). A bit of homological algebra yields

\[ \text{Tor}_{RG}^p \left( RG/I_k, \mathbb{Z} \right) = \text{Tor}_{RG}^p \left( RG/\langle y_1, \ldots, y_n \rangle, \mathbb{Z} \right) = \bigoplus_{2^{n-1}} \mathbb{Z}_{\gcd(y_1, \ldots, y_n)} . \]

Finally, what about higher differentials and extension ambiguities? The dual K-homology spectral sequence is a spectral sequence of algebras under the Pontryagin product. One can use this to show that there are no further differentials, and that all extension ambiguities are trivial. Hence,

\[ \mathcal{K}^*(G) = \bigoplus_{2^{n-1}} \mathbb{Z}_{\gcd(y_1, \ldots, y_n)} . \]

3. SO(3) Wess-Zumino-Witten Model

As an example of a non-simply connected Lie group, let us consider SO(3). This Wess-Zumino-Witten (WZW) model was treated from the boundary conformal field theory side in [7], where it was found that the D-brane charge groups is either \( \mathbb{Z}_2 \oplus \mathbb{Z}_2 \) or \( \mathbb{Z}_4 \) depending on whether \( \kappa \overset{\text{def}}{=} k + 1 \) is odd or even. Interestingly, the charge groups do not grow with the level in this example. This is in contradiction to the usual Atiyah-Hirzebruch spectral sequence, which predicts \( ^kK^*(SO(3)) = \mathbb{Z}_2 \oplus \mathbb{Z}_k \). Our resolution to this paradox is that D-brane charges in the SO(3) WZW model, that is the bosonic SO(3) supersymmetrized with free fermions, correspond to another twisted K-theory. Recall that the possible twists of K-theory actually contain

\[ H^1 \left( SO(3); \mathbb{Z}_2 \right) \oplus H^3 \left( SO(3); \mathbb{Z} \right) \simeq \mathbb{Z}_2 \oplus \mathbb{Z} . \]

The WZW model of [7] corresponds to the \((-\kappa)\) twisted K-theory! We can easily estimate the resulting K-groups from a twisted Atiyah-Hirzebruch spectral sequence

\[ E_2 = -H^p \left( SO(3); K^q(\text{pt.}) \right) \Rightarrow \left( ^{-\kappa} K^{p+q} \left( SO(3) \right) \right) . \]

to be either \( \mathbb{Z}_2 \oplus \mathbb{Z}_2 \) or \( \mathbb{Z}_4 \), depending on an extension ambiguity.

To resolve this ambiguity, we again rewrite the K-groups as certain equivariant K-groups. But since the Künneth theorem fails for non-simply connected groups, we chose to work SU(2) equivariant, and obtain

\[ \mathcal{K}^*(SO(3)) = \mathcal{K}^*_\text{SU}(2) \left( SO(3)^\text{Ad} \times SU(2)^L \right) \]
We found the twisted equivariant $K$-groups $K^*_{SU(2)}(SO(3)^{Ad})$ by a Mayer-Vietoris sequence for a certain cell decomposition, whose details I am going to skip. The result is that

\begin{align}
(-,\kappa)K^0_{SU(2)}(SO(3)) &= 0 \\
(-,\kappa \text{ odd})K^1_{SU(2)}(SO(3)) &= \mathbb{Z}[\Lambda, \sigma]/\langle \Lambda(\sigma-1), \sigma^2-1, p_\kappa(\Lambda) \rangle \\
(-,\kappa \text{ even})K^1_{SU(2)}(SO(3)) &= \mathbb{Z}[\Lambda, \sigma]/\langle \Lambda(\sigma-1), \sigma^2-1, p_\kappa(\Lambda)+(-1)^{\frac{\kappa}{2}}(1+\sigma) \rangle
\end{align}

as $RSU(2) = \mathbb{Z}[\Lambda]$ modules, where $p_\kappa$ are certain degree $\kappa$ polynomials. A bit of homological algebra then shows that only the $Tor^0$ in the equivariant Künneth theorem is nonvanishing, and moreover that

\begin{equation}
(-,\kappa)K^*(SO(3)) = E_2^{0,*} = \begin{cases} 
\mathbb{Z}_2 \oplus \mathbb{Z}_2 & \kappa \text{ odd} \\
\mathbb{Z}_4 & \kappa \text{ even}
\end{cases}
\end{equation}

as predicted by the boundary conformal field theory.

REFERENCES