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The Pruned State-Space System for Non-Linear DSGE Models: Theory and Empirical Applications

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This article studies the pruned state-space system for higher-order perturbation approximations to dynamic stochastic general equilibrium (DSGE) models. We show the stability of the pruned approximation up to third order and provide closed-form expressions for first and second unconditional moments and impulse response functions. Our results introduce generalized method of moments (GMM) estimation and impulse-response matching for DSGE models approximated up to third order and provide a foundation for indirect inference and simulated method of moments (SMM). As an application, we consider a New Keynesian model with Epstein–Zin preferences and two novel feedback effects from long-term bonds to the real economy, allowing us to match the level and variability of the 10-year term premium in the U.S. with a low relative risk aversion of 5.

Key words: Epstein–Zin preferences, Feedback-effects from long-term bonds, Higher-order perturbation approximation, Yield curve.

JEL Codes: C15, C53, E30

1. INTRODUCTION

The perturbation method approximates the solution to dynamic stochastic general equilibrium (DSGE) models by higher-order Taylor series expansions around the steady state (see Judd and Guu (1997) and Schmitt-Grohé and Uribe (2004), among others). These approximations have grown in popularity, mainly because they allow researchers to quickly and accurately solve DSGE models with many state variables and inherent non-linearities to analyse uncertainty shocks or time-varying risk premia (see Fernández-Villaverde *et al.* (2011) and Rudebusch and Swanson (2012), among others).

Although higher-order approximations are intuitive and straightforward to compute, they often generate explosive sample paths even when the corresponding linearized solution is stable. As noted by Kim *et al.* (2008), these explosive sample paths arise because the higher-order terms generate unstable steady states in the approximated system. The presence of explosive behaviour complicates any model evaluation because no unconditional moments exist in this approximation. It also means that any estimation method using unconditional moments, such as generalized method of moments (GMM) or simulated method of moments (SMM), is inapplicable because it relies on finite moments from stationary and ergodic probability distributions.[1]

For second-order approximations, Kim *et al.* (2008) suggest eliminating explosive sample paths by pruning terms of higher-order effects than the considered approximation order when the system is iterated forward in time. This article extends this pruning idea to perturbation approximations of any order and shows how pruning greatly facilitates inference of DSGE models. Special attention is devoted to the widely used second- and third-order approximations. We first show that our pruning method ensures stable sample paths, provided the linearized solution is stable. Next, we provide closed-form solutions for first and second unconditional moments and impulse response functions (IRFs). We also derive conditions for the existence of third and fourth unconditional moments to compute skewness and kurtosis.[2]

While it is hard to show general accuracy results regarding the pruned approximation, Lan and Meyer-Gohde (2013a) and an earlier version of this article (Andreasen *et al.*, 2013) have found that our pruning scheme does not decrease accuracy (and often it improves it) when compared to the unpruned state-space system. We can offer some intuition for this result. Unpruned approximations are subject to what we call microbursts of instability. Often, the simulations are hit by relatively large shocks, which push the simulation towards an explosive path. After a few periods, a large shock of opposite sign typically sends the simulation back into a stable path. During these periods of transitory explosive paths (or microbursts of instability), the Euler equation errors of the unpruned approximation are often fairly large. Even when we define a threshold above which we disregard a sample path as explosive, simulations will have several microbursts of instability that do not reach the threshold and are kept in the sample. In comparison, pruned approximations are not subject to these microbursts.

Our results are significant as most of the existing moment-based estimation methods for linearized DSGE models now carry over to non-linear approximations. For models solved up to third order, this includes GMM estimation based on first and second unconditional moments and matching model-implied IRFs to their empirical counterparts in the tradition of Christiano *et al.* (2005). Our results are also useful when estimating DSGE models using Bayesian methods, for instance, when conducting inference using a limited information likelihood function from unconditional moments, as suggested by Kim (2002), or when doing posterior model evaluations on unconditional moments, as in An and Schorfheide (2007). If simulations are needed to calculate higher-order unconditional moments such as skewness or kurtosis, then our results provide a foundation for SMM as in Duffie and Singleton (1993) and different types of indirect inference as in Smith (1993), Dridi *et al.* (2007), and Creel and Kristensen (2011).[3] Finally, our results are also relevant to researchers who prefer to calibrate their models as in Cooley and Prescott (1995),

1. Ruge-Murcia (2013) reviews the use of GMM in the context of DSGE models. Non-explosive sample paths are also required for likelihood methods, for instance, when using the particle filter outlined in Fernández-Villaverde and Rubio-Ramírez (2007).

2. `Matlab` codes to implement our procedures are available on the authors' home pages; see for instance https://sites.google.com/site/mandreasendk/home-1. Also, Dynare 4.4.0. has implemented our pruning method to simulate models approximated to third order.

3. See also Ruge-Murcia (2012) for a Monte Carlo study and application of SMM based on the neoclassical growth model solved up to third order.

because the unconditional mean of a model solved with higher-order terms generally differs from its steady-state value. Given our results, researchers can now easily correct for these higher-order effects and non-linearly calibrate their models.[4]

The suggested GMM estimation approach, its Bayesian equivalent, non-linear calibration, and IRF matching are promising because we can compute first and second unconditional moments or IRFs in a trivial amount of time for medium-size DSGE models solved up to third order. For the model described in Section 6 with seven state variables, it takes 0.75 second to find all first and second unconditional moments and only 0.08 second to compute the IRFs for 20 periods following a shock using an off-the-shelf laptop.

An application illustrates some of the new techniques this article makes available. We consider a rich New Keynesian economy with Calvo pricing, consumption habits, and Epstein–Zin preferences, which we estimate by GMM using first and second unconditional moments for the U.S. yield curve and five macro variables. Our New Keynesian model introduces two novel mechanisms that help us to improve our understanding of the interactions between financial markets, monetary policy, and the real economy. First, households deposit their savings in a financial intermediary. This financial intermediary invests in short- and long-term bonds and creates a wedge between the policy rate set by the monetary authority and the interest rate on deposits. Second, we augment the standard Taylor rule of the monetary authority to include the excess return on a longer-term bond, which is closely related to term premia. The first mechanism captures frictions in the financial markets that induce differences between the policy rate and the interest rate faced by households. The second mechanism captures the observation that central banks also react to term premia, as seen during the recent financial crisis. Our two mechanisms depend on the degree of precautionary behaviour and, therefore, are only operative when the model is solved at least using a third-order approximation. Thus, the methods derived in the present article are essential for the quantitative analysis of the model.

When introducing the two feedback mechanisms from financial markets to allocations, our model matches the mean and variability of the 10-year term premium with a reasonable risk aversion of 5, while simultaneously matching key moments for standard real macro variables. We also demonstrate the importance of a positive steady-state inflation in driving this result, as it amplifies the non-linearities in the price dispersion index related to Calvo pricing and produces the desired conditional heteroscedasticity in the stochastic discount factor. Notably, an unpruned third-order approximation to our model gives explosive sample paths and is, therefore, unable to "see" this novel channel for term premia volatility, which we uncover when using our pruning method. Thus, our model and our pruning method go a long way in resolving the bond risk premium puzzle described in Rudebusch and Swanson (2008) without postulating highly risk-averse households, as in much of the existing literature.

The rest of the article is structured as follows. Section 2 introduces the problem. Section 3 presents the pruning method and the pruned state-space system for approximated DSGE models. Stability and unconditional moments of the pruned state-space system for second- and third-order approximations are derived in Section 4, with the expressions for the IRFs deferred to Section 5. Section 6 presents our empirical application and Section 7 reports our empirical findings. Section 8 concludes. Detailed derivations and proofs are deferred to the Appendix and a longer Online Appendix available on the authors' home pages or on request.

4. Some papers in the literature have accounted for the difference between the steady state and the mean of the ergodic distribution by simulation; see, for instance, Fernández-Villaverde *et al.* (2011). These simulations are, however, computationally demanding, in particular, for very persistent processes, where a long sample path is required to accurately compute unconditional moments.

2. THE STATE-SPACE SYSTEM

We consider the following class of DSGE models. Let $\mathbf{y}_t \in \mathbb{R}^{n_y}$ be a vector of control variables, $\mathbf{x}_t \in \mathbb{R}^{n_x}$ a vector of state variables, and $\sigma \geq 0$ an auxiliary perturbation parameter. To simplify the notation, $\mathbf{y}_t$ and $\mathbf{x}_t$ are expressed in deviations from their steady state. The exact solution to the DSGE model is given by the state-space system

$$\mathbf{y}_t = \mathbf{g}(\mathbf{x}_t, \sigma), \tag{1}$$

$$\mathbf{x}_{t+1} = \mathbf{h}(\mathbf{x}_t, \sigma) + \sigma \eta \epsilon_{t+1}, \tag{2}$$

where ϵ_{t+1} contains the n_ϵ exogenous zero-mean shocks. We refer to (1) and (2) as the observation and state equations, respectively. Initially, we do not impose a distributional form for ϵ_{t+1}. We only assume that ϵ_{t+1} is independent and identically distributed with finite second moments, denoted by $\epsilon_{t+1} \sim \mathcal{IID}(\mathbf{0}, \mathbf{I})$. Additional moment restrictions will be imposed later. The perturbation parameter σ scales the matrix η having dimension $n_x \times n_\epsilon$.[5]

In general, DSGE models do not have a closed-form solution and the functions $\mathbf{g}(\mathbf{x}_t, \sigma)$ and $\mathbf{h}(\mathbf{x}_t, \sigma)$ cannot be found explicitly. The perturbation method is a popular way to obtain Taylor series expansions to these functions around the steady state ($x_t = x_{t+1} = 0$ and $\sigma = 0$). When the functions $\mathbf{g}(\mathbf{x}_t, \sigma)$ and $\mathbf{h}(\mathbf{x}_t, \sigma)$ are approximated up to first order, the state-space system is approximated by $\mathbf{g_x x}_t$ and $\mathbf{h_x x}_t$ in (1) and (2), respectively. Here, $\mathbf{g_x}$ is an $n_y \times n_x$ matrix with derivatives of $\mathbf{g}(\mathbf{x}_t, \sigma)$ with respect to $\mathbf{x}_t$ and $\mathbf{h_x}$ is an $n_x \times n_x$ matrix with derivatives of $\mathbf{h}(\mathbf{x}_t, \sigma)$ with respect to $\mathbf{x}_t$.[6] Given our assumptions about ϵ_{t+1}, the approximated state-space system has finite first and second unconditional moments if all eigenvalues of $\mathbf{h_x}$ have modulus less than one. Furthermore, the approximated state-space system fluctuates around the steady state, which is the unconditional mean. In this case, it is straightforward to calibrate the structural parameters in the DSGE model from unconditional first and second moments or carry out a formal estimation using Bayesian inference, maximum likelihood, GMM, SMM, etc. (Ruge-Murcia, 2007).

When the functions $\mathbf{g}(\mathbf{x}_t, \sigma)$ and $\mathbf{h}(\mathbf{x}_t, \sigma)$ are approximated beyond linearisation, we could, in principle, apply the same method to construct the approximated state-space system with their higher-order Taylor series expansions. However, the resulting system cannot, in general, be shown to have any finite unconditional moments and may display explosive dynamics. This occurs even when we solve simple versions of the New Keynesian model. Hence, it is hard to use this approximated state-space system to calibrate or estimate model parameters. Consequently, it is useful to construct another approximated state-space system that has well-defined statistical properties when analysing DSGE models solved beyond linearisation. We now explain how this can be done.

3. THE PRUNING METHOD

Kim *et al.* (2008) suggest a pruning method to construct the approximated state-space system for second-order approximations of DSGE models. We will refer to the resulting approximated state-space system as the pruned state-space system. Section 3.1 reviews pruning and explains its logic for the second-order approximation. Section 3.2 extends the method to a third-order approximation. The general procedure for constructing the pruned state-space system for any approximation order is then straightforward, but deferred to Appendix A.1 in the interest of space. We relate our approach to the existing literature in Section 3.3.

5. As shown in Andreasen (2012), the assumption that ϵ_{t+1} enters linearly in (2) is without loss of generality. A few examples are provided in the Online Appendix.

6. The derivatives of $\mathbf{g}(\mathbf{x}_t, \sigma)$ and $\mathbf{h}(\mathbf{x}_t, \sigma)$ with respect to σ are zero (Schmitt-Grohé and Uribe, 2004).

3.1. *Second-order approximation*

The first step when constructing the pruned state-space system for the second-order approximation is to decompose the state variables into first-order effects $\mathbf{x}_t^f$ and second-order effects $\mathbf{x}_t^s$ as follows. We start from the second-order Taylor series expansion of the state equation

$$\mathbf{x}_{t+1}^{(2)} = \mathbf{h_x}\mathbf{x}_t^{(2)} + \frac{1}{2}\mathbf{H_{xx}}\left(\mathbf{x}_t^{(2)} \otimes \mathbf{x}_t^{(2)}\right) + \frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2 + \sigma\eta\epsilon_{t+1}, \tag{3}$$

where $\mathbf{x}_t^{(2)}$ is the unpruned second-order approximation to the state variables.[7] Here, $\mathbf{H_{xx}}$ is an $n_x \times n_x^2$ matrix with the derivatives of $\mathbf{h}(\mathbf{x}_t, \sigma)$ with respect to $(\mathbf{x}_t, \mathbf{x}_t)$ and $\mathbf{h}_{\sigma\sigma}$ is an $n_x \times 1$ matrix containing derivatives taken with respect to (σ, σ).[8] Substituting $\mathbf{x}_t^{(2)}$ with $\mathbf{x}_t^f + \mathbf{x}_t^s$ into the right-hand side of (3) gives

$$\mathbf{h_x}\left(\mathbf{x}_t^f + \mathbf{x}_t^s\right) + \frac{1}{2}\mathbf{H_{xx}}\left(\left(\mathbf{x}_t^f + \mathbf{x}_t^s\right) \otimes \left(\mathbf{x}_t^f + \mathbf{x}_t^s\right)\right) + \frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2 + \sigma\eta\epsilon_{t+1}. \tag{4}$$

A law of motion for $\mathbf{x}_{t+1}^f$ is derived by preserving only first-order effects in (4). We keep the first-order effects from the previous period $\mathbf{h_x}\mathbf{x}_t^f$ and $\sigma\eta\epsilon_{t+1}$ to obtain

$$\mathbf{x}_{t+1}^f = \mathbf{h_x}\mathbf{x}_t^f + \sigma\eta\epsilon_{t+1}. \tag{5}$$

This expression for $\mathbf{x}_{t+1}^f$ is the standard first-order approximation to the state equation. Note that $\mathbf{x}_{t+1}^f$ is a polynomial in $\{\epsilon_s\}_{s=1}^{t+1}$ that only includes first-order terms. The first-order approximation to the observation equation is

$$\mathbf{y}_t^f = \mathbf{g_x}\mathbf{x}_t^f. \tag{6}$$

Accordingly, the pruned state-space system for the first-order approximation is given by (5) and (6), meaning that the pruned and unpruned state-space systems are identical in this case.

A law of motion for $\mathbf{x}_{t+1}^s$ is derived by preserving only second-order effects in (4). Here, we include the second-order effects from the previous period $\mathbf{h_x}\mathbf{x}_t^s$, the squared first-order effects in the previous period $\frac{1}{2}\mathbf{H_{xx}}\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)$, and the correction $\frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2$. Hence,

$$\mathbf{x}_{t+1}^s = \mathbf{h_x}\mathbf{x}_t^s + \frac{1}{2}\mathbf{H_{xx}}\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + \frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2. \tag{7}$$

We exclude terms with $\mathbf{x}_t^f \otimes \mathbf{x}_t^s$ and $\mathbf{x}_t^s \otimes \mathbf{x}_t^s$ because they reflect third- and fourth-order effects, respectively. Note that $\mathbf{x}_{t+1}^s$ is a polynomial in $\{\epsilon_s\}_{s=1}^{t}$ that only includes second-order terms.

The final step in setting up the pruned state-space system is to derive the expression for the observation equation. Using the same approach as above, we start from the second-order Taylor series expansion of the observation equation

$$\mathbf{y}_t^{(2)} = \mathbf{g_x}\mathbf{x}_t^{(2)} + \frac{1}{2}\mathbf{G_{xx}}\left(\mathbf{x}_t^{(2)} \otimes \mathbf{x}_t^{(2)}\right) + \frac{1}{2}\mathbf{g}_{\sigma\sigma}\sigma^2, \tag{8}$$

where $\mathbf{y}_t^{(2)}$ denotes the unpruned second-order approximation to the control variables. Here, $\mathbf{G_{xx}}$ is an $n_y \times n_x^2$ matrix with the corresponding derivatives of $\mathbf{g}(\mathbf{x}_t, \sigma)$ with respect to $(\mathbf{x}_t, \mathbf{x}_t)$ and $\mathbf{g}_{\sigma\sigma}$

7. We adopt the standard assumption that the model has a unique stable first-order approximation, which implies that all higher-order terms are also unique (Judd and Guu, 1997; Lan and Meyer-Gohde, 2014).

8. The derivatives of $\mathbf{g}(\mathbf{x}_t, \sigma)$ and $\mathbf{h}(\mathbf{x}_t, \sigma)$ with respect to $(\mathbf{x}_t, \sigma)$ are zero.

is an $n_y \times 1$ matrix containing derivatives with respect to (σ,σ). We only want to preserve effects up to second order, meaning that the pruned approximation to the control variables is given by

$$\mathbf{y}_t^s = \mathbf{g_x}\left(\mathbf{x}_t^f + \mathbf{x}_t^s\right) + \frac{1}{2}\mathbf{G_{xx}}\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + \frac{1}{2}\mathbf{g}_{\sigma\sigma}\sigma^2. \tag{9}$$

Here, we leave out terms with $\mathbf{x}_t^f \otimes \mathbf{x}_t^s$ and $\mathbf{x}_t^s \otimes \mathbf{x}_t^s$ because they reflect third- and fourth-order effects, respectively. To simplify notation, we treat $\mathbf{y}_t^s$ as the sum of the first- and second-order effects, while $\mathbf{x}_t^s$ only contains the second-order effects. Hence, $\mathbf{y}_t^s$ is a polynomial in $\{\epsilon_s\}_{s=1}^t$ that includes all first- and second-order terms.[9]

Accordingly, the pruned state-space system for the second-order approximation is given by (5), (7), and (9). The state vector in this system is thus extended to $\left[\left(\mathbf{x}_t^f\right)' \left(\mathbf{x}_t^s\right)'\right]'$ as we separately track first- and second-order effects. For completeness, the unpruned state-space system for the second-order approximation is given by (3) and (8).

3.2. *Third-order approximation*

We now construct the pruned state-space system for the third-order approximation. Following the steps outlined above, we start by decomposing the state variables into first-order effects $\mathbf{x}_t^f$, second-order effects $\mathbf{x}_t^s$, and third-order effects $\mathbf{x}_t^{rd}$. The laws of motion for $\mathbf{x}_t^f$ and $\mathbf{x}_t^s$ are the same as in the previous section, and only the recursion for $\mathbf{x}_t^{rd}$ remains to be derived. The third-order Taylor series expansion to the state equation is (Ruge-Murcia, 2012)

$$\begin{aligned}\mathbf{x}_{t+1}^{(3)} &= \mathbf{h_x}\mathbf{x}_t^{(3)} + \frac{1}{2}\mathbf{H_{xx}}\left(\mathbf{x}_t^{(3)} \otimes \mathbf{x}_t^{(3)}\right) + \frac{1}{6}\mathbf{H_{xxx}}\left(\mathbf{x}_t^{(3)} \otimes \mathbf{x}_t^{(3)} \otimes \mathbf{x}_t^{(3)}\right) \\ &\quad + \frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2 + \frac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2\mathbf{x}_t^{(3)} + \frac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^3 + \sigma\eta\epsilon_{t+1},\end{aligned} \tag{10}$$

where $\mathbf{x}_t^{(3)}$ represents the unpruned third-order approximation to the state variables. Here, $\mathbf{H_{xxx}}$ denotes an $n_x \times n_x^3$ matrix containing derivatives of $\mathbf{h}(\mathbf{x}_t,\sigma)$ with respect to $(\mathbf{x}_t,\mathbf{x}_t,\mathbf{x}_t)$, $\mathbf{h}_{\sigma\sigma\mathbf{x}}$ is an $n_x \times n_x$ matrix including derivatives with respect to $(\sigma,\sigma,\mathbf{x}_t)$, and $\mathbf{h}_{\sigma\sigma\sigma}$ is an $n_x \times 1$ matrix containing derivatives related to (σ,σ,σ).[10] We adopt the same procedure as before and substitute $\mathbf{x}_t^f + \mathbf{x}_t^s + \mathbf{x}_t^{rd}$ into the right-hand side of (10) to obtain

$$\begin{aligned}&\mathbf{h_x}\left(\mathbf{x}_t^f + \mathbf{x}_t^s + \mathbf{x}_t^{rd}\right) + \frac{1}{2}\mathbf{H_{xx}}\left(\left(\mathbf{x}_t^f + \mathbf{x}_t^s + \mathbf{x}_t^{rd}\right) \otimes \left(\mathbf{x}_t^f + \mathbf{x}_t^s + \mathbf{x}_t^{rd}\right)\right) \\ &\quad + \frac{1}{6}\mathbf{H_{xxx}}\left(\left(\mathbf{x}_t^f + \mathbf{x}_t^s + \mathbf{x}_t^{rd}\right) \otimes \left(\mathbf{x}_t^f + \mathbf{x}_t^s + \mathbf{x}_t^{rd}\right) \otimes \left(\mathbf{x}_t^f + \mathbf{x}_t^s + \mathbf{x}_t^{rd}\right)\right) \\ &\quad + \frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2 + \frac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2\left(\mathbf{x}_t^f + \mathbf{x}_t^s + \mathbf{x}_t^{rd}\right) + \frac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^3 + \sigma\eta\epsilon_{t+1}.\end{aligned} \tag{11}$$

A law of motion for the third-order effects is derived by preserving only third-order terms in (11)

$$\mathbf{x}_{t+1}^{rd} = \mathbf{h_x}\mathbf{x}_t^{rd} + \mathbf{H_{xx}}\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^s\right) + \frac{1}{6}\mathbf{H_{xxx}}\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \mathbf{x}^f\right) + \frac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2\mathbf{x}_t^f + \frac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^3. \tag{12}$$

9. Lan and Meyer-Gohde (2013b) derive a stable non-linear approximation of $\mathbf{g}(\mathbf{x}_t,\sigma)$ and $\mathbf{h}(\mathbf{x}_t,\sigma)$ in terms of $\{\epsilon_s\}_{s=1}^t$. By using (5), (7), and (9), we can also express $\mathbf{y}_t^s$ as an infinite moving average in terms of $\{\epsilon_s\}_{s=1}^t$.

10. The derivatives of $\mathbf{g}(\mathbf{x}_t,\sigma)$ and $\mathbf{h}(\mathbf{x}_t,\sigma)$ with respect to $(\mathbf{x}_t,\mathbf{x}_t,\sigma)$ are zero.

As in the derivation of the law of motion for x_t^s in (7), σ is interpreted as a variable when constructing (12). This means that $\frac{3}{6}h_{\sigma\sigma\mathbf{x}}\sigma^2 x_t^s$ and $\frac{3}{6}h_{\sigma\sigma\mathbf{x}}\sigma^2 x_t^{rd}$ represent fourth- and fifth-order effects, respectively, and are therefore omitted. Note that $\mathbf{x}_{t+1}^{rd}$ is a polynomial in $\{\epsilon_s\}_{s=1}^t$ that only includes third-order terms.

The final step is to set up the expression for the observation equation. Using results in Ruge-Murcia (2012), the third-order Taylor series expansion is given by

$$\mathbf{y}_t^{(3)} = \mathbf{g_x}\mathbf{x}_t^{(3)} + \frac{1}{2}\mathbf{G_{xx}}\left(\mathbf{x}_t^{(3)} \otimes \mathbf{x}_t^{(3)}\right) + \frac{1}{6}\mathbf{G_{xxx}}\left(\mathbf{x}_t^{(3)} \otimes \mathbf{x}_t^{(3)} \otimes \mathbf{x}_t^{(3)}\right)$$
$$+\frac{1}{2}\mathbf{g}_{\sigma\sigma}\sigma^2 + \frac{3}{6}\mathbf{g}_{\sigma\sigma\mathbf{x}}\sigma^2\mathbf{x}_t^{(3)} + \frac{1}{6}\mathbf{g}_{\sigma\sigma\sigma}\sigma^3, \tag{13}$$

where $\mathbf{y}_t^{(3)}$ represents the unpruned third-order approximation to the control variables. In (13), $\mathbf{G_{xxx}}$ denotes an $n_y \times n_x^3$ matrix containing derivatives of $\mathbf{g}(\mathbf{x}_t, \sigma)$ with respect to $(\mathbf{x}_t, \mathbf{x}_t, \mathbf{x}_t)$, $\mathbf{g}_{\sigma\sigma\mathbf{x}}$ is an $n_y \times n_x$ matrix including derivatives with respect to $(\sigma, \sigma, \mathbf{x}_t)$, and $\mathbf{g}_{\sigma\sigma\sigma}$ is an $n_y \times 1$ matrix containing derivatives related to (σ, σ, σ). To simplify notation, we treat $\mathbf{y}_t^{rd}$ as the sum of the first-, second-, and third-order effects, while $\mathbf{x}_t^{rd}$ is only the third-order effect. Hence, preserving effects up to third-order gives

$$\mathbf{y}_t^{rd} = \mathbf{g_x}\left(\mathbf{x}_t^f + \mathbf{x}_t^s + \mathbf{x}_t^{rd}\right) + \frac{1}{2}\mathbf{G_{xx}}\left(\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + 2\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^s\right)\right)$$
$$+\frac{1}{6}\mathbf{G_{xxx}}\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + \frac{1}{2}\mathbf{g}_{\sigma\sigma}\sigma^2 + \frac{3}{6}\mathbf{g}_{\sigma\sigma\mathbf{x}}\sigma^2\mathbf{x}_t^f + \frac{1}{6}\mathbf{g}_{\sigma\sigma\sigma}\sigma^3, \tag{14}$$

which is a polynomial in $\{\epsilon_s\}_{s=1}^t$ that include all first-, second-, and third-order terms.

The pruned state-space system for the third-order approximation is given by (5), (7), (12), and (14). The state vector in this system is further extended to $\left[\left(\mathbf{x}_t^f\right)' \ (\mathbf{x}_t^s)' \ \left(\mathbf{x}_t^{rd}\right)'\right]'$, as we need to separately track first-, second-, and third-order effects. For completeness, the unpruned state-space system for the third-order approximation is given by (10) and (13).

3.3. *Related literature*

Lombardo and Sutherland (2007) pioneered the idea of separately keeping track of first- and second-order effects to solve for the second-order perturbation approximation of a DSGE model. Since the first circulation of our paper, Lombardo and Uhlig (2014) have presented an alternative derivation of our pruned state-space system, but only for models without interaction between the shocks and the state variables. We find, nevertheless, that our approach allows us to easily derive many results and that the lack of interaction between the shocks and the state variables in Lombardo and Uhlig (2014) is too restrictive in many models of interest.

Our pruning approach is analysed in Haan and Wind (2012), who highlight two potential disadvantages of the method. First, pruning induces a larger vector of states than the unpruned approximation. Second, the pruned state-space system for the kth-order approximation cannot fit the exact solution if it happens to be a kth-order polynomial. We do not consider the large state vector to be a problem because it is informative to assess how important each of the second- and third-order effects is relative to the first-order effects. In addition, current computing power makes memory considerations less of a constraint. For instance, Section 6 shows that the pruned state-space system for a third-order approximation to a medium-size DSGE model is easily obtained and stored. We also view the second disadvantage as minor because an exact fit can be obtained by raising the approximation beyond order k, as acknowledged by Haan and Wind (2012).

Our pruning scheme differs from the alternative presented in Haan and Wind (2012) along two dimensions. First, for approximations beyond second order, these authors include terms with higher-order effects than the approximation order. Second, their pruning scheme is expressed around what they refer to as the stochastic steady state, while our pruning scheme is expressed around the steady state. An advantage of our choices (*i.e.* omitting all higher-order effects than the approximation order and approximating around the steady state) is that they allow the derivation of unconditional moments in closed form. Furthermore, approximating around the steady state is consistent with our treatment of σ as a variable.[11]

We conclude by stressing that, if a non-linear perturbation approximation does not preserve monotonicity and convexity of the exact policy function — as seen for extreme calibrations of DSGE models — then pruning will not restore these properties. For small DSGE models, Haan and Wind (2012) propose the perturbation-plus approximation and show that it may restore these properties of the policy function. However, the perturbation-plus algorithm is numerically demanding, even for small models, and does not allow the unconditional moments to be obtained in closed form.

4. STATISTICAL PROPERTIES OF THE PRUNED SYSTEM

This section shows that the pruned state-space system has well-defined statistical properties and presents our closed-form expressions for first and second unconditional moments.[12] We proceed as follows. Section 4.1 extends the analysis in Kim *et al.* (2008) for a second-order approximation, and Section 4.2 conducts a similar analysis for a third-order approximation. Applying the steps below to higher-order approximations is conceptually transparent.

4.1. *Second-order approximation*

In this section, it is convenient to consider a more compact representation of the pruned state-space system than the one in Section 3.1. Therefore, we introduce the vector

$$\mathbf{z}_t^{(2)} \equiv \left[\left(\mathbf{x}_t^f\right)' \; \left(\mathbf{x}_t^s\right)' \; \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)' \right]',$$

where the superscript for $\mathbf{z}_t$ denotes the approximation order. The first n_x elements in $\mathbf{z}_t^{(2)}$ are the first-order effects, while the remaining part of $\mathbf{z}_t^{(2)}$ contains second-order effects. The laws of motion for $\mathbf{x}_t^f$ and $\mathbf{x}_t^s$ are stated above and the evolution for $\mathbf{x}_t^f \otimes \mathbf{x}_t^f$ is easily derived from (5). This allows us to write the laws of motion for the first- and second-order effects in (5) and (7) by the linear law of motion in $\mathbf{z}_t^{(2)}$

$$\mathbf{z}_{t+1}^{(2)} = \mathbf{A}^{(2)}\mathbf{z}_t^{(2)} + \mathbf{B}^{(2)}\xi_{t+1}^{(2)} + \mathbf{c}^{(2)}, \tag{15}$$

11. After we circulated the first version of our paper, Francisco Ruge-Murcia directed our attention to his unpublished work on pruning at third order (Kim and Ruge-Murcia, 2009; Ruge-Murcia, 2012). His approach is similar to the one in Haan and Wind (2012), but with additional approximations imposed to compute unconditional moments.

12. Although not explicitly considered in this paper, it is straightforward to compute conditional moments for the state and control variables based on the expressions provided below.

where

$$\xi_{t+1}^{(2)} \equiv \left[(\epsilon_{t+1})' \ (\epsilon_{t+1} \otimes \epsilon_{t+1} - vec\left(\mathbf{I}_{n_e}\right))' \ (\epsilon_{t+1} \otimes \mathbf{x}_t^f)' \ (\mathbf{x}_t^f \otimes \epsilon_{t+1})' \right]', \tag{16}$$

and the law of motion in (9) as

$$\mathbf{y}_t^s = \mathbf{C}^{(2)} \mathbf{z}_t^{(2)} + \mathbf{d}^{(2)}. \tag{17}$$

The expressions for $\mathbf{A}^{(2)}$, $\mathbf{B}^{(2)}$, $\mathbf{c}^{(2)}$, $\mathbf{C}^{(2)}$, and $\mathbf{d}^{(2)}$ are provided in Appendix A.2.

Appendix A.3 shows that the system in (15) is stable if and only if all the eigenvalues of $\mathbf{h_x}$ have modulus less than one. This result might also be directly inferred from (5) and (7) because $\mathbf{x}_t^f$ is stable by assumption, $\mathbf{x}_t^s$ is constructed from a stable process, and the autoregressive part of $\mathbf{x}_t^s$ is stable. The stability of (15) implies that the system has finite unconditional second moments if and only if $\xi_{t+1}^{(2)}$ has finite unconditional second moments, which is equivalent to ϵ_{t+1} having finite unconditional fourth moments; see Appendix A.4. Hence, explosive sample paths do not appear in the pruned state-space system (almost surely). These results also hold for models with deterministic and stochastic trends provided trending variables are appropriately scaled (King and Rebelo (1999)).

The next step is to find the expressions for the first and second unconditional moments. We have from equation (16) that $\mathbb{E}\left[\xi_{t+1}^{(2)}\right] = \mathbf{0}$. Thus, $\mathbb{E}\left[\mathbf{z}_t^{(2)}\right] = \left(\mathbf{I}_{2n_x+n_x^2} - \mathbf{A}^{(2)}\right)^{-1} \mathbf{c}^{(2)}$. To obtain some intuition for the determinants of the mean in the pruned state-space system, we explicitly compute some of the elements in $\mathbb{E}\left[\mathbf{z}_t^{(2)}\right]$. The mean of $\mathbf{x}_t^f$ is easily seen to be zero from (5). Equation (7) implies that

$$\mathbb{E}\left[\mathbf{x}_t^s\right] = (\mathbf{I} - \mathbf{h_x})^{-1} \left(\frac{1}{2} \mathbf{H_{xx}} \mathbb{E}\left[\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right] + \frac{1}{2} \mathbf{h}_{\sigma\sigma} \sigma^2 \right).$$

Adding the mean for the first- and second-order effects, we obtain the mean of the state variables in the pruned second-order approximation $\mathbb{E}\left[\mathbf{x}_t^f + \mathbf{x}_t^s\right] = \mathbb{E}\left[\mathbf{x}_t^f\right] + \mathbb{E}\left[\mathbf{x}_t^s\right]$. These last two equations show that the second-order effects correct the mean of the first-order effects to adjust for uncertainty in the model. The adjustment comes from the second derivative of the perturbation parameter $\mathbf{h}_{\sigma\sigma}$ and $\mathbb{E}\left[\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right]$. The latter can be computed from (5) and is given by $\mathbb{E}\left[\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right] = (\mathbf{I} - \mathbf{h_x} \otimes \mathbf{h_x})^{-1} (\sigma\eta \otimes \sigma\eta) vec\left(\mathbf{I}_{n_e}\right)$.

Since $\mathbb{E}\left[\mathbf{x}_t^f\right] = 0$ and $\mathbb{E}\left[\mathbf{x}_t^s\right] \neq 0$, the mean value of the states in a first-order approximation is their steady state, while the mean of the pruned second-order approximation is corrected by the second moment of ϵ_{t+1}. In other words, the mean of $\mathbf{x}_t$ implied by the pruned state-space system will, in most cases, differ from the steady state. This result is crucial because it shows that we cannot, in general, ignore the term $\mathbb{E}\left[\mathbf{x}_t^s\right]$ and simply use the steady state of the model to calibrate or estimate model parameters.

Let us now consider the unconditional second moments. Standard properties of a VAR(1) system imply that the variance–covariance matrix for $\mathbf{z}_t^{(2)}$ is

$$\mathbb{V}\left(\mathbf{z}_t^{(2)}\right) = \mathbf{A}^{(2)} \mathbb{V}\left(\mathbf{z}_t^{(2)}\right) \left(\mathbf{A}^{(2)}\right)' + \mathbf{B}^{(2)} \mathbb{V}\left(\xi_t^{(2)}\right) \left(\mathbf{B}^{(2)}\right)',$$

because $\mathbf{z}_t^{(2)}$ and $\xi_{t+1}^{(2)}$ are uncorrelated as ϵ_{t+1} is independent across time. Appendix A.4 explains how to calculate $\mathbb{V}\left(\xi_t^{(2)}\right)$. Once $\mathbb{V}\left(\xi_t^{(2)}\right)$ is known, we solve for $\mathbb{V}\left(\mathbf{z}_t^{(2)}\right)$ by standard methods for discrete Lyapunov equations.

Our procedure for computing $\mathbb{V}\left(\mathbf{z}_t^{(2)}\right)$ differs slightly from the one in Kim *et al.* (2008). They suggest using a second-order approximation to $\mathbb{V}\left(\xi_t^{(2)}\right)$ by letting the last n_x^2 elements in $\xi_t^{(2)}$ be zero. This eliminates all third- and fourth-order terms related to ϵ_{t+1} and seems inconsistent with the fact that $\mathbf{A}^{(2)}\otimes\mathbf{A}^{(2)}$ in $vec\left(\mathbf{A}^{(2)}\mathbb{V}\left(\mathbf{z}_t^{(2)}\right)\left(\mathbf{A}^{(2)}\right)'\right)=\left(\mathbf{A}^{(2)}\otimes\mathbf{A}^{(2)}\right)vec\left(\mathbb{V}\left(\mathbf{z}_t^{(2)}\right)\right)$ contains third- and fourth-order terms. We prefer to compute $\mathbb{V}\left(\xi_t^{(2)}\right)$ without further approximations, implying that $\mathbb{V}\left(\mathbf{z}_t^{(2)}\right)$ corresponds to the sample moment in a long simulation of the pruned state-space system.

The variance of the combined first- and second-order effects for the state variables is obtained by taking the variance of $x_t^s+x_t^f$, that is

$$\mathbb{V}\left(\mathbf{x}_t^s+\mathbf{x}_t^f\right)=\mathbb{V}\left(\mathbf{x}_t^f\right)+\mathbb{V}\left(\mathbf{x}_t^s\right)+Cov\left(\mathbf{x}_t^f,\mathbf{x}_t^s\right)+Cov\left(\mathbf{x}_t^s,\mathbf{x}_t^f\right).$$

The auto-covariances for $\mathbf{z}_t^{(2)}$ are $Cov\left(\mathbf{z}_{t+l}^{(2)},\mathbf{z}_t^{(2)}\right)=\left(\mathbf{A}^{(2)}\right)^l\mathbb{V}\left(\mathbf{z}_t^{(2)}\right)$ for $l=1,2,3,\dots$ because $\mathbf{z}_t^{(2)}$ and $\xi_{t+l}^{(2)}$ are uncorrelated for $l=1,2,3,\dots$, given that ϵ_{t+1} is independent across time.

The closed-form expressions for all corresponding unconditional moments related to $\mathbf{y}_t^s$ follow directly from the linear relationship between $\mathbf{y}_t^s$ and $\mathbf{z}_t^{(2)}$ in (17). That is,

$$\mathbb{E}\left[\mathbf{y}_t^s\right]=\mathbf{C}^{(2)}\mathbb{E}\left[\mathbf{z}_t^{(2)}\right]+\mathbf{d}^{(2)},\ \mathbb{V}\left[\mathbf{y}_t^s\right]=\mathbf{C}^{(2)}\mathbb{V}[\mathbf{z}_t]\left(\mathbf{C}^{(2)}\right)',\text{ and}$$

$$Cov\left(\mathbf{y}_{t+l}^s,\mathbf{y}_t^s\right)=\mathbf{C}^{(2)}Cov\left(\mathbf{z}_{t+l}^{(2)},\mathbf{z}_t^{(2)}\right)\left(\mathbf{C}^{(2)}\right)'\quad\text{for } l=1,2,3,\dots$$

The representation in (15) and (17) allows the derivation of additional statistical properties for the pruned state-space system. If the system is stable, the system has finite unconditional third and fourth moments if and only if $\xi_{t+1}^{(2)}$ has finite unconditional third and fourth moments, which is the case if and only if ϵ_{t+1} has finite unconditional sixth and eighth moments; see Appendix A.5.

4.2. *Third-order approximation*

As we did for the second-order approximation, we start by deriving a more compact representation for the pruned state-space system than the one in Section 3.2. We define

$$\mathbf{z}_t^{(3)}\equiv\left[\left(\mathbf{x}_t^f\right)'\ \left(\mathbf{x}_t^s\right)'\ \left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\ \left(\mathbf{x}_t^{rd}\right)'\ \left(\mathbf{x}_t^f\otimes\mathbf{x}_t^s\right)'\ \left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right]',$$

where the first part reproduces $\mathbf{z}_t^{(2)}$ and the last three components denote third-order effects. The law of motion for $\mathbf{x}_t^{rd}$ was derived in Section 3.2, and recursions for $\mathbf{x}_t^f\otimes\mathbf{x}_t^s$ and $\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f$ follow from (5) and (7). Hence, the law of motion for $\mathbf{x}_t^f$, $\mathbf{x}_t^s$, and $\mathbf{x}_t^{rd}$ in (5), (7), and (12), respectively, can be represented by the linear law of motion in $\mathbf{z}_t^{(3)}$

$$\mathbf{z}_{t+1}^{(3)}=\mathbf{A}^{(3)}\mathbf{z}_t^{(3)}+\mathbf{B}^{(3)}\xi_{t+1}^{(3)}+\mathbf{c}^{(3)}.\tag{18}$$

We also have that the control variables are linear in $\mathbf{z}_t^{(3)}$ as

$$\mathbf{y}_t^{rd}=\mathbf{C}^{(3)}\mathbf{z}_t^{(3)}+\mathbf{d}^{(3)}.\tag{19}$$

The expressions for $\mathbf{A}^{(3)}$, $\mathbf{B}^{(3)}$, $\xi_{t+1}^{(3)}$, $\mathbf{c}^{(3)}$, $\mathbf{C}^{(3)}$, and $\mathbf{d}^{(3)}$ are provided in Appendix A.6.

Appendix A.7 shows that the system in (18) is stable if and only if all the eigenvalues of $\mathbf{A}^{(3)}$ have modulus less than one. This follows from the fact that the new component of the state vector $\mathbf{x}_t^{rd}$ is constructed from stable processes and its autoregressive component is also stable. The stability of $\mathbf{x}_t^{rd}$ relies on σ being treated as a variable in the pruned state-space system. If, instead, we had interpreted σ as a constant and included the term $\frac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2\mathbf{x}_t^{rd}$ in the law of motion for $\mathbf{x}_{t+1}^{rd}$, then $\mathbf{x}_{t+1}^{rd}$ would have the autoregressive matrix $\mathbf{h_x}+\frac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2$, which may imply eigenvalues with modulus greater than one even when $\mathbf{h_x}$ is stable. The stability of (18) implies that the system has finite unconditional second moments, which is equivalent to ϵ_{t+1} having finite unconditional sixth moments; see Appendix A.8.

The next step is to compute the first and second unconditional moments. The $\xi_{t+1}^{(3)}$ in (18) is a function of $\mathbf{x}_t^f$, $\mathbf{x}_t^s$, ϵ_{t+1}, $\epsilon_{t+1}\otimes\epsilon_{t+1}$, and $\epsilon_{t+1}\otimes\epsilon_{t+1}\otimes\epsilon_{t+1}$. Thus, $\mathbb{E}\left[\xi_{t+1}^{(3)}\right]=\mathbf{0}$ and $\mathbb{E}\left[\mathbf{z}_t^{(3)}\right]=\left(\mathbf{I}_{3n_x+2n_x^2+n_x^3}-\mathbf{A}^{(3)}\right)^{-1}\mathbf{c}^{(3)}$. It is interesting to explore the value of $\mathbb{E}\left[\mathbf{x}_t^{rd}\right]$ as it may change the mean of the state variables. From (12), we have

$$\mathbb{E}\left[\mathbf{x}_t^{rd}\right]=\left(\mathbf{I}_{n_x}-\mathbf{h_x}\right)^{-1}\left(\mathbf{H_{xx}}\mathbb{E}\left[\mathbf{x}_t^f\otimes\mathbf{x}_t^s\right]+\frac{1}{6}\mathbf{H_{xxx}}\mathbb{E}\left[\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right]+\frac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^3\right),$$

and simple algebra gives $\mathbb{E}\left[\mathbf{x}_t^f\otimes\mathbf{x}_t^s\right]=\left(\mathbf{I}_{n_x^2}-(\mathbf{h_x}\otimes\mathbf{h_x})\right)^{-1}\left(\mathbf{h_x}\otimes\frac{1}{2}\mathbf{H_{xx}}\right)\mathbb{E}\left[\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right]$ and $\mathbb{E}\left[\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right]=\left(\mathbf{I}_{n_x^3}-(\mathbf{h_x}\otimes\mathbf{h_x}\otimes\mathbf{h_x})\right)^{-1}(\sigma\eta\otimes\sigma\eta\otimes\sigma\eta)\mathbb{E}\left[\epsilon_{t+1}\otimes\epsilon_{t+1}\otimes\epsilon_{t+1}\right]$. Adding the mean for the first-, second- and third-order effects, we obtain $\mathbb{E}\left[\mathbf{x}_t^f\right]+\mathbb{E}\left[\mathbf{x}_t^s\right]+\mathbb{E}\left[\mathbf{x}_t^{rd}\right]$. If we next consider the case where ϵ_{t+1} has symmetric probability distributions, then $\mathbb{E}\left[\epsilon_{t+1}\otimes\epsilon_{t+1}\otimes\epsilon_{t+1}\right]=\mathbf{0}$, which in turn implies $\mathbb{E}\left[\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right]=\mathbf{0}$ and $\mathbb{E}\left[\mathbf{x}_t^f\otimes\mathbf{x}_t^s\right]=\mathbf{0}$. Furthermore, based on the results in Andreasen (2012), $\mathbf{h}_{\sigma\sigma\sigma}$ and $\mathbf{g}_{\sigma\sigma\sigma}$ are also zero when ϵ_{t+1} has a symmetric probability distribution. Thus, $\mathbb{E}\left[\mathbf{x}_t^{rd}\right]=\mathbf{0}$ and the unconditional mean of the state vector is not further corrected by the third-order effects when ϵ_{t+1} has zero third moments. A similar property holds for the control variables because they are a linear function of $\mathbf{x}_t^{rd}$, $\mathbf{x}_t^f\otimes\mathbf{x}_t^s$, and $\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f$. This result is useful when calibrating or estimating DSGE models with symmetric probability distributions. On the other hand, if one or several components of ϵ_{t+1} have non-symmetric probability distributions, then $\mathbf{h}_{\sigma\sigma\sigma}$ and $\mathbf{g}_{\sigma\sigma\sigma}$ may be non-zero and $\mathbb{E}\left[\mathbf{x}_t^{rd}\right]\neq\mathbf{0}$, implying that the unconditional mean has an additional uncertainty correction compared to a second-order approximation.

Let us now consider the unconditional second moments. The expression for the variance–covariance matrix of $\mathbf{z}_t^{(3)}$ is slightly more complicated than the one for $\mathbf{z}_t^{(2)}$ because $\mathbf{z}_t^{(3)}$ is correlated with $\xi_{t+1}^{(3)}$. This correlation arises from terms of the form $\mathbf{x}_t^f\otimes\epsilon_{t+1}\otimes\epsilon_{t+1}$ in $\xi_{t+1}^{(3)}$ that are correlated with elements in $\mathbf{z}_t^{(3)}$. Hence,

$$\begin{aligned}\mathbb{V}\left(\mathbf{z}_t^{(3)}\right) = {} & \mathbf{A}^{(3)}\mathbb{V}\left(\mathbf{z}_t^{(3)}\right)\left(\mathbf{A}^{(3)}\right)'+\mathbf{B}^{(3)}\mathbb{V}\left(\xi_t^{(3)}\right)\left(\mathbf{B}^{(3)}\right)' \\ & +\mathbf{A}^{(3)}Cov\left(\mathbf{z}_t^{(3)},\xi_{t+1}^{(3)}\right)\left(\mathbf{B}^{(3)}\right)'+\mathbf{B}^{(3)}Cov\left(\xi_{t+1}^{(3)},\mathbf{z}_t^{(3)}\right)\left(\mathbf{A}^{(3)}\right)'.\end{aligned}$$

The expressions for $\mathbb{V}\left(\xi_t^{(3)}\right)$ and $Cov\left(\xi_{t+1}^{(3)},\mathbf{z}_t^{(3)}\right)$ are provided in Appendix A.8. The variance of the combined first-, second- and third-order effects for the state variables is given by

$$\begin{aligned}\mathbb{V}\left(\mathbf{x}_t^s+\mathbf{x}_t^f+\mathbf{x}_t^{rd}\right) &= \mathbb{V}\left(\mathbf{x}_t^f\right)+\mathbb{V}\left(\mathbf{x}_t^s\right)+\mathbb{V}\left(\mathbf{x}_t^{rd}\right)+Cov\left(\mathbf{x}_t^f,\mathbf{x}_t^s\right)+Cov\left(\mathbf{x}_t^f,\mathbf{x}_t^{rd}\right)\\ &\quad+Cov\left(\mathbf{x}_t^s,\mathbf{x}_t^f\right)+Cov\left(\mathbf{x}_t^s,\mathbf{x}_t^{rd}\right)+Cov\left(\mathbf{x}_t^{rd},\mathbf{x}_t^f\right)+Cov\left(\mathbf{x}_t^{rd},\mathbf{x}_t^s\right).\end{aligned}$$

The auto-covariances for $\mathbf{z}_t^{(3)}$ are

$$Cov\left(\mathbf{z}_{t+s}^{(3)},\mathbf{z}_t^{(3)}\right)=\left(\mathbf{A}^{(3)}\right)^s\mathbb{V}\left[\mathbf{z}_t^{(3)}\right]+\sum_{j=0}^{s-1}\left(\mathbf{A}^{(3)}\right)^{s-1-j}\mathbf{B}^{(3)}Cov\left(\xi_{t+1+j}^{(3)},\mathbf{z}_t^{(3)}\right)$$

for $s=1,2,3,\ldots$

The closed-form expressions for all corresponding unconditional moments related to $\mathbf{y}_t^{rd}$ follow from the linear relationship between $\mathbf{y}_t^{rd}$ and $\mathbf{z}_t^{(3)}$ in (19) and are given by $\mathbb{E}\left[\mathbf{y}_t^{rd}\right]=\mathbf{C}^{(3)}\mathbb{E}\left[\mathbf{z}_t^{(3)}\right]+\mathbf{d}^{(3)}$, $\mathbb{V}\left[\mathbf{y}_t^{rd}\right]=\mathbf{C}^{(3)}\mathbb{V}\left[\mathbf{z}_t^{(3)}\right]\left(\mathbf{C}^{(3)}\right)'$, and

$$Cov\left(\mathbf{y}_{t+l}^{rd},\mathbf{y}_t^{rd}\right)=\mathbf{C}^{(3)}Cov\left(\mathbf{z}_{t+l}^{(3)},\mathbf{z}_t^{(3)}\right)\left(\mathbf{C}^{(3)}\right)' \quad \text{for } l=1,2,3,..$$

Finally, the representation in (18) and (19) of the pruned state-space system allow us to derive additional properties for the pruned state-space system. For instance, if the system is stable, the system has finite unconditional third and fourth moments if and only if $\xi_{t+1}^{(3)}$ has finite unconditional third and fourth moments. For a stable system, $\xi_{t+1}^{(3)}$ has finite unconditional third and fourth moments if and only if ϵ_{t+1} has finite unconditional ninth and twelfth moments; see Appendix A.9.

5. GENERALIZED IMPULSE RESPONSE FUNCTIONS

Another fruitful way to study the properties of DSGE models is to look at their IRFs. For the first-order approximation, these functions have simple expressions where the effects of shocks are scalable, symmetric, and independent of the state of the economy. For higher-order approximations, no closed-form expressions currently exist for these functions and simulation is, therefore, required. This section shows that the pruned state-space system allows us to derive closed-form solutions for these functions and avoid the use of simulation.

We consider the generalized impulse response function (GIRF) proposed by Koop *et al.* (1996). The GIRF for any variable in the model **var** (either a state or control variable) in period $t+l$ following a disturbance to the ith shock of size ν_i in period $t+1$ is defined as

$$GIRF_{\mathbf{var}}(l,\nu_i,\mathbf{w}_t)=\mathbb{E}\left[\mathbf{var}_{t+l}|\mathbf{w}_t,\epsilon_{i,t+1}=\nu_i\right]-\mathbb{E}\left[\mathbf{var}_{t+l}|\mathbf{w}_t\right],$$

where $\mathbf{w}_t$ denotes the required state variables in period t. As we will see below, the content of $\mathbf{w}_t$ depends on the approximation order.[13] Using this definition, the GIRFs for the first-order effects

13. The expressions we derive below for the GIRFs may also be used for studying the joint effects of more than one disturbance to the economy. Further details are provided in the Online Appendix.

have the well-known expressions

$$GIRF_{\mathbf{x}^f}(l,\nu_i)=\mathbb{E}\left[\mathbf{x}^f_{t+l}|\mathbf{x}^f_t,\epsilon_{i,t+1}=\nu_i\right]-\mathbb{E}\left[\mathbf{x}^f_{t+l}|\mathbf{x}^f_t\right]=\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\eta\nu \tag{20}$$

and $GIRF_{\mathbf{y}^f}(l,\nu_i)=\mathbf{g}_{\mathbf{x}}GIRF_{\mathbf{x}^f}(l,\nu_i)$, where ν has dimension $n_\epsilon\times 1$, $\nu(i,1)=\nu_i$, and $\nu(k,1)=0$ for $k\neq i$. Here, $GIRF_{\mathbf{x}^f}$ and $GIRF_{\mathbf{y}^f}$ are scalable, symmetric, and independent of the state of the economy because the state vector $\mathbf{x}^f_t$ enters symmetrically in the two conditional expectations for computing each of these GIRFs. Momentarily, we will see how the GIRFs for second- and third-order effects will not be scalable, symmetric, and independent of the state of the economy.

5.1. *Second-order approximation*

For the second-order effects $\mathbf{x}^s_t$, we have from (7) that

$$\mathbf{x}^s_{t+l}=\mathbf{h}_{\mathbf{x}}^l\mathbf{x}^s_t+\sum_{j=1}^{l-1}\mathbf{h}_{\mathbf{x}}^{l-1-j}\frac{1}{2}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}^f_{t+j}\otimes\mathbf{x}^f_{t+j}\right)+\frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\sum_{j=0}^{l-1}\mathbf{h}_{\mathbf{x}}^{l-1-j}. \tag{21}$$

The GIRF for $\mathbf{x}^f_t\otimes\mathbf{x}^f_t$ is derived in Appendix A.10, showing that

$$\begin{aligned}GIRF_{\mathbf{x}^f\otimes\mathbf{x}^f}\left(l,\nu_i,\mathbf{x}^f_t\right) &= \mathbf{h}_{\mathbf{x}}^l\mathbf{x}^f_t\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\eta\nu+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\eta\nu\otimes\mathbf{h}_{\mathbf{x}}^l\mathbf{x}^f_t\\ &\quad+\left(\mathbf{h}_{\mathbf{x}}^{l-1}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\right)(\sigma\eta\nu\otimes\sigma\eta\nu+\Lambda),\end{aligned} \tag{22}$$

where

$$\Lambda\equiv((\sigma\eta(\mathbf{I}-\mathbf{S})\otimes\sigma\eta(\mathbf{I}-\mathbf{S}))-(\sigma\eta\otimes\sigma\eta))\,vec(\mathbf{I}). \tag{23}$$

Here, $\mathbf{S}$ is an $n_\epsilon\times n_\epsilon$ diagonal matrix with $\mathbf{S}(i,i)=1$ and $\mathbf{S}(k,k)=0$ for $k\neq i$. Using this expression and (21), we get the GIRF for the second-order effects

$$GIRF_{\mathbf{x}^s}\left(l,\nu_i,\mathbf{x}^f_t\right)=\sum_{j=1}^{l-1}\mathbf{h}_{\mathbf{x}}^{l-1-j}\frac{1}{2}\mathbf{H}_{\mathbf{xx}}GIRF_{\mathbf{x}^f\otimes\mathbf{x}^f}\left(j,\nu_i,\mathbf{x}^f_t\right). \tag{24}$$

The expressions in (22) to (24) reveal three implications about the GIRF for the second-order effects. First, it is not scalable as $GIRF_{\mathbf{x}^f\otimes\mathbf{x}^f}\left(l,\tau\times\nu_i,\mathbf{x}^f_t\right)\neq\tau\times GIRF_{\mathbf{x}^f\otimes\mathbf{x}^f}\left(l,\nu_i,\mathbf{x}^f_t\right)$ for $\tau\in\mathbb{R}$. Second, the term $(\sigma\eta\nu\otimes\sigma\eta\nu)$ means that the GIRF is not symmetric in positive and negative shocks. Third, it depends on the first-order effects of the state variables. Adding the GIRFs for the first- and second-order effects in (20) and (24), we obtain the pruned GIRF for the state variables in a second-order approximation.

Finally, the pruned GIRF for the control variables is easily derived from (9) and previous results

$$\begin{aligned}GIRF_{\mathbf{y}^s}\left(l,\nu_i,\mathbf{x}^f_t\right) &= \mathbf{g}_{\mathbf{x}}\left(GIRF_{\mathbf{x}^f}\left(l,\nu_i,\mathbf{x}^f_t\right)+GIRF_{\mathbf{x}^s}\left(l,\nu_i,\mathbf{x}^f_t\right)\right)\\ &\quad+\frac{1}{2}\mathbf{G}_{\mathbf{xx}}GIRF_{\mathbf{x}^f\otimes\mathbf{x}^f}\left(l,\nu_i,\mathbf{x}^f_t\right).\end{aligned}$$

Another interesting result from our analytical expressions relates to the IRFs in a linearised solution for a positive or negative one-standard-deviation shock computed at the steady state.

As shown in Appendix A.11, these IRFs coincide with the GIRFs in a pruned second-order approximation because $GIRF_{\mathbf{x}^f \otimes \mathbf{x}^f}\left(j, \nu_i, \mathbf{x}_t^f\right) = \mathbf{0}$, implying that these IRFs in a linearised solution are actually second-order accurate.

5.2. *Third-order approximation*

Using (12), we first note that for the third-order effects $\mathbf{x}_t^{rd}$

$$\mathbf{x}_{t+l}^{rd} = \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^{rd} + \sum_{j=0}^{l-1}\mathbf{h}_{\mathbf{x}}^{l-1-j}\left[\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}_{t+j}^f \otimes \mathbf{x}_{t+j}^s\right) + \frac{1}{6}\mathbf{H}_{\mathbf{xxx}}\left(\mathbf{x}_{t+j}^f \otimes \mathbf{x}_{t+j}^f \otimes \mathbf{x}_{t+j}^f\right)\right]$$
$$+\sum_{j=0}^{l-1}\mathbf{h}_{\mathbf{x}}^{l-1-j}\left[\frac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2\mathbf{x}_{t+j}^f + \frac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^3\right].$$

Simple algebra implies

$$GIRF_{\mathbf{x}^{rd}}\left(l, \nu_i, \left(\mathbf{x}_t^f, \mathbf{x}_t^s\right)\right) = \sum_{j=1}^{l-1}\mathbf{h}_{\mathbf{x}}^{l-1-j}\mathbf{H}_{\mathbf{xx}}GIRF_{\mathbf{x}^f \otimes \mathbf{x}^s}\left(j, \nu_i, \left(\mathbf{x}_t^f, \mathbf{x}_t^s\right)\right)$$
$$+\sum_{j=1}^{l-1}\mathbf{h}_{\mathbf{x}}^{l-1-j}\frac{1}{6}\mathbf{H}_{\mathbf{xxx}}GIRF_{\mathbf{x}^f \otimes \mathbf{x}^f \otimes \mathbf{x}^f}\left(j, \nu_i, \mathbf{x}_t^f\right)$$
$$+\sum_{j=1}^{l-1}\mathbf{h}_{\mathbf{x}}^{l-1-j}\frac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2 GIRF_{\mathbf{x}^f}(j, \nu_i).$$

All terms are known except for $GIRF_{\mathbf{x}^f \otimes \mathbf{x}^s}\left(j, \nu_i, \left(\mathbf{x}_t^f, \mathbf{x}_t^s\right)\right)$ and $GIRF_{\mathbf{x}^f \otimes \mathbf{x}^f \otimes \mathbf{x}^f}\left(j, \nu_i, \mathbf{x}_t^f\right)$, which are derived in Appendix A.12. As was the case for the second-order effect, the GIRF for the third-order effect is not scalable, not symmetric, and depends on the first-order effects of the state variables $\mathbf{x}_t^f$. In addition, the GIRF for the third-order effects also depends on $\mathbf{x}_t^s$. Adding the GIRF for the first-, second-, and third-order effects, we obtain the pruned GIRF for the state variables in a third-order approximation.

The pruned GIRF for the control variables in a third-order approximation is

$$GIRF_{\mathbf{y}^{rd}}\left(l, \nu_i, \left(\mathbf{x}_t^f, \mathbf{x}_t^s\right)\right) = \mathbf{g}_{\mathbf{x}}\left(GIRF_{\mathbf{x}^f}(l, \nu_i) + GIRF_{\mathbf{x}^s}\left(l, \nu_i, \mathbf{x}_t^f\right)\right)$$
$$+\mathbf{g}_{\mathbf{x}}GIRF_{\mathbf{x}^{rd}}\left(l, \nu_i, \left(\mathbf{x}_t^f, \mathbf{x}_t^s\right)\right)$$
$$+\frac{1}{2}\mathbf{G}_{\mathbf{xx}}\left(GIRF_{\mathbf{x}^f \otimes \mathbf{x}^f}\left(l, \nu_i, \mathbf{x}_t^f\right) + 2GIRF_{\mathbf{x}^f \otimes \mathbf{x}^s}\left(l, \nu_i, \left(\mathbf{x}_t^f, \mathbf{x}_t^s\right)\right)\right)$$
$$+\frac{1}{6}\mathbf{G}_{\mathbf{xxx}}GIRF_{\mathbf{x}^f \otimes \mathbf{x}^f \otimes \mathbf{x}^f}\left(l, \nu_i, \mathbf{x}_t^f\right) + \frac{3}{6}\mathbf{g}_{\sigma\sigma\mathbf{x}}\sigma^2 GIRF_{\mathbf{x}^f}(l, \nu_i), \quad (25)$$

where all terms are known.

5.3. *Conditional impulse response functions*

Given that the expressions for the GIRFs in Sections 5.1 and 5.2 depend on the values of the state variables, we can use them to analyse how the responses to shocks depend on the business cycle. For example, in a simple stochastic neoclassical growth model, the economy may respond differently to a positive technological shock when capital is high than when it is low. The challenge is that in most DSGE models, the values of the state variables are often hard to interpret and hence it is difficult to assign them relevant values, except for the obvious benchmark given by the unconditional mean. To address this challenge, we suggest conditioning the GIRFs on the set $\mathcal{A}$ with a clear economic interpretation, such as the economy being in a recession (*i.e.* negative output growth). More concretely, consider a conditional GIRF of the form

$$GIRF_{\mathbf{var}}(l, \nu_i, \mathcal{A}) = \int 1_{\mathcal{A}}(\mathbf{w}_t) f(\mathbf{w}_t) GIRF_{\mathbf{var}}(l, \nu_i, \mathbf{w}_t) d\mathbf{w}_t, \tag{26}$$

where $1_{\mathcal{A}}(\mathbf{w}_t)$ is an indicator function and $f(\mathbf{w}_t)$ is the unconditional density of $\mathbf{w}_t$. Note that $\mathbf{w}_t$ in (26) may contain $\left(\mathbf{x}_t^f, \mathbf{x}_t^s\right)$ and sufficient lags as required for evaluating $1_{\mathcal{A}}(\mathbf{w}_t)$. The integral in (26) can be evaluated by Monte Carlo integration, where draws of the states are obtained from a long simulated sample path of the pruned state-space system. The advantage of a conditional GIRF is that it is defined on the set $\mathcal{A}$ with a clear economic interpretation, in contrast to the GIRFs provided in Sections 5.1 and 5.2.

An example illustrates the procedure. Imagine we are working with the stochastic neoclassical growth model and we want to calculate the GIRF of $\mathbf{y}_t^{rd}$ conditional on output growth being above a given threshold, say, 2% annualized. Then $\mathcal{A}$ is the set of $\left(\mathbf{x}_t^f, \mathbf{x}_t^s, \mathbf{x}_{t-1}^f, \mathbf{x}_{t-1}^s\right)$, which implies that annualized output growth in period t is above 2%.

6. AN APPLICATION

We now present an application that illustrates some of the tools that our article makes available. We postulate a New Keynesian model with two novel features. First, we introduce a financial intermediary that trades short- and long-term government bonds. The intermediary generates a wedge between the policy rate chosen by the monetary authority and the interest rate faced by households. This wedge depends on the time-variation in the conditional second moments of the stochastic discount factor. A similar premium appears in models with a banking sector due to steady-state frictions (Bernanke *et al.*, 1999; Gertler and Karadi, 2011). Thus, our model combines the macro-finance literature focusing on stochastic discount factors with the recent work on financial intermediation in DSGE models. Our second innovation is to consider a central bank that sets the policy rate based not only on the inflation and output gaps, but also on a measure of term premia. This extension is motivated by the recent efforts of central banks to lower term premia through large asset purchases (Gagnon *et al.*, 2011). In summary, our model displays two feedback effects from long-term bonds to the real economy: (1) a wedge between the policy rate and the interest rate faced by households and (2) a policy rate that depends on a measure of term premia. Each of these feedback effects helps our model to overcome the counterintuitive result of Tallarini (2000) that real allocations are essentially unaffected by the amount of risk in the economy. We also note that these features create a richer environment for monetary policy than the standard New Keynesian model and make our application of relevance on its own. We proceed by outlining the model in Sections 6.1–6.5 and describing our solution and estimation method in Sections 6.6 and 6.7, respectively.

6.1. *Households*

We consider a representative household with recursive preferences as in Epstein and Zin (1989). Following Rudebusch and Swanson (2012), we write the value function V_t of the household as

$$V_t \equiv \begin{cases} u_t + \beta\left(\mathbb{E}_t\left[V_{t+1}^{1-\phi_3}\right]\right)^{\frac{1}{1-\phi_3}} & \text{if } u_t > 0 \text{ for all } t \\ u_t - \beta\left(\mathbb{E}_t\left[\left(-V_{t+1}\right)^{1-\phi_3}\right]\right)^{\frac{1}{1-\phi_3}} & \text{if } u_t < 0 \text{ for all } t \end{cases}, \tag{27}$$

where $\mathbb{E}_t$ is the conditional expectation given information in period t and $\beta \in (0,1)$ is the discount factor. For higher values of $\phi_3 \in \mathbb{R}\setminus\{1\}$, these preferences imply higher levels of risk aversion if the utility kernel u_t is always positive, and vice versa for $u_t < 0$.[14] When $\phi_3 \neq 0$, Epstein–Zin preferences disentangle risk aversion from the intertemporal elasticity of substitution (IES); otherwise, (27) simplifies to standard expected utility.

The utility kernel displays separability between consumption c_t and hours worked h_t

$$u_t \equiv d_t\left[\frac{1}{1-\phi_2}\left(\left(\frac{c_t - bc_{t-1}}{z_t^*}\right)^{1-\phi_2} - 1\right) + \phi_0\frac{(1-h_t)^{1-\phi_1}}{1-\phi_1}\right], \tag{28}$$

where b controls the degree of internal habit formation.[15] The variable $d_t \equiv \exp\left\{\sigma_d \epsilon_{d,t}\right\}$ with $\epsilon_{d,t} \sim \mathcal{NID}(0,1)$ is a preference shock. The AR(1) term in d_t is omitted to ensure that variations in long-term yields and term premia are primarily explained in our model by consumption and inflation dynamics and not by persistent preference shocks. We simultaneously include habit formation and Epstein–Zin preferences because the New Keynesian model needs both to jointly match macro and financial moments (see Hordahl *et al.*, 2008; Binsbergen *et al.*, 2012).

The budget constraint at time t reads

$$c_t + \frac{i_t}{\Upsilon_t} + b_t + T_t = w_t h_t + r_t^k k_t + \frac{b_{t-1}\exp\left\{r_{t-1}^b\right\}}{\pi_t} + div_t^h. \tag{29}$$

Resources are spent on consumption, investment i_t, a one-period deposit b_t in the financial intermediary at the net nominal risk-free deposit rate r_t^b, and a lump-sum tax T_t. The variable Υ_t denotes a deterministic trend in the real relative price of investment: $\log\Upsilon_{t+1} = \log\Upsilon_t + \log\mu_{\Upsilon,ss}$. Letting w_t be the real wage and r_t^k the real price of capital k_t, resources consist of labour income $w_t h_t$, income from capital services sold to firms $r_t^k k_t$, real returns from deposits in the previous period (where $\pi_t \equiv P_t/P_{t-1}$ is gross inflation), and dividends to households div_t^h. These dividends come from the profits of the monopolistic competitors and the profits of the financial intermediator. Since firms and the financial operator could operate at a loss, these dividends may be negative.

The law of motion for k_t is

$$k_{t+1} = (1-\delta)k_t + i_t - \frac{\kappa}{2}\left(\frac{i_i}{k_t} - \psi\right)^2 k_t, \tag{30}$$

where $\kappa \geq 0$ introduces capital adjustment costs as in Jermann (1998). The constant ψ ensures that these adjustment costs are zero along the balanced growth path of the economy.

14. As shown in Table 1, for all estimated models, the steady-state value of u_t, u_{ss}, is substantially below zero.

15. The constant $-1/(1-\phi_2)$ guarantees a stable level of u_t and V_t in the steady state when ϕ_2 is close to one. Utility from habit-adjusted consumption is expressed relative to the deterministic trend in the economy z_t^* (to be defined later) to guarantee the existence of a balanced growth path.

TABLE 1
Estimation results

	No feedback	With feedback			
	$\mathcal{M}_0$	$\mathcal{M}_{FB}$	$\mathcal{M}_{FB}^{RRA}$	$\mathcal{M}_{FB,Taylor}$	$\mathcal{M}_{FB,Taylor}^{RRA}$
β	0.9995 (0.0001)	0.9972 (0.0006)	0.9970 (0.0006)	0.9974 (0.0007)	0.9969 (0.0004)
b	0.6720 (0.0526)	0.6973 (0.0363)	0.7054 (0.0413)	0.7094 (0.0373)	0.7152 (0.0257)
h_{ss}	0.3427 (0.0011)	0.3395 (0.0008)	0.3392 (0.0006)	0.3396 (0.0008)	0.3373 (0.0013)
ϕ_2	0.9757 (0.2668)	0.7082 (0.1307)	0.7387 (0.1824)	0.6097 (0.1707)	0.4753 (0.0906)
RRA	615.7 (26.96)	28.78 (24.10)	5	29.323 (27.8351)	5
κ	5.3986 (0.8263)	10.129 (1.1215)	11.828 (1.3219)	9.5938 (1.3390)	10.714 (1.4113)
α	0.8101 (0.0066)	0.7951 (0.0093)	0.8349 (0.0081)	0.7923 (0.0093)	0.7692 (0.0121)
ρ_r	0.6491 (0.0288)	0.8585 (0.0219)	0.7944 (0.0411)	0.8642 (0.0240)	0.8780 (0.0296)
β_π	1.2668 (0.1512)	2.0891 (0.1674)	3.0414 (0.2290)	2.0778 (0.1683)	3.1426 (0.4243)
β_y	0.0315 (0.0257)	0.2246 (0.0467)	0.2786 (0.0337)	0.2034 (0.0504)	0.2899 (0.0543)
$\mu_{\Upsilon,ss}$	1.0012 (0.0011)	1.0012 (0.0013)	1.0010 (0.0013)	1.0012 (0.0013)	1.0007 (0.0010)
$\mu_{z,ss}$	1.0052 (0.0005)	1.0053 (0.0006)	1.0054 (0.0005)	1.0054 (0.0006)	1.0053 (0.0005)
ρ_a	0.7450 (0.0557)	0.7918 (0.0216)	0.7622 (0.0241)	0.7855 (0.0226)	0.6691 (0.0365)
ρ_G	0.8033 (0.0950)	0.8122 (0.0497)	0.8545 (0.0535)	0.8325 (0.0505)	0.8281 (0.0401)
g_{ss}/y_{ss}	0.2062 (0.0029)	0.2071 (0.0029)	0.2071 (0.0031)	0.2083 (0.0034)	0.2180 (0.0016)
σ_a	0.0161 (0.0020)	0.0121 (0.0018)	0.0146 (0.0017)	0.0124 (0.0017)	0.0131 (0.0016)
σ_G	0.0422 (0.0122)	0.0534 (0.0105)	0.0479 (0.0104)	0.0511 (0.0106)	0.0540 (0.0075)
σ_d	0.0131 (0.0020)	0.0093 (0.0015)	0.0094 (0.0018)	0.0083 (0.0019)	0.0062 (0.0011)
π_{ss}	1.0121 (0.0006)	1.0118 (0.0005)	1.0104 (0.0005)	1.0127 (0.0009)	1.0181 (0.0012)
ω	–	**0.8542** (0.1632)	**0.9906** (0.0054)	0.8505 (0.1795)	0.9909 (0.0070)
β_{xhr}	–	–		**−0.1721** (0.1664)	**−1.5829** (0.272)
Memo					
IES	0.053	0.062	0.056	0.065	0.080
u_{ss}	−2.273	−1.766	−1.837	−1.620	−1.441
ϕ_3	−1466.0	−83.05	−13.04	−94.00	−17.44

The reported estimates are from the second step in GMM using the optimal weighting matrix with ten lags in the Newey–West estimator, with standard errors shown in parenthesis. For $\mathcal{M}_{FB}^{RRA}$ and $\mathcal{M}_{FB,Taylor}^{RRA}$, the value of the *RRA* is restricted to five and not estimated.

6.2. *The financial intermediary*

The representative household makes one-period deposits b_t in a perfectly competitive financial intermediary, which invests these funds in short- and long-term government bonds. The household may overdraw this deposit (*i.e.* $b_t < 0$), in which case the financial intermediary may short the bonds.

We make two assumptions about how the financial intermediary operates. First, the financial intermediary invests a fraction $\omega \in [0,1]$ of the deposits it receives in bonds of maturity $L > 1$ and the remaining fraction $1-\omega$ in one-period bonds. We set L to 10 years, but other maturities could be considered. The value of ω is determined by factors exogenous to the model. In our estimation, we will treat ω as a free parameter to be inferred from the data.

Second, the financial intermediary is risk neutral. This assumption, together with free entry and perfect competition, means that the intermediary pays a deposit rate r_t^b equal to the ex-ante holding period return on the invested bond portfolio

$$r_t^b \equiv (1-\omega)\times hr_{t,1} + \omega \times hr_{t,L}. \tag{31}$$

To parse this expression, let the ex-ante holding period return on the kth bond be

$$hr_{t,k} \equiv \mathbb{E}_t\left[\log P_{t+1,k-1} - \log P_{t,k}\right], \tag{32}$$

where $P_{t,k}$ is the nominal price in period t of a zero-coupon bond maturing in period $t+k$. The excess holding period return is $xhr_{t,k} \equiv hr_{t,k} - r_t$, where r_t is the one-period nominal policy rate set by the central bank. Since $hr_{t,1} = r_t$ and $xhr_{t,1} = 0$, we get

$$r_t^b \equiv (1-\omega)\times hr_{t,1} + \omega \times hr_{t,L} = r_t + \omega \times xhr_{t,L}. \tag{33}$$

To understand (33), suppose for a moment that $\omega = 0$. In this case, the financial intermediary only holds the one-period bond and (33) simplifies to $r_t^b = r_t$. Hence, our framework recovers the specification in most New Keynesian models where the deposit rate equals the one-period policy rate set by the central bank. When $\omega > 0$, there is a feedback effect from long-term bonds to the real economy as the excess holding period return affects r_t^b and the household's consumption decision. For instance, an increase in $xhr_{t,L}$ due to a higher term premium during a recession will raise the deposit rate and encourage the household to postpone consumption. Given that $xhr_{t,L}$ is non-zero because of uncertainty, this feedback effect from long-term bonds to the real economy operates through precautionary saving. We will exploit this point below to derive an efficient perturbation solution to our model.

The price of government bonds with a maturity exceeding one period is determined in a standard way using the household stochastic discount factor

$$P_{t,k} = \mathbb{E}_t\left[\beta \frac{\lambda_{t+1}}{\lambda_t}\frac{1}{\pi_{t+1}} P_{t+1,k-1}\right], \tag{34}$$

for $k = 2, 3..., \mathcal{K}$ with $P_{t,1} = \exp\{-r_t\}$. The nominal yield curve with continuous compounding is then given by $r_{t,k} = -\frac{1}{k}\log P_{t,k}$ for $k = 2, 3, ..., \mathcal{K}$.

To keep the model simple, we impose the previous assumptions without explicit microfoundations. However, extra effort and heavier notation could flesh out some of the structure behind them. For example, with respect to ω, regulation forces financial institutions to keep large shares of their portfolios in short maturities, regardless of their preferred investment strategies. Endogenizing the factors determining ω is beyond the scope of this article.

With respect to bond pricing, we could formulate a segmented markets model where the household can trade in bonds with a maturity exceeding one period, but it cannot invest in a one-period bond or trade in longer maturity bonds to replicate a one-period bond. Nevertheless, the household still requires deposits in the financial intermediary for liquidity services (*i.e.* payments are settled through the deposit account). In this setup, the representative household will price bonds using (34) for $k = 2, 3..., \mathcal{K}$. See Chien *et al.* (2014) for an example of a general framework for writing such a model. Market segmentation can be motivated by the observation that large financial institutions bid up the price of one-period bonds to use them for repo collateral or other transaction services (see also the next paragraph). The no one-period bond replication can occur if the transaction costs (including tax obligations) of short-term bond trading are high.

Finally, as in much of the literature (see, *e.g*. Binsbergen *et al.* 2012), the pricing condition for the one-period bond within our model is entirely determined by the policy rate set by the monetary authority. This assumption is motivated by the crucial role of liquidity considerations in the demand for the one-quarter bond, which suppress its implied yield (Krishnamurthy and Vissing-Jorgensen, 2012). This effect is not present if the pricing condition (34) is used for $k=1$.

Fisher (2015) provides a structural interpretation of a similar environment by introducing a liquidity demand for short-term Treasuries. Furthermore, Fisher shows how such a microfoundation does not materially affect the predictions of the more reduced-form model.

Our framework is also related to the risk-premium shocks in Smets and Wouters (2007), where an exogenous shock drives a wedge between the policy rate and the interest rate faced by the households. When equation (33) is substituted into the consumption Euler equation, that is, $\mathbb{E}_t\left[\beta\lambda_{t+1}\exp\left\{r_t^b\right\}/\pi_{t+1}\right]=\lambda_t$ with λ_t, denoting the marginal utility of habit-adjusted consumption, we obtain a similar wedge, except that our wedge is endogenously generated. Also, if we were to follow Smets and Wouters (2007) and use a log-linear approximation to our model, then $xhr_{t,L}=0$ for all t and (33) would reduce to the specification where $r_t^b=r_t$, even when $\omega>0$.

6.3. *Firms*

A perfectly competitive representative firm produces final output y_t by aggregating a continuum of intermediate goods $y_{i,t}$ using the technology $y_t=\left(\int_0^1 y_{i,t}^{\frac{\eta-1}{\eta}}di\right)^{\frac{\eta}{\eta-1}}$ with $\eta>1$. If we denote $P_{i,t}$ as the price of the ith good, we can find the demand function $y_{i,t}=\left(\frac{P_{i,t}}{P_t}\right)^{-\eta}y_t$, with aggregate price level $P_t\equiv\left[\int_0^1 P_{i,t}^{1-\eta}di\right]^{\frac{1}{1-\eta}}$.

The intermediate good i is produced by a monopolistic competitor given $y_{i,t}=a_t k_{i,t}^{\theta}\left(z_t h_{i,t}\right)^{1-\theta}$. Here, z_t is a deterministic trend that follows $\log z_{t+1}=\log z_t+\log\mu_{z,ss}$, and $\log a_{t+1}=\rho_a\log a_t+\sigma_a\epsilon_{a,t+1}$ where $\epsilon_{a,t}\sim\mathcal{NID}(0,1)$. We define $z_t^*\equiv\Upsilon_t^{\frac{\theta}{1-\theta}}z_t$, which denotes the technological trend in the economy.

The intermediate firms maximize the net present value of real profit with respect to capital, labour, and prices given a nominal rigidity. We consider price-setting à la Calvo (1983), where contracts expire with probability $1-\alpha$ in each period. Whenever a contract expires, firms set their optimal nominal prices, which otherwise are equal to past prices, that is, $P_{i,t}=P_{i,t-1}$.

6.4. *Monetary and fiscal policy*

A central bank sets the policy rate r_t based on a desire to stabilize the inflation gap $\log(\pi_t/\pi_{ss})$ and the output gap $\log\left(y_t/\left(z_t^*Y_{ss}\right)\right)$. Here, π_{ss} refers to steady-state inflation. The output gap is measured in terms of deviation from the deterministic trend in output, which equals z_t^* times production in the normalized steady state Y_{ss}.

Since in our model r_t^b depends on $xhr_{t,L}$, the central bank may also find it useful to account for the latter when setting its policy rate. For instance, a central bank may consider a larger reduction in r_t to offset the impact from a negative shock to economic activity and a subsequent increase in $xhr_{t,L}$ than would be required with $xhr_{t,L}=0$. More concretely, we postulate that monetary policy follows an augmented Taylor rule

$$\begin{aligned} r_t = & (1-\rho_r) r_{ss} + \rho_r r_{t-1} + (1-\rho_r)\left(\beta_\pi \log\left(\frac{\pi_t}{\pi_{ss}}\right) + \beta_y \log\left(\frac{y_t}{z_t^* Y_{ss}}\right)\right) \\ & + (1-\rho_r)\beta_{xhr}\left(xhr_{t,L} - \mathbb{E}\left[xhr_{t,L}\right]\right), \end{aligned} \tag{35}$$

where we omit monetary policy shocks as the literature has documented that they have a tiny effect on term premia.[16]

When implementing (35), we approximate $\mathbb{E}\left[xhr_{t,L}\right]$ by $\mathbb{E}_t\left[(1-\gamma)\sum_{l=0}^{\infty}\gamma^l xhr_{t+l,L}\right] \equiv X_{t,L}$ with $\gamma = 0.9999$. This variable has the convenient representation $X_{t,L} = (1-\gamma) xhr_{t,L} + \gamma \mathbb{E}_t\left[X_{t+1,L}\right]$. In contrast, the steady-state value of $xhr_{t,L}$ equals zero and is a poor approximation of $\mathbb{E}\left[xhr_{t,L}\right] \neq 0$. Finally, if we were to solve our model by a log-linearisation, then $xhr_{t,L} = 0$ for all t and (35) would reduce to the standard Taylor rule even if $\beta_{xhr} \neq 0$.

Government consumption $g_t \equiv G_t z_t^*$ grows with the economy as in Rudebusch and Swanson (2012), where

$$\log\left(\frac{G_{t+1}}{G_{ss}}\right) = \rho_G \log\left(\frac{G_t}{G_{ss}}\right) + \sigma_G \epsilon_{G,t+1}$$

and $\epsilon_{G,t+1} \sim \mathcal{NID}(0,1)$. Government consumption and the service on government debt are paid with lump-sum taxes. Given that a version of Ricardian equivalence holds in our economy, we do not need to specify the timing of these taxes and simply write the resource constraint of the economy as $y_t = c_t + i_t \Upsilon_t^{-1} + g_t$. Also because of Ricardian equivalence, it is always possible for the government to issue the required number of bonds to clear the bond market (although, in equilibrium, the total amount of deposits is zero and those issuances are therefore not required).

6.5. *Model aggregation*

The aggregated resource constraint in the goods market is $a_t k_t^\theta (z_t h_t)^{1-\theta} = s_{t+1} y_t$, where s_t is the price dispersion index. The dynamic of this endogenous state variable is

$$s_{t+1} = (1-\alpha)\tilde{p}_t^{-\eta} + \alpha \pi_t^\eta s_t, \tag{36}$$

where $\tilde{p}_t \equiv \tilde{P}_t / P_t$ and $\tilde{P}_t$ denotes the optimal nominal price in period t. The relation between inflation and the newly optimized prices is

$$1 = (1-\alpha)\tilde{p}_t^{1-\eta} + \alpha \pi_t^{\eta-1}. \tag{37}$$

There is an alternative, yet equivalent, representation of our model with complete markets described in Appendix A.13. In that representation of the model, the households do not need to rely on the financial intermediary to invest their savings in government bonds, but the Taylor rule depends instead on $\omega * xhr_{t,L}$ and $\omega * \rho_r * xhr_{t-1,L}$.

6.6. *An efficient perturbation approximation*

To solve the model, we first induce stationarity by eliminating trending variables with appropriate transformations; see Appendix A.14. The desired policy functions that characterize

16. This is illustrated in Rudebusch and Swanson (2012), who also show that it is the systematic part of monetary policy that has a large impact on term premia. Below, we report a similar finding.

the equilibrium dynamics of the model are then obtained by employing a third-order perturbation approximation. We require at least a third-order approximation to generate variation in $xhr_{t,L}$ and capture the feedback effects from long-term government bonds to the real economy. Our quarterly model with L set to reflect the 10-year yield has seven state variables and fifty-four control variables. The large number of control variables is required to compute all bond prices within the 10-year maturity range.

The standard approach to efficiently compute a higher-order perturbation approximation to DSGE models with a yield curve exploits the fact that bond prices beyond the policy rate do not affect allocations and prices (Binsbergen *et al.*, 2012; Andreasen and Zabczyk, 2015). Taking advantage of that property, the models are approximated by a two-step procedure, where the first step solves the model *without* bond prices exceeding one period, after which, in a second step, all remaining bond prices are computed recursively based on (34). This two-step procedure reduces the size of the simultaneous equation systems to be solved and, therefore, substantially reduces the computational burden of the approximation.

We cannot apply this two-step procedure to our model when $\omega > 0$ or $\beta_{xhr} \neq 0$ because the long-term bond price affects the deposit rate r_t^b and the policy rate through $xhr_{t,L}$ and, hence, allocations and prices. Fortunately, the terms associated with the perfect foresight solution of our model — that is $(\mathbf{g_x}, \mathbf{G_{xx}}, \mathbf{G_{xxx}})$ and $(\mathbf{h_x}, \mathbf{H_{xx}}, \mathbf{H_{xxx}})$ — can be found with the standard two-step procedure even when $\omega > 0$ or $\beta_{xhr} \neq 0$ because $xhr_{t,L}$ is equal to zero under perfect foresight. Once we have computed these terms, we only need to find the derivatives involving the perturbation parameter σ using the full model. This three-step procedure is formally described in Appendix A.15 and constitutes a new numerical contribution to the literature. Our three-step procedure allows us to compute a third-order solution to our model in just 3.7 seconds, whereas it takes 6.2 seconds when using the standard one-step perturbation algorithm of Binning (2013). This 40% improvement in speed greatly facilitates the estimation, as the perturbation approximation must be computed for many different parameter values.[17]

6.7. *Data and moments for GMM*

We employ the following quarterly time series to estimate our model: (1) consumption growth Δc_t, (2) investment growth Δi_t, (3) inflation π_t, (4) the one-quarter nominal yield r_t, (5) the 10-year nominal yield $r_{t,40}$, (6) the 10-year ex post excess holding period return $xhr_{t,40} \equiv \log\left(P_{t,39}/P_{t-1,40}\right) - r_{t-1}$, (7) the log ratio of government spending to GDP $\log(g_t/y_t)$, and (8) the log of hours $\log h_t$. The short- and long-term yields capture the slope of the yield curve, whereas the excess holding period return is included as a noisy proxy for the 10-year term premium. All series are stored in $\mathbf{data}_t$ with dimension 8×1. Our sample goes from 1961.Q3 to 2007.Q4. The end date is set to avoid the complications created by the zero lower bound of the nominal interest rate. See Appendix A.16 for a description of the data series.

We want to explore whether our model can match the mean, the variance, the contemporaneous covariances, and the persistence in the data. Hence, we let

$$\mathbf{q}_t \equiv \begin{bmatrix} \mathbf{data}_t \\ diag\left(\mathbf{data}_t \mathbf{data}_t'\right) \\ vech\left(\widetilde{\mathbf{data}}_t \widetilde{\mathbf{data}}_t'\right) \\ diag\left(\mathbf{data}_t \mathbf{data}_{t-1}'\right) \end{bmatrix}, \tag{38}$$

17. These computations are done in `Matlab` 2014a on a Fujitsu laptop with an Intel(R) Core(TM) i5-4200M CPU @ 2.50 GHz.

where $diag(\cdot)$ denotes the diagonal elements of a matrix and $\widetilde{\mathbf{data}}_t$ refers to the first six elements of $\mathbf{data}_t$. We omit moments on the contemporaneous correlation relating to $\log(g_t/y_t)$ and $\log h_t$ due to the parsimonious specification of government spending and the labor market in our model. Letting θ contain the structural parameters, our GMM estimator is given by

$$\hat{\theta}_{GMM} = \underset{\theta \in \Theta}{\arg\min} \left(\frac{1}{T} \sum\nolimits_{t=1}^{T} \mathbf{q}_t - \mathbb{E}[\mathbf{q}_t(\theta)] \right)' \mathbf{W} \left(\frac{1}{T} \sum\nolimits_{t=1}^{T} \mathbf{q}_t - \mathbb{E}[\mathbf{q}_t(\theta)] \right).$$

Here, $\mathbf{W}$ is a positive definite weighting matrix and $\mathbb{E}\left[\mathbf{q}_t\left(\theta\right)\right]$ contains the model-implied moments computed in closed form using the above formulas. We use the conventional two-step implementation of GMM by letting $\mathbf{W}_T = diag\left(\hat{\mathbf{S}}_{mean}^{-1}\right)$ in a preliminary first step to obtain $\hat{\theta}^{step1}$, where $\hat{\mathbf{S}}_{mean}$ denotes the long-run variance of $\frac{1}{T}\sum_{t=1}^{T}\mathbf{q}_t$ when re-centered around its sample mean. Our final estimates $\hat{\theta}^{step2}$ are obtained using the optimal weighting matrix $\mathbf{W}_T = \hat{\mathbf{S}}_{\hat{\theta}^{step1}}^{-1}$, where $\hat{\mathbf{S}}_{\hat{\theta}^{step1}}$ denotes the long-run variance of our moments re-centered around $\mathbb{E}\left[\mathbf{q}_t\left(\hat{\theta}^{step1}\right)\right]$. The long-run variances in both steps are estimated by the Newey–West estimator using 10 lags, but our results are robust to using more lags.

We estimate all structural parameters in our model except for a few poorly identified ones. We let $\delta = 0.025$ and $\theta = 0.36$ as typically considered for the U.S. economy. We also impose $\eta = 6$ to get an average markup of 20%, and we let $\phi_1 = 4$ to obtain a Frisch labour supply elasticity in the neighborhood of 0.5.[18]

7. RESULTS

7.1. *Estimation results I: the benchmark model*

As a benchmark, we first estimate our model without feedback effects from long-term bonds by imposing $\omega = 0$ and $\beta_{xhr} = 0$. We call this version of the model $\mathcal{M}_0$. The estimated parameters in Table 1 are standard, with large investment adjustment costs $(\hat{\kappa} = 5.40)$, little curvature in the utility of consumption $\left(\hat{\phi}_2 = 0.98\right)$, and sizeable habits $\left(\hat{b} = 0.67\right)$. The latter implies a low steady-state intertemporal elasticity of substitution ($IES_{ss} = 0.053$), which in the presence of internal habits is

$$IES_{ss} = \frac{1}{\phi_2} \left[\frac{\left(1 - \frac{b}{\mu_{z^*,ss}}\right)(\mu_{z^*,ss} - \beta b)}{\mu_{z^*,ss} + b\beta + \beta b^2 \mu_{z^*,ss}^{-1}} \right], \tag{39}$$

or $IES_{ss} \approx \frac{1}{\phi_2} \frac{(1-b)^2}{1+b+b^2}$ with $\mu_{z^*,ss} \approx 1$ and $\beta \approx 1$. As in much of the macro-finance literature, we find extreme relative risk aversion ($\widehat{RRA} = 615.7$), even when accounting for a variable labour supply as in Swanson (2012). Using the formulae provided in Swanson (2013), (28) implies

$$RRA = \frac{\phi_2}{\frac{1-b\mu_{z^*,ss}^{-1}}{1-\beta b} + \frac{\phi_2}{\phi_1}\frac{W_{ss}(1-h_{ss})}{C_{ss}}} + \phi_3 \frac{1-\phi_2}{\frac{1-b\mu_{z^*,ss}^{-1}}{1-\beta b} - \frac{\left(1-b\mu_{z^*,ss}^{-1}\right)^{\phi_2}}{1-\beta b} C_{ss}^{\phi_2 - 1} + \frac{W_{ss}(1-h_{ss})}{C_{ss}}\frac{1-\phi_2}{1-\phi_1}}, \tag{40}$$

18. The Frisch labour supply in our model is $\frac{1}{\phi_1}\left(\frac{1}{h_{ss}} - 1\right)$ and hence is affected by the steady-state labour supply h_{ss}, which is close to 1/3 (Table 1).

where W_{ss} and C_{ss} are the real wage and consumption in the normalized steady state.[19] Our estimated level of risk aversion is clearly too high to be consistent with the micro-evidence. For instance, Barsky *et al.* (1997) find an *RRA* between 3.8 and 15.7 in surveys, and Mehra and Prescott (1985) argue that a plausible level of relative risk aversion should not exceed 10. We also find moderate nominal frictions, with prices being re-optimized roughly every fifth quarter $\left(\hat{\alpha}=0.81\right)$, and a central bank assigning more weight to stabilize inflation than output ($\hat{\beta}_\pi=1.27$ versus $\hat{\beta}_y=0.03$), subject to smoothing changes in the policy rate ($\hat{\rho}_r=0.65$).

Table 2 shows that $\mathcal{M}_0$ matches all means, in particular the short- and long-term interest rates of 5.6% and 7.0%, respectively. We only match the mean inflation rate of 3.8% due to a large precautionary saving correction that lowers the annual steady-state inflation rate of $4\log\pi_{ss}=4.8\%$ to obtain a model-implied inflation rate of 3.4%. $\mathcal{M}_0$ also successfully matches the variability in the data (except for a too low standard deviation in the 10-year excess holding period return), the first-order autocorrelations, and the contemporaneous correlations.

The implied term premia in $\mathcal{M}_0$ are particularly interesting. As in Rudebusch and Swanson (2012), we define term premia as $TP_{t,k}=r_{t,k}-\tilde{r}_{t,k}$, where $\tilde{r}_{t,k}$ is the yield-to-maturity on a zero-coupon bond $\tilde{P}_{t,k}$ under a risk-neutral valuation, that is, $\tilde{P}_{t,k}=e^{-r_t}\mathbb{E}_t\left[\tilde{P}_{t+1,k-1}\right]$. The mean of the 10-year term premium $TP_{t,40}$ in $\mathcal{M}_0$ is 145 basis points, which is close to the average slope of the yield curve (139 basis points) that serves as an observable proxy for the average term premium. The standard deviation of the 10-year term premium in $\mathcal{M}_0$ is 116 basis points. This is in line with the variability in the 10-year term premium obtained in Gaussian affine term structure models for our sample: (1) the three- and four-factor models of Andreasen and Meldrum (2014) with bias-adjusted factor dynamics have a standard deviation of 105 and 115 basis points, respectively, and (2) the five-factor model of Adrian *et al.* (2013) has a standard deviation of 121 basis points. Finally, the variation in $TP_{t,k}$ within $\mathcal{M}_0$ is also consistent with another noisy measure of term premia variability, namely the standard deviation of the slope for the 10-year yield curve, which equals 139 basis points.

7.2. *Understanding the volatility of the term premium*

Although high risk aversion increases the mean term premium, it does not necessarily imply a highly volatile term premium. Instead, the volatility of $TP_{t,40}$ is directly related to the degree of heteroscedasticity in the stochastic discount factor $M_{t,t+1}\equiv\beta\lambda_{t+1}/\left(\lambda_t\pi_{t+1}\right)$, that is, the variation of $\mathbb{V}_t\left(M_{t,t+1}\right)$. Since the three shocks in our model are homoscedastic, $\mathcal{M}_0$ must be generating endogenous heteroscedasticity in $M_{t,t+1}$. But what is the source of this large heteroscedasticity?

Consider the effect of the price dispersion index s_t. Combining (36) and (37), we get

$$s_{t+1}=(1-\alpha)^{\frac{1}{1-\eta}}\left[1-\alpha\pi_t^{\eta-1}\right]^{\frac{\eta}{\eta-1}}+\alpha\pi_t^{\eta}s_t,$$

which is highly non-linear to ensure $s_t\geq 1$, as shown by Schmitt-Grohe and Uribe (2007). The first term in the expression for s_{t+1} does not generate much heteroscedasticity with $1-\alpha\pi_t^{\eta-1}$ being well below one given our estimates. The second term $\alpha\pi_t^{\eta}s_t$, on the other hand, may generate extreme levels of heteroscedasticity because $s_t\geq 1$ and we generally also have $\pi_t\geq 1$. In addition, the degree of heteroscedasticity is increasing in the mean of both variables. A higher value of π_{ss}

19. Household wealth is measured by the present value of lifetime consumption in (40). Given that W_{ss} and C_{ss} are unaffected by ϕ_3, we use (40) to back out the value of ϕ_3 for a given value of the *RRA*. See also Rudebusch and Swanson (2012) for a discussion of why $\mathcal{M}_0$ requires high risk aversion to match post-war U.S. data.

TABLE 2
Model fit

		No feedback	With feedback			
	Data	$\mathcal{M}_0$	$\mathcal{M}_{FB}$	$\mathcal{M}_{FB}^{RRA}$	$\mathcal{M}_{FB,Taylor}$	$\mathcal{M}_{FB,Taylor}^{RRA}$
Means						
$\Delta c_t \times 100$	2.439	2.350	2.376	2.367	2.404	2.283
$\Delta i_t \times 100$	3.105	2.847	2.852	2.756	2.867	2.547
$\pi_t \times 100$	3.757	3.404	3.348	3.308	3.314	3.484
$r_t \times 100$	5.605	5.567	5.505	5.433	5.483	5.576
$r_{t,40} \times 100$	6.993	6.924	6.937	6.817	6.860	6.989
$xhr_{t,40} \times 100$	1.724	2.090	1.519	1.403	1.466	1.442
$\log(g_t/y_t)$	−1.575	−1.578	−1.576	−1.577	-1.576	−1.579
$\log h_t$	−1.084	−1.083	−1.083	−1.083	−1.083	−1.083
Stds (in pct)						
Δc_t	2.685	2.701	2.669	2.620	2.698	2.680
Δi_t	8.914	8.687	8.941	8.566	8.921	8.700
π_t	2.481	2.669	2.723	2.540	2.702	2.665
r_t	2.701	2.520	2.558	2.508	2.527	2.598
$r_{t,40}$	2.401	2.282	2.055	2.134	2.056	2.134
$xhr_{t,40}$	22.978	12.930	13.088	9.763	12.704	9.116
$\log g_t/y_t$	8.546	8.264	9.330	9.944	9.502	10.192
$\log h_t$	1.676	2.396	1.855	2.260	1.801	1.924
Auto-correlations						
$corr(\Delta c_t, \Delta c_{t-1})$	0.254	0.238	0.326	0.342	0.347	0.359
$corr(\Delta i_t, \Delta i_{t-1})$	0.506	0.355	0.125	0.078	0.141	0.125
$corr(\pi_t, \pi_{t-1})$	0.859	0.824	0.877	0.912	0.863	0.786
$corr(r_t, r_{t-1})$	0.942	0.966	0.989	0.974	0.989	0.978
$corr\left(r_{t,40}, r_{t-1,40}\right)$	0.963	0.989	0.987	0.993	0.987	0.994
$corr\left(xhr_{t,40}, xhr_{t-1,40}\right)$	−0.024	−0.006	−0.005	−0.005	−0.006	−0.006
$corr(\log g_t/y_t, \log g_{t-1}/y_{t-1})$	0.9922	0.888	0.855	0.895	0.871	0.874
$corr(\log h_t, \log h_{t-1})$	0.792	0.543	0.561	0.609	0.536	0.471
$corr(\Delta c_t, \Delta i_t)$	0.594	0.518	0.518	0.546	0.529	0.568
$corr(\Delta c_t, \pi_t)$	−0.362	−0.313	−0.304	-0.225	−0.318	−0.304
$corr(\Delta c_t, r_t)$	−0.278	−0.212	−0.183	-0.210	−0.187	−0.193
$corr\left(\Delta c_t, r_{t,40}\right)$	−0.178	−0.111	−0.135	-0.107	−0.141	−0.114
$corr\left(\Delta c_t, xhr_{t,40}\right)$	0.271	0.495	0.371	0.482	0.356	0.475
$corr(\Delta i_t, \pi_t)$	−0.242	−0.452	−0.343	-0.219	−0.362	−0.409
$corr(\Delta i_t, r_t)$	−0.265	−0.151	−0.104	-0.098	−0.101	−0.086
$corr\left(\Delta i_t, r_{t,40}\right)$	−0.153	−0.057	−0.052	-0.044	−0.051	−0.035
$corr\left(\Delta i_t, xhr_{t,40}\right)$	0.021	0.706	0.249	0.630	0.253	0.711
$corr(\pi_t, r_t)$	0.628	0.938	0.838	0.898	0.832	0.793
$corr\left(\pi_t, r_{t,40}\right)$	0.479	0.822	0.892	0.906	0.880	0.829
$corr\left(\pi_t, xhr_{t,40}\right)$	−0.249	−0.379	−0.203	-0.209	−0.198	−0.235
$corr\left(r_t, r_{t,40}\right)$	0.861	0.847	0.831	0.821	0.830	0.821
$corr\left(r_t, xhr_{t,40}\right)$	−0.233	−0.150	−0.069	−0.124	−0.063	−0.123
$corr\left(r_{t,40}, xhr_{t,40}\right)$	−0.121	−0.053	−0.114	−0.073	−0.107	−0.064

All variables are expressed in annualised terms, except for $\log(g_t/y_t)$ and $\log h_t$.

increases π_t, but also the steady state of s_t, given that $\partial s_{ss}/\partial \pi_{ss} \geq 0$ for $\pi_{ss} \geq 1$. Figure 1 illustrates this effect by plotting a sample path from $\mathcal{M}_0$ with positive steady-state inflation $\left(\pi_{ss} = \hat{\pi}_{ss}^{GMM}\right)$ and one without $(\pi_{ss} = 1)$. When $\pi_{ss} > 1$ (left panels), we see more extreme observations and hence more heteroscedasticity than when $\pi_{ss} = 1$ (right panels). Note also how the capital stock and the price dispersion when $\pi_{ss} > 1$ attain very low and high values, respectively, just before observation 9,000, at which point the unpruned state-space system explodes. A similar divergence in the sample path does not appear for $\pi_{ss} = 1$, where the two approximations induce nearly identical time series. Thus, positive steady-state inflation serves as a channel to generate heteroscedasticity

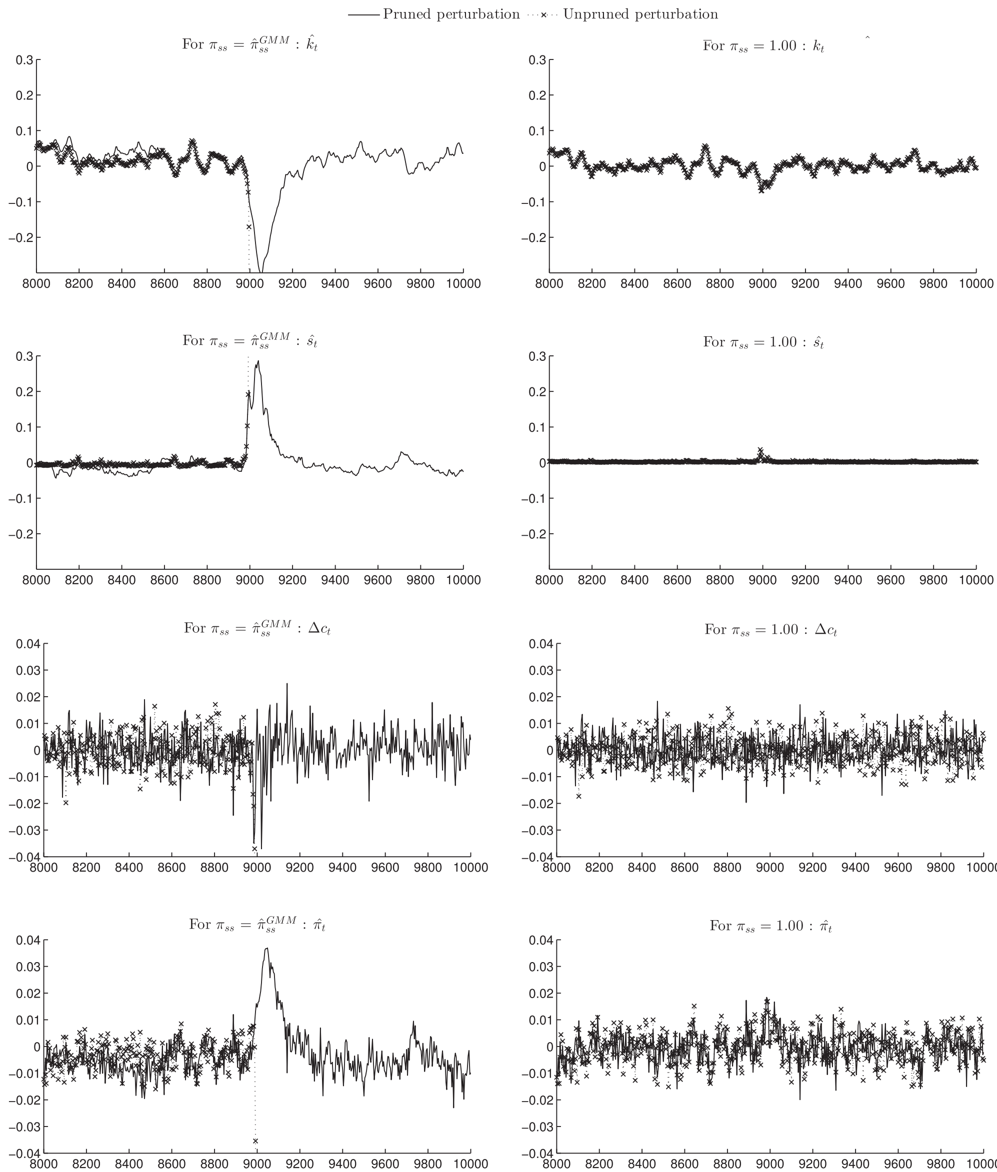

FIGURE 1

Simulated sample path.

The capital stock, the price dispersion index, and the inflation rate are expressed in deviation from the deterministic steady state, whereas consumption growth is de-meaned. Unless stated otherwise, all parameters are from $\mathcal{M}_0$. All variables are expressed at a quarterly level (we only display every fourth observation in each of the series to facilitate the plotting).

in s_t and, hence, variation in $\mathbb{V}_t\left(M_{t,t+1}\right)$, as required to produce the volatile 10-year term premium in our model.[20]

20. Swanson (2015) also emphasizes the importance of the price dispersion index as a source of heteroscedasticity in the New Keynesian model, but without pointing out the importance of positive steady-state inflation.

The first column in Table 4 shows the large effect of this channel. The standard deviation of the 10-year term premium falls from 116 basis points with $\hat{\pi}_{ss}^{GMM}$ to just 1.42 basis points when $\pi_{ss}=1.00$. To further analyse the effects of $\pi_{ss}>1$, we decompose risk premia into the market price of risk MPR_t times the quantity of risk. As in Cochrane (2001), we let $MPR_t = \mathbb{V}_t\left(M_{t,t+1}\right)/\mathbb{E}_t\left[M_{t,t+1}\right]$, implying that the quantity of risk $QoR_{t,k}$ equals $TP_{t,k}/MPR_t$. Table 4 shows that omitting positive steady-state inflation lowers the standard deviation in the MPR_t by a factor of 100, whereas the standard deviation of $QoR_{t,k}$ falls by a factor of 10^5. Hence, $\pi_{ss}>1$ causes a volatile term premium in our model mainly by increasing the variability in the quantity of risk. Table 4 shows that $\pi_{ss}>1$ also affects the mean term premium, which falls from 145 basis points to just 32 basis points when $\pi_{ss}=1.00$, although the RRA equals 615.7! This fall is due to a reduction in the mean of $\mathbb{V}_t\left(M_{t,t+1}\right)$, which lowers the MPR_t, whereas the level for the $QoR_{t,k}$ is nearly unaffected.

Thus, accounting for positive steady-state inflation serves as a key new channel to endogenously generate heteroscedasticity in the New Keynesian model and produce a 10-year term premium with the desired level and variability. Importantly, an unpruned third-order approximation to our model results in explosive sample paths and is unable to "detect" this novel channel, which we uncover by using our pruning scheme for a third-order perturbation.

7.3. *Estimation results II: the first feedback effect*

Our next step is to introduce the first feedback effect from long-term bonds by allowing $\omega \geq 0$, while still maintaining that the central bank does not respond to the excess holding period return ($\beta_{xhr}=0$). We call this version of the model $\mathcal{M}_{FB}$. Table 1 shows that the financial intermediary is estimated to allocate a large fraction of its investments to long-term bonds with $\hat{\omega}=0.85$. A standard $t-$test rejects the null hypothesis of $\omega=0$ at conventional significance levels, which provides support for our first feedback channel. Another important property of $\mathcal{M}_{FB}$ is the estimated RRA, which is only 28.8, and thus much lower than in $\mathcal{M}_0$. For the remaining parameters, we find minor changes compared to $\mathcal{M}_0$, except for the policy rule and investment adjustment costs.

Table 2 shows that $\mathcal{M}_{FB}$ fits the considered moments despite its lower risk aversion. To quantify the performance of $\mathcal{M}_{FB}$ compared to $\mathcal{M}_0$, Table 3 reports objective functions from our two-step GMM procedure. Only the objective functions from the first step use the same weighting matrix and are, therefore, comparable across models. They show that $\mathcal{M}_{FB}$ fits the data better than $\mathcal{M}_0$ (14.676 versus 16.929).[21]

However, risk aversion in $\mathcal{M}_{FB}$ is estimated very imprecisely with a large standard error of 24, and it is likely that the RRA can be lowered further with only a minor reduction in the goodness of fit. Consistent with the micro-evidence provided in Barsky *et al.* (1997), we restrict the RRA to 5 and re-estimate our model with the first feedback effect. Table 2 verifies our conjecture as this restricted model $\mathcal{M}_{FB}^{RRA}$ with low risk aversion provides nearly the same fit as the unrestricted model. In particular, $\mathcal{M}_{FB}^{RRA}$ matches the slope of the yield curve, while simultaneously fitting key moments for the five macro variables. We also note from Table 3 that $\mathcal{M}_{FB}^{RRA}$ provides a better overall fit to the data than $\mathcal{M}_0$. Here, we report the P-value from the J-test for model misspecification, showing that we are unable to reject $\mathcal{M}_{FB}^{RRA}$ (and all the other models). However, this finding should be interpreted with caution as the J- test has low power due to our short sample ($T=186$). Finally, $\mathcal{M}_{FB}^{RRA}$ has a realistic 10-year term premium with a mean of 146 basis points and a standard deviation of 91 basis points, as seen from Table 4.

21. The estimates in step 1 are very similar to those reported for step 2 in Table 1, implying that the objective functions in step 1 serve as a good metric for model comparison.

TABLE 3
Model specification test

	No feedback	With feedback			
	$\mathcal{M}_0$	$\mathcal{M}_{FB}$	$\mathcal{M}_{FB}^{RRA}$	$\mathcal{M}_{FB,Taylor}$	$\mathcal{M}_{FB,Taylor}^{RRA}$
Objective function: Q^{step1}	16.929	14.676	16.868	14.674	16.558
Objective function: Q^{step2}	0.0887	0.0865	0.0835	0.0847	0.0863
Number of moments	39	39	39	39	39
Number of parameters	19	20	19	21	20
P-value	0.685	0.651	0.745	0.610	0.654

The objective function in step 1, Q^{step1}, is computed with the weighting matrix $\mathbf{W}_T = diag\left(\hat{\mathbf{S}}_{mean}^{-1}\right)$, where $\hat{\mathbf{S}}_{mean}$ denotes the variance of the sample moments as computed by the Newey–West estimator using ten lags. The objective function in step 2, Q^{step2}, is computed using the optimal weighting matrix with ten lags in the Newey–West estimator. The P-value is for the J-test for model misspecification based on the objective function in step 2.

TABLE 4
Decomposing the 10-year term premium

	$TP_{t,40}$	$\mathbb{V}_t\left(M_{t,t+1}\right)$	Market price of risk	Quantity of risk
Means ($\pi_{ss} = \hat{\pi}_{ss}^{GMM}$)				
$\mathcal{M}_0$	145.20	0.0321	0.0327	0.2881
$\mathcal{M}_{FB}$	151.45	0.0011	0.0012	3.2220
$\mathcal{M}_{FB}^{RRA}$	145.66	2.17×10^{-4}	2.21×10^{-4}	16.6781
$\mathcal{M}_{FB,Taylor}$	145.10	0.0011	0.0012	3.0662
$\mathcal{M}_{FB,Taylor}^{RRA}$	144.80	1.34×10^{-4}	1.37×10^{-4}	19.2431
Means ($\pi_{ss} = 1.00$)				
$\mathcal{M}_0$	32.19	0.0030	0.0030	0.2702
$\mathcal{M}_{FB}$	39.55	0.0003	0.0003	3.1839
$\mathcal{M}_{FB}^{RRA}$	88.95	1.17×10^{-4}	1.18×10^{-4}	18.8987
$\mathcal{M}_{FB,Taylor}$	36.84	3.01×10^{-4}	3.03×10^{-4}	3.0578
$\mathcal{M}_{FB,Taylor}^{RRA}$	85.22	6.88×10^{-5}	6.94×10^{-5}	30.9294
Stds ($\pi_{ss} = \hat{\pi}_{ss}^{GMM}$)				
$\mathcal{M}_0$	115.88	0.0503	0.0515	137.86
$\mathcal{M}_{FB}$	118.40	0.0012	0.0013	583.31
$\mathcal{M}_{FB}^{RRA}$	91.12	1.37×10^{-4}	1.41×10^{-4}	359.01
$\mathcal{M}_{FB,Taylor}$	117.18	0.0013	0.0013	339.50
$\mathcal{M}_{FB,Taylor}^{RRA}$	124.67	1.03×10^{-4}	1.06×10^{-4}	3948.46
Stds ($\pi_{ss} = 1.00$)				
$\mathcal{M}_0$	1.42	3.39×10^{-4}	3.55×10^{-4}	0.0219
$\mathcal{M}_{FB}$	1.16	2.76×10^{-5}	2.88×10^{-5}	0.2189
$\mathcal{M}_{FB}^{RRA}$	5.35	1.05×10^{-5}	1.09×10^{-5}	0.7277
$\mathcal{M}_{FB,Taylor}$	1.00	2.67×10^{-5}	2.78×10^{-5}	0.2119
$\mathcal{M}_{FB,Taylor}^{RRA}$	3.25	7.28×10^{-6}	7.50×10^{-6}	2.3241

Moments for the 10-year term premium $TP_{t,40}$ are reported in annualised basis points, whereas moments for the remaining variables are at a quarterly frequency and unscaled. Moments for the quantity of risk cannot be computed directly by the perturbation method (because $\mathbb{V}_t\left(M_{t,t+1}\right)$ and, hence, the market price of risk is zero in the steady state). Thus, we compute these moments from simulated sample paths of 1,000,000 observations for $TP_{t,40}$ and the market price of risk.

Thus, our first feedback effect goes a long way in resolving the bond risk premium puzzle described in Rudebusch and Swanson (2008) without postulating highly risk-averse households as in much of the existing literature (see Andreasen, 2012; Binsbergen *et al.*, 2012; Rudebusch and Swanson, 2012, among others).

7.4. *Understanding the first feedback effect*

We now explore the mechanisms that enable $\mathcal{M}_{FB}$ and $\mathcal{M}_{FB}^{RRA}$ to generate a large and variable term premium without relying on high risk aversion. Suppose $xhr_{t,L} > 0$ for $\omega = 0$ and consider increasing ω to some positive value less than one. This increase in ω lowers $\mathbb{E}_t\left[M_{t,t+1}\right]$ as $\mathbb{E}_t\left[M_{t,t+1}\right] = e^{-r_t - \omega \times xhr_{t,L}}$ and, hence, the current bond price because

$$P_{t,L} = \mathbb{E}_t\left[M_{t,t+1}\right]\mathbb{E}_t\left[P_{t+1,L-1}\right] + Cov_t\left(M_{t,t+1}, P_{t+1,L-1}\right).$$

This fall in $P_{t,L}$ increases $xhr_{t,L}$ according to (32). But a higher $xhr_{t,L}$ induces a further fall in $\mathbb{E}_t\left[M_{t,t+1}\right]$ and the current bond price $P_{t,L}$, which generates an even larger increase in $xhr_{t,L}$. That is, $\omega > 0$ causes a feedback multiplication effect that amplifies the level and variability in $xhr_{t,L}$. A way to see this feedback loop is to use a first-order approximation of the logarithmic and exponential function in (32) to obtain (see Appendix A.17):

$$xhr_{t,L} \approx \frac{1}{1 - \omega \frac{\mathbb{E}_t\left[P_{t+1,L-1}\right]}{P_{ss,L-1}}} \left[\left(\frac{\mathbb{E}_t\left[P_{t+1,L-1}\right]}{P_{ss,L-1}} - 1 \right)(r_t - r_{ss}) - Cov_t\left(\frac{M_{t,t+1}}{M_{ss,ss+1}}, \frac{P_{t+1,L-1}}{P_{ss,L-1}} \right) \right]. \tag{41}$$

The first term in (41) is the risk-neutral component of $xhr_{t,L}$, whereas $Cov_t\left(\frac{M_{t,t+1}}{M_{ss,ss+1}}, \frac{P_{t+1,L-1}}{P_{ss,L-1}}\right)$ is the required compensation by the risk-averse household for carrying risk. The expression in (41) demonstrates that both terms in $xhr_{t,L}$ are amplified by the factor $1/\left(1 - \omega \frac{\mathbb{E}_t\left[P_{t+1,L-1}\right]}{P_{ss,L-1}}\right)$ when $\omega > 0$. Hence, our model requires lower volatility in $M_{t,t+1}$ and, correspondingly, lower risk aversion, to match short- and long-term interest rates. Finally, extending the expression for the term premium in Rudebusch and Swanson (2012) to our model, it follows that

$$\begin{aligned} TP_{t,k} \approx & -\frac{1}{kP_{ss,k}} \mathbb{E}_t \left[\sum_{j=0}^{k-1} e^{\left\{ -\sum_{m=0}^{j-1} (r_{t+m} + \omega \times xhr_{t+m,L}) \right\}} Cov_{t+j}\left(M_{t+j,t+j+1}, P_{t+j+1,k-j-1}\right) \right] \\ & - \frac{1}{kP_{ss,k}} \mathbb{E}_t \left[e^{\left\{ -\sum_{m=0}^{k-1} \omega \times xhr_{t+m,L} \right\}} - 1 \right]. \end{aligned} \tag{42}$$

This expression shows how a higher level and variability in $xhr_{t,L}$ translates into a larger and more volatile term premium.

To illustrate the magnitude of the multiplication effect from long-term bonds on term premia, we momentarily set $\omega = 0$ in $\mathcal{M}_{FB}^{RRA}$, whereas all remaining parameters are as reported in Table 1. Omitting the first feedback channel reduces the mean of the 10-year term premium from 146 to 10 basis points and lowers the standard deviation from 91 to 4 basis points. Table 4 documents that the feedback channel from long-term bonds reduces the mean and standard deviation of $\mathbb{V}_t\left[M_{t,t+1}\right]$ by a factor of 100 in $\mathcal{M}_{FB}$ and $\mathcal{M}_{FB}^{RRA}$ compared to $\mathcal{M}_0$. This, in turn, leads to a similar reduction in the corresponding moments for the MPR_t. Hence, $\mathcal{M}_{FB}$ and $\mathcal{M}_{FB}^{RRA}$ generate a high and volatile term premium by increasing the quantity of risk. As observed for $\mathcal{M}_0$, Table 4 shows that $\pi_{ss} > 1$ is essential for $\mathcal{M}_{FB}$ and $\mathcal{M}_{FB}^{RRA}$ to generate the desired level and variability of the term premium even with the first feedback effect from long-term bonds. Hence, it would be hard to discover this

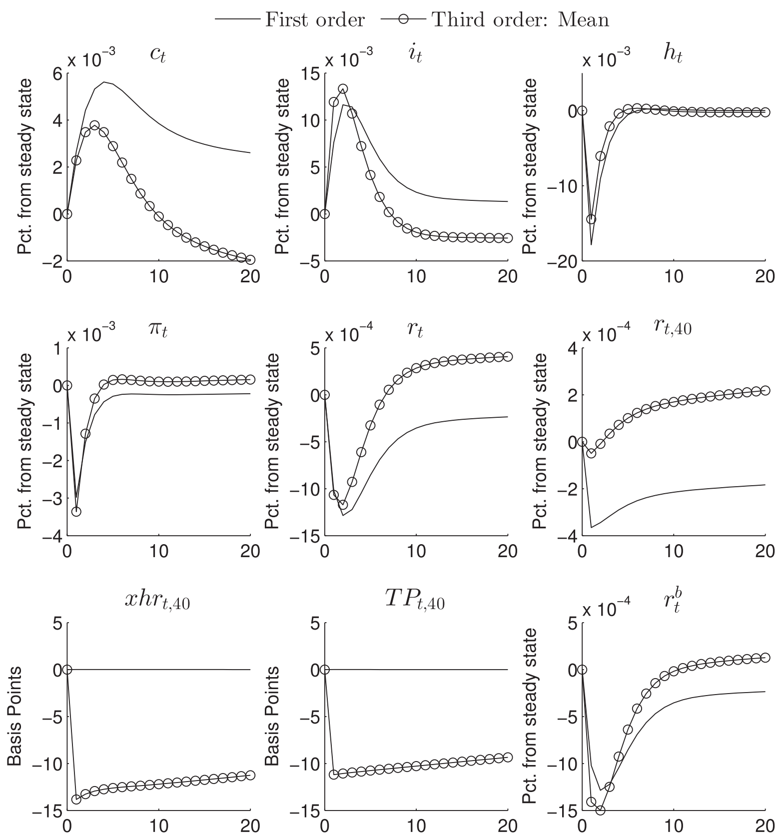

FIGURE 2

GIRFs: technology shock.

GIRFs following a positive one-standard-deviation shock to technology. The GIRFs are computed at the unconditional mean of the states using the estimated parameters for $\mathcal{M}_{FB,Taylor}^{RRA}$. All GIRFs are expressed in deviation from the steady state, except for the excess holding period return and term premium, which are expressed in annualised basis points from their unconditional means.

novel feedback effect without our pruning method, as an unpruned state-space system induces explosive sample paths when $\pi_{ss} > 1$.

7.5. *Estimation results III: the first and second feedback effects*

We now introduce our second feedback effect from long-term bonds by allowing the central bank to respond to the variation in $xhr_{t,L}$, which is closely related to term premia as shown in (41) and (42). That is, we let $\beta_{xhr} \neq 0$ and refer to this model as $\mathcal{M}_{FB,Taylor}$. Table 1 shows that this second feedback effect from long-term bonds has a small effect in the model as $\hat{\beta}_{xhr} = -0.17$, a point estimate not sufficiently far from zero to be statistically significant. However, our second feedback effect is larger when the *RRA* is restricted to 5 in $\mathcal{M}_{FB,Taylor}^{RRA}$. Here, $\hat{\beta}_{xhr} = -1.58$ and the response of the central bank to $xhr_{t,L}$ is now significant given a standard error of 0.27 for $\hat{\beta}_{xhr}$. That is, the model implies a reduction in the policy rate when term premia and $xhr_{t,L}$ increase, as

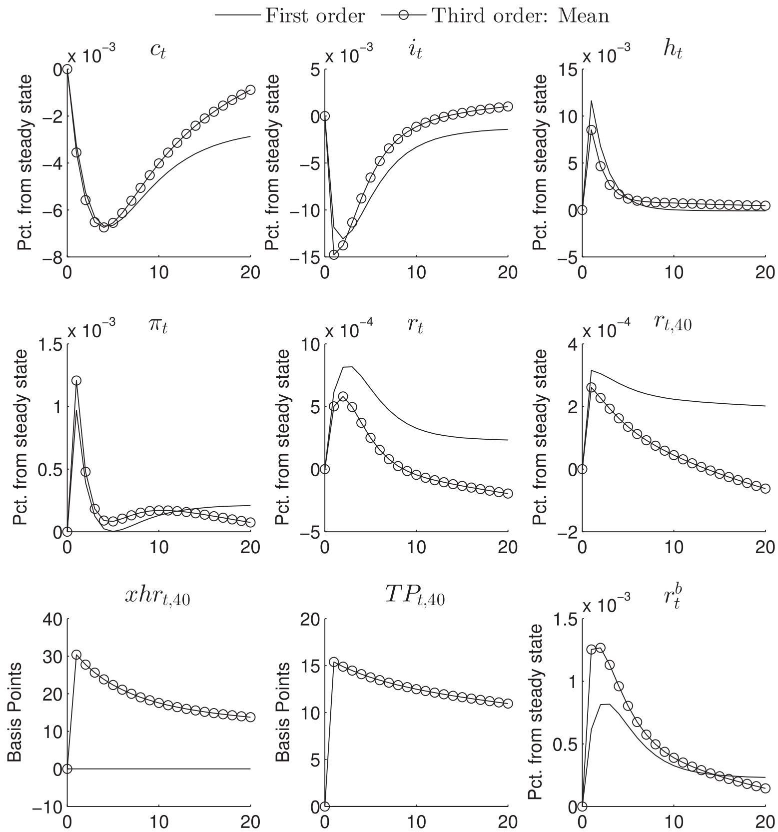

FIGURE 3

GIRFs: government shock.

GIRFs following a positive one standard deviation shock to government spending. The GIRFs are computed at the unconditional mean of the states using the estimated parameters for $\mathcal{M}_{FB,Taylor}^{RRA}$. All GIRFs are expressed in deviation from the steady state, except for the excess holding period return and term premium, which are expressed in annualised basis points from their unconditional means.

the central bank tries to offset the rise in the deposit rate with a lower policy rate. Although we end our sample in 2007.Q4, this finding is consistent with monetary policy during the recent financial crisis, where the Federal Reserve undertook vigorous policy measures to stimulate economic activity in response to elevated levels of term premia.

Table 2 shows that $\mathcal{M}_{FB,Taylor}^{RRA}$ matches most of the moments considered, in particular all mean values and the slope of the yield curve. Table 3 documents how $\mathcal{M}_{FB,Taylor}^{RRA}$ outperforms both $\mathcal{M}_{FB}^{RRA}$ and $\mathcal{M}_0$ in terms of overall goodness of fit, although $\mathcal{M}_{FB,Taylor}$ with unrestricted risk aversion does somewhat better than $\mathcal{M}_{FB,Taylor}^{RRA}$. The term premium is consistent with empirical moments, as $\mathcal{M}_{FB,Taylor}^{RRA}$ generates a 10-year term premium with a mean of 145 basis points and a standard deviation of 125 basis points (Table 4). As before, $\pi_{ss} > 1$ is essential for $\mathcal{M}_{FB,Taylor}$ and $\mathcal{M}_{FB,Taylor}^{RRA}$ to generate the desired level and variability in the term premium by activating the two feedback effects from long-term bonds to the real economy considered in this article.

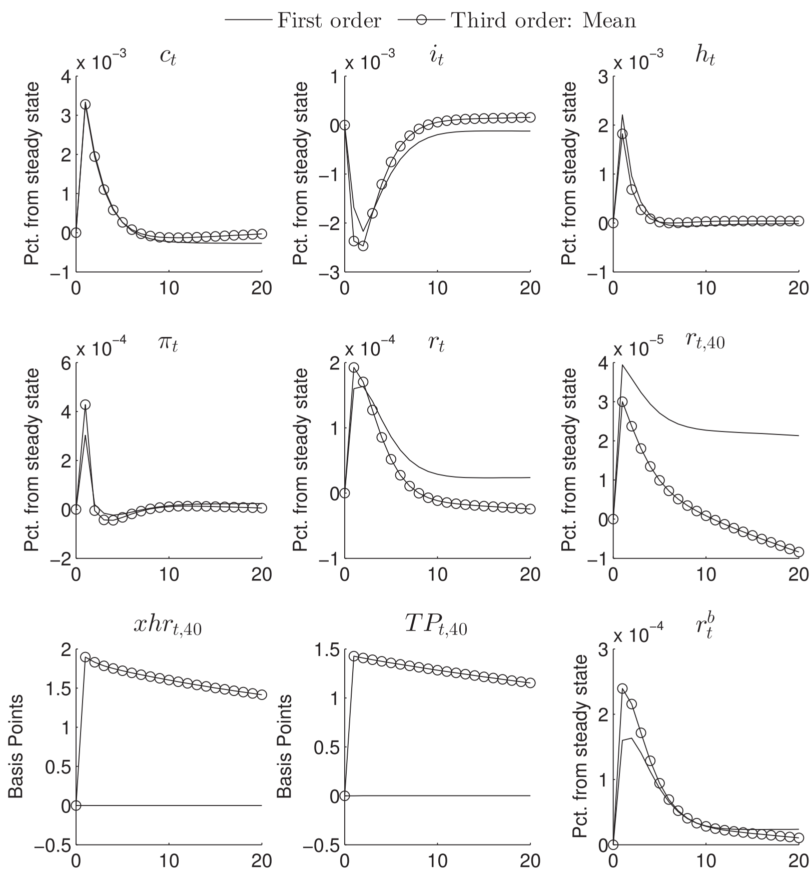

FIGURE 4

GIRFs: preference shock.

GIRFs following a positive one standard deviation shock to preferences. The GIRFs are computed at the unconditional mean of the states using the estimated parameters for $\mathcal{M}^{RRA}_{FB,Taylor}$. All GIRFs are expressed in deviation from the steady state, except for the excess holding period return and term premium, which are expressed in annualised basis points from their unconditional means.

7.6. *GIRFs and conditional GIRFs*

We now report the GIRFs following positive one-standard-deviation shocks in $\mathcal{M}^{RRA}_{FB,Taylor}$ to technology, government spending, and preferences (Figures 2–4). These functions are computed for a log-linearized solution and a third-order approximation using (25) with the relevant state variables at their unconditional means. Since all the GIRFs have the expected pattern, we direct attention to the effects of higher-order terms, that is, the differences between the marked and unmarked lines. Shocks to technology and government spending have substantial non-linear effects on consumption and investment, mainly because these shocks generate considerable variation in the ex ante excess holding period return and the term premium. This finding reveals that higher-order effects, and hence the amount of risk in the economy, affect real allocations in our model and, therefore, overturns the result of Tallarini (2000) that risk does not matter for these allocations.

A key advantage of computing second- and third-order approximations is that we can analyse the effects of different shocks conditional on the state of the economy. This is illustrated in

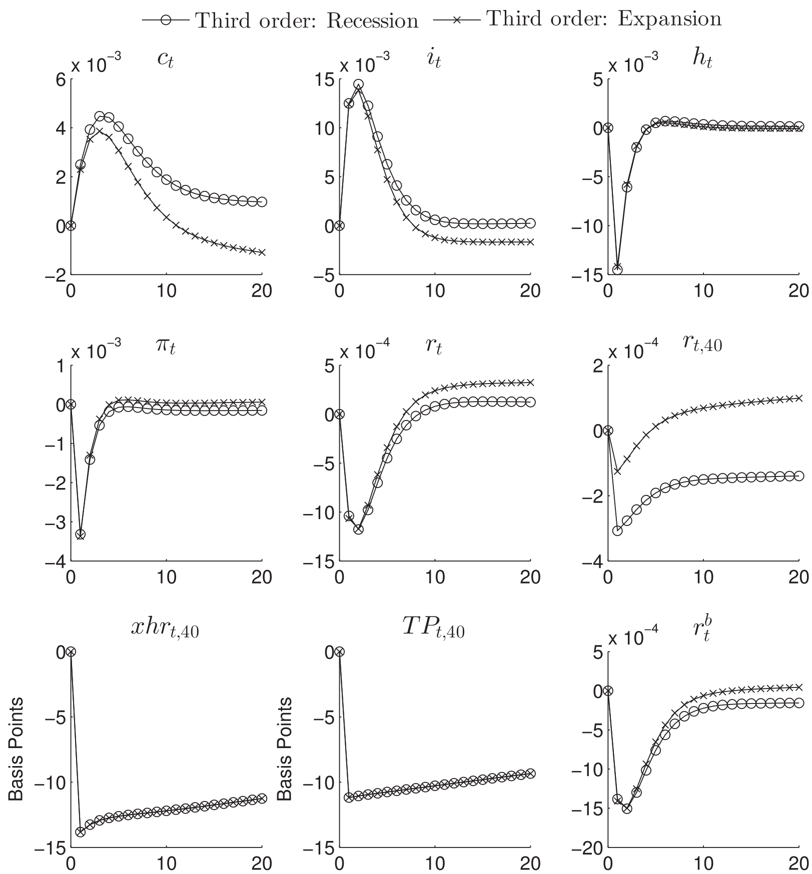

FIGURE 5

Conditional GIRFs: expansions vs. recessions.

GIRFs following a positive one standard deviation shock to technology using the estimated parameters for $\mathcal{M}^{RRA}_{FB,Taylor}$. The state values representing recessions are defined from episodes in a simulated sample path with negative output growth in the current and the previous two periods; otherwise, the economy is in expansion. The GIRFs are computed as the average across 500 draws from expansions and recessions. All GIRFs are expressed in deviation from the steady state, except for the excess holding period return and term premium, which are expressed in annualised basis points from their unconditional means.

Figure 5, where we show how the response of consumption, investment, and $r_{t,40}$ to a positive one-standard-deviation shock to technology is larger when the economy is in a recession than when it is not.[22] When the economy is in a recession, consumption and capital tend to be low, and hence the marginal utility of extra consumption and the marginal return of additional investment are higher than usual. A similar exercise is done in Figure 6, except that now we compare the situation where we condition on high versus low inflation.[23] When inflation is high, consumption, investment, and interest rates respond more vigorously than when inflation is low. Nominal rigidities are particularly damaging when inflation is high, since firms that are not able to change

22. We define a recession as negative output growth in the current and previous two periods. Otherwise, the economy is in expansion.

23. We define the high inflation regime as episodes where π_t is larger than the mean of inflation plus two standard deviations; otherwise, the economy is in a low inflation regime. The conditional GIRFs for government spending and preference shocks are omitted in the interest of space.

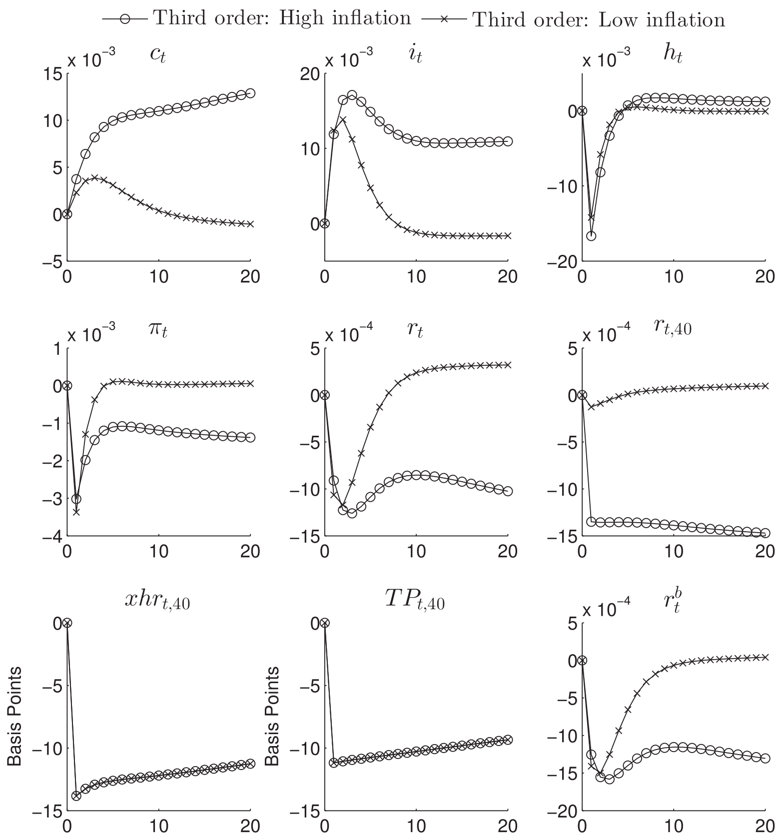

FIGURE 6

Conditional GIRFs: high vs. low inflation.

GIRFs following a positive one standard deviation shock to technology using the estimated parameters for $\mathcal{M}^{RRA}_{FB,Taylor}$. The state values representing high inflation are defined from episodes with inflation larger than the mean of inflation plus two standard deviations in a simulated sample path; otherwise, the economy is in a low inflation regime. The GIRFs are computed as the average across 500 draws from regimes of high and low inflation. All GIRFs are expressed in deviation from the steady state, except for the excess holding period return and term premium, which are expressed in annualised basis points from their unconditional means.

their prices are far from their optimal unconstrained price. A positive productivity shock translates into lower inflation through lower marginal costs and, hence, alleviates some of these pernicious effects of nominal rigidities. When inflation is low, nominal rigidities are less of a constraint on firm behaviour and a positive technology shock is, therefore, less useful for firms. The asymmetries in responses to shocks reported in Figures 5 and 6 document how the methods we present in our article allow researchers to probe deeper into the behaviour of their models and uncover economic mechanisms that would have otherwise remained hidden.

8. CONCLUSION

This article extends the pruning method by Kim *et al.* (2008) to third- and higher-order approximations, with special attention devoted to models solved up to third order. Conditions for the existence of first and second unconditional moments are derived, and their values are provided in closed form. The existence of higher-order unconditional moments in the form of skewness and kurtosis is also established. We also analyse GIRFs and provide simple closed-form expressions for these functions.

Our findings are significant as most of the existing moment-based estimation methods for linearized DSGE models now carry over to non-linear approximations. For approximations up to third order, this includes GMM estimation based on first and second unconditional moments and matching model-implied GIRFs to their empirical counterparts. When simulations are needed, our analysis also provides a foundation for indirect inference and SMM. These results are also relevant for Bayesian inference, as the moment conditions in optimal GMM estimation may be used to build a limited information likelihood function.

To illustrate one of these new estimation methods, we revisit the term structure implications of the New Keynesian model. We first demonstrate a new channel to amplify the level and time variation in term premia by accounting for positive steady-state inflation. Given this more realistic term premium, we then introduce two feedback effects from long-term bonds to the real economy, and we show that they enable the New Keynesian model to generate a high and variable term premium with the same low risk aversion as found in the micro-evidence. Our pruning scheme has greatly facilitated the discovery of these new channels and, hence, helped us to address the long-standing bond premium puzzle.

A. APPENDIX

A.1. *Pruned state-space beyond third order*

The pruned state-space system for the kth-order approximation based on the kth-order Taylor series expansions of $\mathbf{g}(\mathbf{x}_t,\sigma)$ and $\mathbf{h}(\mathbf{x}_t,\sigma)$ is obtained by: (1) decomposing the state variables into first-, second-, … , and kth-order effects, (2) setting up laws of motion for the state variables capturing only first-, second-, … , and kth-order effects, and (3) constructing the expression for control variables by preserving only effects up to kth-order. In comparison, the unpruned state-space system for the kth-order approximation is given by the kth-order Taylor series expansions of $\mathbf{g}(\mathbf{x}_t,\sigma)$ and $\mathbf{h}(\mathbf{x}_t,\sigma)$.

A.2. *Coefficients for the pruned state-space system at second order*

$$\mathbf{A}^{(2)} \equiv \begin{bmatrix} \mathbf{h_x} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{h_x} & \frac{1}{2}\mathbf{H_{xx}} \\ \mathbf{0} & \mathbf{0} & \mathbf{h_x}\otimes\mathbf{h_x} \end{bmatrix},$$

$$\mathbf{B}^{(2)} \equiv \begin{bmatrix} \sigma\eta & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \sigma\eta\otimes\sigma\eta & \sigma\eta\otimes\mathbf{h_x} & \mathbf{h_x}\otimes\sigma\eta \end{bmatrix},$$

$$\mathbf{c}^{(2)} \equiv \begin{bmatrix} \mathbf{0} \\ \frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2 \\ (\sigma\eta\otimes\sigma\eta)\, vec\left(\mathbf{I}_{n_e}\right) \end{bmatrix},$$

$$\mathbf{C}^{(2)} \equiv \begin{bmatrix} \mathbf{g_x} & \mathbf{g_x} & \frac{1}{2}\mathbf{G_{xx}} \end{bmatrix},$$

and

$$\mathbf{d}^{(2)} \equiv \frac{1}{2}\mathbf{g}_{\sigma\sigma}\sigma^2.$$

A.3. *Second order*: *stability*

First, note that all eigenvalues of $\mathbf{A}^{(2)}$ are strictly less than one. To see this, we work with

$$\begin{aligned}
p(\lambda) &= \left|\mathbf{A}-\lambda\mathbf{I}_{2n_x+n_x^2}\right| \\
&= \left|\begin{bmatrix} \mathbf{h_x}-\lambda\mathbf{I}_{n_x} & \mathbf{0}_{n_x\times n_x} & \mathbf{0}_{n_x\times n_x^2} \\ \mathbf{0}_{n_x\times n_x} & \mathbf{h_x}-\lambda\mathbf{I}_{n_x} & \frac{1}{2}\mathbf{H_{xx}} \\ \mathbf{0}_{n_x^2\times n_x} & \mathbf{0}_{n_x^2\times n_x} & \mathbf{h_x}\otimes\mathbf{h_x}-\lambda\mathbf{I}_{n_x^2} \end{bmatrix}\right| \\
&= \begin{vmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} \\ \mathbf{B}_{21} & \mathbf{B}_{22} \end{vmatrix} \\
&= |\mathbf{B}_{11}|\,|\mathbf{B}_{22}|,
\end{aligned}$$

where we let

$$\begin{aligned}
\mathbf{B}_{11} &\equiv \begin{bmatrix} \mathbf{h_x}-\lambda\mathbf{I}_{n_x} & \mathbf{0}_{n_x\times n_x} \\ \mathbf{0}_{n_x\times n_x} & \mathbf{h_x}-\lambda\mathbf{I}_{n_x} \end{bmatrix}, \\
\mathbf{B}_{12} &\equiv \begin{bmatrix} \mathbf{0}_{n_x\times n_x^2} \\ \frac{1}{2}\mathbf{H_{xx}} \end{bmatrix}, \\
\mathbf{B}_{21} &\equiv \begin{bmatrix} \mathbf{0}_{n_x^2\times n_x} & \mathbf{0}_{n_x^2\times n_x} \end{bmatrix},
\end{aligned}$$

and

$$\mathbf{B}_{22} \equiv \mathbf{h_x}\otimes\mathbf{h_x}-\lambda\mathbf{I}_{n_x^2}$$

and we use the fact that

$$\begin{vmatrix} \mathbf{U} & \mathbf{C} \\ \mathbf{0} & \mathbf{Y} \end{vmatrix} = |\mathbf{U}|\,|\mathbf{Y}|,$$

where $\mathbf{U}$ is an $m\times m$ matrix and $\mathbf{Y}$ is an $n\times n$ matrix. Hence,

$$p(\lambda) = \left|\begin{bmatrix} \mathbf{h_x}-\lambda\mathbf{I}_{n_x} & \mathbf{0}_{n_x\times n_x} \\ \mathbf{0}_{n_x\times n_x} & \mathbf{h_x}-\lambda\mathbf{I}_{n_x} \end{bmatrix}\right| \left|\mathbf{h_x}\otimes\mathbf{h_x}-\lambda\mathbf{I}_{n_x^2}\right| = \left|\mathbf{h_x}-\lambda\mathbf{I}_{n_x}\right| \left|\mathbf{h_x}-\lambda\mathbf{I}_{n_x}\right| \left|\mathbf{h_x}\otimes\mathbf{h_x}-\lambda\mathbf{I}_{n_x^2}\right|.$$

The eigenvalues are determined from $\left|\mathbf{h_x}-\lambda\mathbf{I}_{n_x}\right|=0$ or $\left|\mathbf{h_x}\otimes\mathbf{h_x}-\lambda\mathbf{I}_{n_x^2}\right|=0$. The absolute values of all eigenvalues of the first problem are strictly less than one by assumption. That is, $|\lambda_i|<1$ $i=1,2,...,n_x$. This is also the case for the second problem because the eigenvalues of $\mathbf{h_x}\otimes\mathbf{h_x}$ are $\lambda_i\lambda_j$ for $i=1,2,...,n_x$ and $j=1,2,...,n_x$.

A.4. *Second order*: *unconditional second moments*

For the variance, we have

$$\mathbb{V}\left(\mathbf{z}_{t+1}^{(2)}\right) = \mathbf{A}^{(2)}\mathbb{V}\left(\mathbf{z}_t^{(2)}\right)\left(\mathbf{A}^{(2)}\right)' + \mathbf{B}^{(2)}\mathbb{V}\left(\xi_{t+1}^{(2)}\right)\left(\mathbf{B}^{(2)}\right)'$$

as

$$\mathbb{E}\left[\mathbf{z}_t^{(2)}\left(\xi_{t+1}^{(2)}\right)'\right] = \mathbb{E}\begin{bmatrix} \mathbf{x}_t^f\epsilon_{t+1}' & \mathbf{x}_t^f(\epsilon_{t+1}\otimes\epsilon_{t+1}-vec(\mathbf{I}_{ne}))' & \mathbf{x}_t^f\left(\epsilon_{t+1}\otimes\mathbf{x}_t^f\right)' & \mathbf{x}_t^f\left(\mathbf{x}_t^f\otimes\epsilon_{t+1}\right)' \\ \mathbf{x}_t^s\epsilon_{t+1}' & \mathbf{x}_t^s(\epsilon_{t+1}\otimes\epsilon_{t+1}-vec(\mathbf{I}_{ne}))' & \mathbf{x}_t^s\left(\epsilon_{t+1}\otimes\mathbf{x}_t^f\right)' & \mathbf{x}_t^s\left(\mathbf{x}_t^f\otimes\epsilon_{t+1}\right)' \\ \left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\epsilon_{t+1}' & \left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)(\epsilon_{t+1}\otimes\epsilon_{t+1}-vec(\mathbf{I}_{ne}))' & \left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\epsilon_{t+1}\otimes\mathbf{x}_t^f\right)' & \left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\otimes\epsilon_{t+1}\right)' \end{bmatrix} = \mathbf{0}.$$

Now, we only need to compute $\mathbb{V}\left(\xi_{t+1}^{(2)}\right)$:

$$\mathbb{V}\left(\xi_{t+1}^{(2)}\right) = \mathbb{E}\left[\begin{bmatrix} \epsilon_{t+1} \\ \epsilon_{t+1}\otimes\epsilon_{t+1} - vec\left(\mathbf{I}_{n_e}\right) \\ \epsilon_{t+1}\otimes\mathbf{x}_t^f \\ \mathbf{x}_t^f\otimes\epsilon_{t+1} \end{bmatrix}\begin{bmatrix} \epsilon_{t+1} \\ \epsilon_{t+1}\otimes\epsilon_{t+1} - vec\left(\mathbf{I}_{n_e}\right) \\ \epsilon_{t+1}\otimes\mathbf{x}_t^f \\ \mathbf{x}_t^f\otimes\epsilon_{t+1} \end{bmatrix}'\right]$$

$$= \begin{bmatrix} \mathbf{I}_{n_e} & \mathbb{E}\left[\epsilon_{t+1}\left(\epsilon_{t+1}\otimes\epsilon_{t+1}\right)'\right] & \mathbf{0} & \mathbf{0} \\ \mathbb{E}\left[\left(\epsilon_{t+1}\otimes\epsilon_{t+1}\right)\epsilon_{t+1}'\right] & \mathbb{E}\left[\left(\epsilon_{t+1}\otimes\epsilon_{t+1} - vec\left(\mathbf{I}_{n_e}\right)\right)\left(\epsilon_{t+1}\otimes\epsilon_{t+1} - vec\left(\mathbf{I}_{n_e}\right)\right)'\right] & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbb{E}\left[\left(\epsilon_{t+1}\otimes\mathbf{x}_t^f\right)\left(\epsilon_{t+1}\otimes\mathbf{x}_t^f\right)'\right] & \mathbb{E}\left[\left(\epsilon_{t+1}\otimes\mathbf{x}_t^f\right)\left(\mathbf{x}_t^f\otimes\epsilon_{t+1}\right)'\right] \\ \mathbf{0} & \mathbf{0} & \mathbb{E}\left[\left(\mathbf{x}_t^f\otimes\epsilon_{t+1}\right)\left(\epsilon_{t+1}\otimes\mathbf{x}_t^f\right)'\right] & \mathbb{E}\left[\left(\mathbf{x}_t^f\otimes\epsilon_{t+1}\right)\left(\mathbf{x}_t^f\otimes\epsilon_{t+1}\right)'\right] \end{bmatrix}.$$

This variance is finite when ϵ_{t+1} has a finite fourth moment. All elements in this matrix can be computed element-by-element.

A.5. *Second order: unconditional third and fourth moments*

We consider the system $\mathbf{x}_{t+1} = \mathbf{a} + \mathbf{A}\mathbf{x}_t + \mathbf{v}_{t+1}$, where $\mathbf{A}$ is stable and $\mathbf{v}_{t+1}$ are mean-zero innovations. Thus, the pruned state-space representation of DSGE models belong to this class. For notational convenience, the system is expressed in deviation from its mean as $\mathbf{a} = (\mathbf{I} - \mathbf{A})\mathbb{E}[\mathbf{x}_t]$. Therefore

$$\mathbf{x}_{t+1} = (\mathbf{I} - \mathbf{A})\mathbb{E}[\mathbf{x}_t] + \mathbf{A}\mathbf{x}_t + \mathbf{v}_{t+1} \Rightarrow$$

$$\mathbf{x}_{t+1} - \mathbb{E}[\mathbf{x}_t] = \mathbf{A}(\mathbf{x}_t - E[\mathbf{x}_t]) + \mathbf{v}_{t+1} \Rightarrow$$

$$\mathbf{z}_{t+1} = \mathbf{A}\mathbf{z}_t + \mathbf{v}_{t+1}.$$

We then have

$$\begin{aligned} \mathbf{z}_{t+1}\otimes\mathbf{z}_{t+1} &= (\mathbf{A}\mathbf{z}_t + \mathbf{v}_{t+1})\otimes(\mathbf{A}\mathbf{z}_t + \mathbf{v}_{t+1}) \\ &= \mathbf{A}\mathbf{z}_t\otimes\mathbf{A}\mathbf{z}_t + \mathbf{A}\mathbf{z}_t\otimes\mathbf{v}_{t+1} + \mathbf{v}_{t+1}\otimes\mathbf{A}\mathbf{z}_t + \mathbf{v}_{t+1}\otimes\mathbf{v}_{t+1}, \end{aligned}$$

$$\begin{aligned} \mathbf{z}_{t+1}\otimes\mathbf{z}_{t+1}\otimes\mathbf{z}_{t+1} = {} & \mathbf{A}\mathbf{z}_t\otimes\mathbf{A}\mathbf{z}_t\otimes\mathbf{A}\mathbf{z}_t + \mathbf{A}\mathbf{z}_t\otimes\mathbf{A}\mathbf{z}_t\otimes\mathbf{v}_{t+1} \\ & + \mathbf{A}\mathbf{z}_t\otimes\mathbf{v}_{t+1}\otimes\mathbf{A}\mathbf{z}_t + \mathbf{A}\mathbf{z}_t\otimes\mathbf{v}_{t+1}\otimes\mathbf{v}_{t+1} \\ & + \mathbf{v}_{t+1}\otimes\mathbf{A}\mathbf{z}_t\otimes\mathbf{A}\mathbf{z}_t + \mathbf{v}_{t+1}\otimes\mathbf{A}\mathbf{z}_t\otimes\mathbf{v}_{t+1} \\ & + \mathbf{v}_{t+1}\otimes\mathbf{v}_{t+1}\otimes\mathbf{A}\mathbf{z}_t + \mathbf{v}_{t+1}\otimes\mathbf{v}_{t+1}\otimes\mathbf{v}_{t+1} \end{aligned}$$

and

$$\begin{aligned} \mathbf{z}_{t+1}\otimes\mathbf{z}_{t+1}\otimes\mathbf{z}_{t+1}\otimes\mathbf{z}_{t+1} = {} & \mathbf{A}\mathbf{z}_t\otimes\mathbf{A}\mathbf{z}_t\otimes\mathbf{A}\mathbf{z}_t\otimes\mathbf{A}\mathbf{z}_t + \mathbf{A}\mathbf{z}_t\otimes\mathbf{A}\mathbf{z}_t\otimes\mathbf{A}\mathbf{z}_t\otimes\mathbf{v}_{t+1} \\ & + \mathbf{A}\mathbf{z}_t\otimes\mathbf{A}\mathbf{z}_t\otimes\mathbf{v}_{t+1}\otimes\mathbf{A}\mathbf{z}_t + \mathbf{A}\mathbf{z}_t\otimes\mathbf{A}\mathbf{z}_t\otimes\mathbf{v}_{t+1}\otimes\mathbf{v}_{t+1} \\ & + \mathbf{A}\mathbf{z}_t\otimes\mathbf{v}_{t+1}\otimes\mathbf{A}\mathbf{z}_t\otimes\mathbf{A}\mathbf{z}_t + \mathbf{A}\mathbf{z}_t\otimes\mathbf{v}_{t+1}\otimes\mathbf{A}\mathbf{z}_t\otimes\mathbf{v}_{t+1} \\ & + \mathbf{A}\mathbf{z}_t\otimes\mathbf{v}_{t+1}\otimes\mathbf{v}_{t+1}\otimes\mathbf{A}\mathbf{z}_t + \mathbf{A}\mathbf{z}_t\otimes\mathbf{v}_{t+1}\otimes\mathbf{v}_{t+1}\otimes\mathbf{v}_{t+1} \\ & + \mathbf{v}_{t+1}\otimes\mathbf{A}\mathbf{z}_t\otimes\mathbf{A}\mathbf{z}_t\otimes\mathbf{A}\mathbf{z}_t + \mathbf{v}_{t+1}\otimes\mathbf{A}\mathbf{z}_t\otimes\mathbf{A}\mathbf{z}_t\otimes\mathbf{v}_{t+1} \\ & + \mathbf{v}_{t+1}\otimes\mathbf{A}\mathbf{z}_t\otimes\mathbf{v}_{t+1}\otimes\mathbf{A}\mathbf{z}_t + \mathbf{v}_{t+1}\otimes\mathbf{A}\mathbf{z}_t\otimes\mathbf{v}_{t+1}\otimes\mathbf{v}_{t+1} \\ & + \mathbf{v}_{t+1}\otimes\mathbf{v}_{t+1}\otimes\mathbf{A}\mathbf{z}_t\otimes\mathbf{A}\mathbf{z}_t + \mathbf{v}_{t+1}\otimes\mathbf{v}_{t+1}\otimes\mathbf{A}\mathbf{z}_t\otimes\mathbf{v}_{t+1} \\ & + \mathbf{v}_{t+1}\otimes\mathbf{v}_{t+1}\otimes\mathbf{v}_{t+1}\otimes\mathbf{A}\mathbf{z}_t + \mathbf{v}_{t+1}\otimes\mathbf{v}_{t+1}\otimes\mathbf{v}_{t+1}\otimes\mathbf{v}_{t+1}. \end{aligned}$$

Thus, to solve for $\mathbb{E}[\mathbf{z}_{t+1} \otimes \mathbf{z}_{t+1} \otimes \mathbf{z}_{t+1}]$, the innovations need to have a finite third moment. At second order, $\mathbf{v}_{t+1}$ depends on $\epsilon_{t+1} \otimes \epsilon_{t+1}$, meaning that ϵ_{t+1} must have a finite sixth moment. Similarly, to solve for $\mathbb{E}[\mathbf{z}_{t+1} \otimes \mathbf{z}_{t+1} \otimes \mathbf{z}_{t+1} \otimes \mathbf{z}_{t+1}]$, the innovations need to have finite fourth moments. At second order, $\mathbf{v}_{t+1}$ depends on $\epsilon_{t+1} \otimes \epsilon_{t+1}$, meaning that ϵ_{t+1} must have a finite eighth moment.

A.6. *Coefficients for the pruned state-space system at third order*

$$
\mathbf{A}^{(3)} \equiv \begin{bmatrix}
\mathbf{h_x} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{h_x} & \frac{1}{2}\mathbf{H_{xx}} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{h_x} \otimes \mathbf{h_x} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\frac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2 & \mathbf{0} & \mathbf{0} & \mathbf{h_x} & \mathbf{H_{xx}} & \frac{1}{6}\mathbf{H_{xxx}} \\
\mathbf{h_x} \otimes \frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{h_x} \otimes \mathbf{h_x} & \mathbf{h_x} \otimes \frac{1}{2}\mathbf{H_{xx}} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{h_x} \otimes \mathbf{h_x} \otimes \mathbf{h_x}
\end{bmatrix},
$$

$$
\mathbf{B}^{(3)} \equiv \left[\begin{matrix}
\sigma\eta & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \sigma\eta \otimes \sigma\eta & \sigma\eta \otimes \mathbf{h_x} & \mathbf{h_x} \otimes \sigma\eta \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\sigma\eta \otimes \frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}
\end{matrix}\right.
$$

$$
\begin{matrix}
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\sigma\eta \otimes \mathbf{h_x} & \sigma\eta \otimes \frac{1}{2}\mathbf{H_{xx}} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \sigma\eta \otimes \mathbf{h_x} \otimes \mathbf{h_x} & \mathbf{h_x} \otimes \mathbf{h_x} \otimes \sigma\eta & \mathbf{h_x} \otimes \sigma\eta \otimes \mathbf{h_x}
\end{matrix}
$$

$$
\left.\begin{matrix}
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{h_x} \otimes \sigma\eta \otimes \sigma\eta & \sigma\eta \otimes \mathbf{h_x} \otimes \sigma\eta & \sigma\eta \otimes \sigma\eta \otimes \mathbf{h_x} & \sigma\eta \otimes \sigma\eta \otimes \sigma\eta
\end{matrix}\right],
$$

$$
\xi_{t+1}^{(3)} \equiv \begin{bmatrix}
\epsilon_{t+1} \\
\epsilon_{t+1} \otimes \epsilon_{t+1} - vec\left(\mathbf{I}_{n_e}\right) \\
\epsilon_{t+1} \otimes \mathbf{x}_t^f \\
\mathbf{x}_t^f \otimes \epsilon_{t+1} \\
\epsilon_{t+1} \otimes \mathbf{x}_t^s \\
\epsilon_{t+1} \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f \\
\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \epsilon_{t+1} \\
\mathbf{x}_t^f \otimes \epsilon_{t+1} \otimes \mathbf{x}_t^f \\
\mathbf{x}_t^f \otimes \epsilon_{t+1} \otimes \epsilon_{t+1} \\
\epsilon_{t+1} \otimes \mathbf{x}_t^f \otimes \epsilon_{t+1} \\
\epsilon_{t+1} \otimes \epsilon_{t+1} \otimes \mathbf{x}_t^f \\
\left(\epsilon_{t+1} \otimes \epsilon_{t+1} \otimes \epsilon_{t+1}\right) - \mathbb{E}\left[\left(\epsilon_{t+1} \otimes \epsilon_{t+1} \otimes \epsilon_{t+1}\right)\right]
\end{bmatrix},
$$

$$
\mathbf{c}^{(3)} \equiv \begin{bmatrix}
\mathbf{0}_{n_x \times 1} \\
\frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2 \\
(\sigma\eta \otimes \sigma\eta)\, vec\left(\mathbf{I}_{n_e}\right) \\
\frac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^3 \\
\mathbf{0}_{n_x^2 \times 1} \\
(\sigma\eta \otimes \sigma\eta \otimes \sigma\eta)\mathbb{E}\left[\left(\epsilon_{t+1} \otimes \epsilon_{t+1} \otimes \epsilon_{t+1}\right)\right]
\end{bmatrix},
$$

$\mathbf{C}^{(3)} \equiv \left[\ \mathbf{g_x} + \frac{3}{6}\mathbf{g}_{\sigma\sigma\mathbf{x}}\sigma^2 \ \ \mathbf{g_x} \ \ \frac{1}{2}\mathbf{G_{xx}} \ \ \mathbf{g_x} \ \ \mathbf{G_{xx}} \ \ \frac{1}{6}\mathbf{G_{xxx}}\ \right],$
and
$\mathbf{d}^{(3)} \equiv \frac{1}{2}\mathbf{g}_{\sigma\sigma}\sigma^2 + \frac{1}{6}\mathbf{g}_{\sigma\sigma\sigma}\sigma^3.$

A.7. *Third order*: *stability*

To prove stability:

$$\begin{aligned}
p(\lambda) &= \left|\mathbf{A}^{(3)} - \lambda\mathbf{I}\right| \\
&= \left|\begin{bmatrix} \mathbf{h_x} - \lambda\mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{h_x} - \lambda\mathbf{I} & \frac{1}{2}\mathbf{H_{xx}} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{h_x}\otimes\mathbf{h_x} - \lambda\mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \frac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2 & \mathbf{0} & \mathbf{0} & \mathbf{h_x} - \lambda\mathbf{I} & \mathbf{H_{xx}} & \frac{1}{6}\mathbf{H_{xxx}} \\ \mathbf{h_x}\otimes\frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{h_x}\otimes\mathbf{h_x} - \lambda\mathbf{I} & \mathbf{h_x}\otimes\frac{1}{2}\mathbf{H_{xx}} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{h_x}\otimes\mathbf{h_x}\otimes\mathbf{h_x} - \lambda\mathbf{I} \end{bmatrix}\right| \\
&= \left|\begin{bmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} \\ \mathbf{B}_{21} & \mathbf{B}_{22} \end{bmatrix}\right|,
\end{aligned}$$

where

$$\mathbf{B}_{11} \equiv \begin{bmatrix} \mathbf{h_x} - \lambda\mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{h_x} - \lambda\mathbf{I} & \frac{1}{2}\mathbf{H_{xx}} \\ \mathbf{0} & \mathbf{0} & \mathbf{h_x}\otimes\mathbf{h_x} - \lambda\mathbf{I} \end{bmatrix} \ \mathbf{B}_{12} \equiv \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix},$$

$$\mathbf{B}_{21} \equiv \begin{bmatrix} \frac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2 & \mathbf{0} & \mathbf{0} \\ \mathbf{h_x}\otimes\frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \ \mathbf{B}_{22} \equiv \begin{bmatrix} \mathbf{h_x} - \lambda\mathbf{I} & \mathbf{H_{xx}} & \frac{1}{6}\mathbf{H_{xxx}} \\ \mathbf{0} & \mathbf{h_x}\otimes\mathbf{h_x} - \lambda\mathbf{I} & \mathbf{h_x}\otimes\frac{1}{2}\mathbf{H_{xx}} \\ \mathbf{0} & \mathbf{0} & \mathbf{h_x}\otimes\mathbf{h_x}\otimes\mathbf{h_x} - \lambda\mathbf{I} \end{bmatrix},$$

$= |\mathbf{B}_{11}||\mathbf{B}_{22}|$

$= |\mathbf{h_x} - \lambda\mathbf{I}||\mathbf{h_x} - \lambda\mathbf{I}||\mathbf{h_x}\otimes\mathbf{h_x} - \lambda\mathbf{I}||\mathbf{B}_{22}|$

(using the result from the proof in Appendix A.3)

$= |\mathbf{h_x} - \lambda\mathbf{I}||\mathbf{h_x} - \lambda\mathbf{I}||\mathbf{h_x}\otimes\mathbf{h_x} - \lambda\mathbf{I}||\mathbf{h_x} - \lambda\mathbf{I}||\mathbf{h_x}\otimes\mathbf{h_x} - \lambda\mathbf{I}||\mathbf{h_x}\otimes\mathbf{h_x}\otimes\mathbf{h_x} - \lambda\mathbf{I}|$

(using the rule on block determinants repeatedly on $\mathbf{B}_{22}$).

The eigenvalue λ solves $p(\lambda) = 0$, which implies:

$$|\mathbf{h_x} - \lambda\mathbf{I}| = 0 \text{ or } |\mathbf{h_x}\otimes\mathbf{h_x} - \lambda\mathbf{I}| = 0 \text{ or } |(\mathbf{h_x}\otimes\mathbf{h_x}\otimes\mathbf{h_x}) - \lambda\mathbf{I}| = 0.$$

The absolute values of all eigenvalues of the first problem are strictly less than one by assumption. That is $|\lambda_i| < 1$, $i = 1, 2, ..., n_x$. This is also the case for the second problem, because the eigenvalues of $\mathbf{h_x}\otimes\mathbf{h_x}$ are $\lambda_i\lambda_j$ for $i = 1, 2, ..., n_x$ and $j = 1, 2, ..., n_x$. The same argument ensures that the absolute values of all eigenvalues of the third problem are also less than one. This shows that all eigenvalues of $\mathbf{A}^{(3)}$ have modulus less than one.

A.8. *Third order*: *unconditional second moments*

For the variance, we have

$$\begin{aligned}
\mathbb{V}\left[\mathbf{z}_{t+1}^{(3)}\right] = {} & \mathbf{A}^{(3)}\mathbb{V}\left[\mathbf{z}_t^{(3)}\right]\left(\mathbf{A}^{(3)}\right)' + \mathbf{B}^{(3)}\mathbb{V}\left[\xi_{t+1}^{(3)}\right]\left(\mathbf{B}^{(3)}\right)' \\
& + \mathbf{A}^{(3)}Cov\left[\mathbf{z}_t^{(3)}, \xi_{t+1}^{(3)}\right]\left(\mathbf{B}^{(3)}\right)' + \mathbf{B}^{(3)}Cov\left[\xi_{t+1}^{(3)}, \mathbf{z}_t^{(3)}\right]\left(\mathbf{A}^{(3)}\right)'.
\end{aligned}$$

Contrary to a second-order approximation, $Cov\left[\xi_{t+1}^{(3)}, \mathbf{z}_t^{(3)}\right] \neq 0$. This is seen as follows:

$$
\mathbb{E}\left[\mathbf{z}_t^{(3)}\left(\xi_{t+1}^{(3)}\right)'\right]=\mathbb{E}\left[\begin{bmatrix}\mathbf{x}_t^f\\ \mathbf{x}_t^s\\ \mathbf{x}_t^f\otimes\mathbf{x}_t^f\\ \mathbf{x}_t^{rd}\\ \mathbf{x}_t^f\otimes\mathbf{x}_t^s\\ \mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\end{bmatrix}\right.
$$

$$
\times\left[\ \epsilon_{t+1}'\ \ \left(\epsilon_{t+1}\otimes\epsilon_{t+1}-vec\left(\mathbf{I}_{n_e}\right)\right)'\ \left(\epsilon_{t+1}\otimes\mathbf{x}_t^f\right)'\ \left(\mathbf{x}_t^f\otimes\epsilon_{t+1}\right)'\ \left(\epsilon_{t+1}\otimes\mathbf{x}_t^s\right)'\ \left(\epsilon_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'\right.
$$

$$
\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\epsilon_{t+1}\right)'\ \left(\mathbf{x}_t^f\otimes\epsilon_{t+1}\otimes\mathbf{x}_t^f\right)'\ \left(\mathbf{x}_t^f\otimes\epsilon_{t+1}\otimes\epsilon_{t+1}\right)'\ \left(\epsilon_{t+1}\otimes\mathbf{x}_t^f\otimes\epsilon_{t+1}\right)'
$$

$$
\left.\left(\epsilon_{t+1}\otimes\epsilon_{t+1}\otimes\mathbf{x}_t^f\right)'\ \left(\left(\epsilon_{t+1}\otimes\epsilon_{t+1}\otimes\epsilon_{t+1}\right)-E\left[\left(\epsilon_{t+1}\otimes\epsilon_{t+1}\otimes\epsilon_{t+1}\right)\right]\right)'\ \right]
$$

$$
=\begin{bmatrix}
0_{n_x\times n_e} & 0_{n_x\times n_e^2} & 0_{n_x\times n_e n_x} & 0_{n_x\times n_x n_e} & 0_{n_x\times n_x n_e} & 0_{n_x\times n_e n_x^2} & 0_{n_x\times n_x^2 n_e} & 0_{n_x\times n_x^2 n_e} & R_{1,1} & R_{1,2} & R_{1,3} & 0_{n_x\times n_e^3}\\
0_{n_x\times n_e} & 0_{n_x\times n_e^2} & 0_{n_x\times n_e n_x} & 0_{n_x\times n_x n_e} & 0_{n_x\times n_x n_e} & 0_{n_x\times n_e n_x^2} & 0_{n_x\times n_x^2 n_e} & 0_{n_x\times n_x^2 n_e} & R_{2,1} & R_{2,2} & R_{2,3} & 0_{n_x\times n_e^3}\\
0_{n_x^2\times n_e} & 0_{n_x^2\times n_e^2} & 0_{n_x^2\times n_e n_x} & 0_{n_x^2\times n_x n_e} & 0_{n_x^2\times n_x n_e} & 0_{n_x^2\times n_e n_x^2} & 0_{n_x^2\times n_x^2 n_e} & 0_{n_x^2\times n_x^2 n_e} & R_{3,1} & R_{3,2} & R_{3,3} & 0_{n_x^2\times n_e^3}\\
0_{n_x\times n_e} & 0_{n_x\times n_e^2} & 0_{n_x\times n_e n_x} & 0_{n_x\times n_x n_e} & 0_{n_x\times n_x n_e} & 0_{n_x\times n_e n_x^2} & 0_{n_x\times n_x^2 n_e} & 0_{n_x\times n_x^2 n_e} & R_{4,1} & R_{4,2} & R_{4,3} & 0_{n_x\times n_e^3}\\
0_{n_x^2\times n_e} & 0_{n_x^2\times n_e^2} & 0_{n_x^2\times n_e n_x} & 0_{n_x^2\times n_x n_e} & 0_{n_x^2\times n_x n_e} & 0_{n_x^2\times n_e n_x^2} & 0_{n_x^2\times n_x^2 n_e} & 0_{n_x^2\times n_x^2 n_e} & R_{5,1} & R_{5,2} & R_{5,3} & 0_{n_x^2\times n_e^3}\\
0_{n_x^3\times n_e} & 0_{n_x^3\times n_e^2} & 0_{n_x^3\times n_e n_x} & 0_{n_x^3\times n_x n_e} & 0_{n_x^3\times n_x n_e} & 0_{n_x^3\times n_e n_x^2} & 0_{n_x^3\times n_x^2 n_e} & 0_{n_x^3\times n_x^2 n_e} & R_{6,1} & R_{6,2} & R_{6,3} & 0_{n_x^3\times n_e^3}
\end{bmatrix}
$$

$$
=\left[\,\mathbf{0}\ \mathbf{R}\ \mathbf{0}\,\right].
$$

The $\mathbf{R}$ matrix can easily be computed element-by-element. To compute $\mathbb{V}\left[\xi_{t+1}^{(3)}\right]$, we consider

$$
\mathbb{E}\left[\xi_{t+1}^{(3)}\left(\xi_{t+1}^{(3)}\right)'\right]=\mathbb{E}\left[\begin{bmatrix}
\epsilon_{t+1}\\
\epsilon_{t+1}\otimes\epsilon_{t+1}-vec\left(\mathbf{I}_{n_e}\right)\\
\epsilon_{t+1}\otimes\mathbf{x}_t^f\\
\mathbf{x}_t^f\otimes\epsilon_{t+1}\\
\epsilon_{t+1}\otimes\mathbf{x}_t^s\\
\epsilon_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\\
\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\epsilon_{t+1}\\
\mathbf{x}_t^f\otimes\epsilon_{t+1}\otimes\mathbf{x}_t^f\\
\mathbf{x}_t^f\otimes\epsilon_{t+1}\otimes\epsilon_{t+1}\\
\epsilon_{t+1}\otimes\mathbf{x}_t^f\otimes\epsilon_{t+1}\\
\epsilon_{t+1}\otimes\epsilon_{t+1}\otimes\mathbf{x}_t^f\\
\left(\epsilon_{t+1}\otimes\epsilon_{t+1}\otimes\epsilon_{t+1}\right)-\mathbb{E}\left[\left(\epsilon_{t+1}\otimes\epsilon_{t+1}\otimes\epsilon_{t+1}\right)\right]
\end{bmatrix}\right.
$$

$$\times\Big[\,\epsilon'_{t+1}\ \left(\epsilon_{t+1}\otimes\epsilon_{t+1}-vec\left(\mathbf{I}_{n_e}\right)\right)'\ \left(\epsilon_{t+1}\otimes\mathbf{x}_t^f\right)'\ \left(\mathbf{x}_t^f\otimes\epsilon_{t+1}\right)'\ \left(\epsilon_{t+1}\otimes\mathbf{x}_t^s\right)'\ \left(\epsilon_{t+1}\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)'$$
$$\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\epsilon_{t+1}\right)'\ \left(\mathbf{x}_t^f\otimes\epsilon_{t+1}\otimes\mathbf{x}_t^f\right)'\ \left(\mathbf{x}_t^f\otimes\epsilon_{t+1}\otimes\epsilon_{t+1}\right)'\ \left(\epsilon_{t+1}\otimes\mathbf{x}_t^f\otimes\epsilon_{t+1}\right)'$$
$$\left(\epsilon_{t+1}\otimes\epsilon_{t+1}\otimes\mathbf{x}_t^f\right)'\ \left(\left(\epsilon_{t+1}\otimes\epsilon_{t+1}\otimes\epsilon_{t+1}\right)-\mathbb{E}\left[\left(\epsilon_{t+1}\otimes\epsilon_{t+1}\otimes\epsilon_{t+1}\right)\right]\right)'\Big].$$

Note that $\mathbb{V}\left[\xi_{t+1}^{(3)}\right]$ contains $(\epsilon_{t+1}\otimes\epsilon_{t+1}\otimes\epsilon_{t+1})$ squared, meaning that ϵ_{t+1} must have a finite sixth moment for $\mathbb{V}\left[\xi_{t+1}^{(3)}\right]$ to be finite. Again, all elements in $\mathbb{V}\left[\xi_{t+1}^{(3)}\right]$ can be computed element-by-element. For further details, we refer to the article's Online Appendix, which also discusses how $V\left[\xi_{t+1}^{(3)}\right]$ can be computed in a more memory-efficient manner.

For the auto-covariance, we have

$$\begin{aligned}Cov\left(\mathbf{z}_{t+1}^{(3)},\mathbf{z}_t^{(3)}\right) &= Cov\left(\mathbf{c}^{(3)}+\mathbf{A}^{(3)}\mathbf{z}_t^{(3)}+\mathbf{B}^{(3)}\xi_{t+1}^{(3)},\mathbf{z}_t^{(3)}\right)\\ &= \mathbf{A}^{(3)}Cov\left(\mathbf{z}_t^{(3)},\mathbf{z}_t^{(3)}\right)+\mathbf{B}^{(3)}Cov\left(\xi_{t+1}^{(3)},\mathbf{z}_t^{(3)}\right)\end{aligned}$$

and

$$\begin{aligned}Cov\left(\mathbf{z}_{t+2}^{(3)},\mathbf{z}_t^{(3)}\right) &= Cov\left(\mathbf{c}^{(3)}+\mathbf{A}^{(3)}\mathbf{z}_{t+1}^{(3)}+\mathbf{B}^{(3)}\xi_{t+2}^{(3)},\mathbf{z}_t^{(3)}\right)\\ &= Cov\left(\mathbf{c}^{(3)}+\mathbf{A}^{(3)}\left(\mathbf{c}^{(3)}+\mathbf{A}^{(3)}\mathbf{z}_t^{(3)}+\mathbf{B}^{(3)}\xi_{t+1}^{(3)}\right)+\mathbf{B}^{(3)}\xi_{t+2}^{(3)},\mathbf{z}_t^{(3)}\right)\\ &= Cov\left(\mathbf{c}^{(3)}+\mathbf{A}^{(3)}\mathbf{c}^{(3)}+\left(\mathbf{A}^{(3)}\right)^2\mathbf{z}_t^{(3)}+\mathbf{A}^{(3)}\mathbf{B}^{(3)}\xi_{t+1}^{(3)}+\mathbf{B}^{(3)}\xi_{t+2}^{(3)},\mathbf{z}_t^{(3)}\right)\\ &= Cov\left(\left(\mathbf{A}^{(3)}\right)^2\mathbf{z}_t^{(3)},\mathbf{z}_t^{(3)}\right)+Cov\left(\mathbf{A}^{(3)}\mathbf{B}^{(3)}\xi_{t+1}^{(3)},\mathbf{z}_t^{(3)}\right)+Cov\left(\mathbf{B}^{(3)}\xi_{t+2}^{(3)},\mathbf{z}_t^{(3)}\right)\\ &= \left(\mathbf{A}^{(3)}\right)^2Cov\left(\mathbf{z}_t^{(3)},\mathbf{z}_t^{(3)}\right)+\mathbf{A}^{(3)}\mathbf{B}^{(3)}Cov\left(\xi_{t+1}^{(3)},\mathbf{z}_t^{(3)}\right)+\mathbf{B}^{(3)}Cov\left(\xi_{t+2}^{(3)},\mathbf{z}_t^{(3)}\right).\end{aligned}$$

So, for $s=1,2,3...$

$$Cov\left(\mathbf{z}_{t+s}^{(3)},\mathbf{z}_t^{(3)}\right)=\left(\mathbf{A}^{(3)}\right)^s\mathbb{V}\left[\mathbf{z}_t^{(3)}\right]+\sum_{j=0}^{s-1}\left(\mathbf{A}^{(3)}\right)^{s-1-j}\mathbf{B}^{(3)}Cov\left(\xi_{t+1+j}^{(3)},\mathbf{z}_t^{(3)}\right)$$

and we therefore only need to compute $Cov\left(\xi_{t+1+j}^{(3)},\mathbf{z}_t^{(3)}\right)$:

$$\mathbb{E}\left[\mathbf{z}_t^{(3)}\left(\xi_{t+1+j}^{(3)}\right)'\right]=\mathbb{E}\left[\begin{bmatrix}\mathbf{x}_t^f\\ \mathbf{x}_t^s\\ \mathbf{x}_t^f\otimes\mathbf{x}_t^f\\ \mathbf{x}_t^{rd}\\ \mathbf{x}_t^f\otimes\mathbf{x}_t^s\\ \mathbf{x}_t^f\otimes\mathbf{x}_t^f\otimes\mathbf{x}_t^f\end{bmatrix}\right.$$
$$\times\Big[\,\epsilon'_{t+1+j}\ \left(\epsilon_{t+1+j}\otimes\epsilon_{t+1+j}-vec\left(\mathbf{I}_{n_e}\right)\right)'\ \left(\epsilon_{t+1+j}\otimes\mathbf{x}_{t+j}^f\right)'$$
$$\left(\mathbf{x}_{t+j}^f\otimes\epsilon_{t+1+j}\right)'\ \left(\epsilon_{t+1+j}\otimes\mathbf{x}_{t+j}^s\right)'\ \left(\epsilon_{t+1+j}\otimes\mathbf{x}_{t+j}^f\otimes\mathbf{x}_{t+j}^f\right)'$$
$$\left(\mathbf{x}_{t+j}^f\otimes\mathbf{x}_{t+j}^f\otimes\epsilon_{t+1+j}\right)'\ \left(\mathbf{x}_{t+j}^f\otimes\epsilon_{t+1+j}\otimes\mathbf{x}_{t+j}^f\right)'$$

$$\left(\mathbf{x}^f_{t+j}\otimes\epsilon_{t+1+j}\otimes\epsilon_{t+1+j}\right)'\left(\epsilon_{t+1+j}\otimes\mathbf{x}^f_{t+j}\otimes\epsilon_{t+1+j}\right)'$$
$$\left(\epsilon_{t+1+j}\otimes\epsilon_{t+1+j}\otimes\mathbf{x}^f_{t+j}\right)'\left(\left(\epsilon_{t+1+j}\otimes\epsilon_{t+1+j}\otimes\epsilon_{t+1+j}\right)-E\left[\left(\epsilon_{t+1}\otimes\epsilon_{t+1}\otimes\epsilon_{t+1}\right)\right]\right)'\Big]$$
$$=\begin{bmatrix}
0_{n_x\times n_e} & 0_{n_x\times n_e^2} & 0_{n_x\times n_e n_x} & 0_{n_x\times n_x n_e} & 0_{n_x\times n_x n_e} & 0_{n_x\times n_e n_x^2} & 0_{n_x\times n_x^2 n_e} & 0_{n_x\times n_x^2 n_e} & R^j_{1,1} & R^j_{1,2} & R^j_{1,3} & 0_{n_x\times n_e^3}\\
0_{n_x\times n_e} & 0_{n_x\times n_e^2} & 0_{n_x\times n_e n_x} & 0_{n_x\times n_x n_e} & 0_{n_x\times n_x n_e} & 0_{n_x\times n_e n_x^2} & 0_{n_x\times n_x^2 n_e} & 0_{n_x\times n_x^2 n_e} & R^j_{2,1} & R^j_{2,2} & R^j_{2,3} & 0_{n_x\times n_e^3}\\
0_{n_x^2\times n_e} & 0_{n_x^2\times n_e^2} & 0_{n_x^2\times n_e n_x} & 0_{n_x^2\times n_x n_e} & 0_{n_x^2\times n_x n_e} & 0_{n_x^2\times n_e n_x^2} & 0_{n_x^2\times n_x^2 n_e} & 0_{n_x^2\times n_x^2 n_e} & R^j_{3,1} & R^j_{3,2} & R^j_{3,3} & 0_{n_x^2\times n_e^3}\\
0_{n_x\times n_e} & 0_{n_x\times n_e^2} & 0_{n_x\times n_e n_x} & 0_{n_x\times n_x n_e} & 0_{n_x\times n_x n_e} & 0_{n_x\times n_e n_x^2} & 0_{n_x\times n_x^2 n_e} & 0_{n_x\times n_x^2 n_e} & R^j_{4,1} & R^j_{4,2} & R^j_{4,3} & 0_{n_x\times n_e^3}\\
0_{n_x^2\times n_e} & 0_{n_x^2\times n_e^2} & 0_{n_x^2\times n_e n_x} & 0_{n_x^2\times n_x n_e} & 0_{n_x^2\times n_x n_e} & 0_{n_x^2\times n_e n_x^2} & 0_{n_x^2\times n_x^2 n_e} & 0_{n_x^2\times n_x^2 n_e} & R^j_{5,1} & R^j_{5,2} & R^j_{5,3} & 0_{n_x^2\times n_e^3}\\
0_{n_x^3\times n_e} & 0_{n_x^3\times n_e^2} & 0_{n_x^3\times n_e n_x} & 0_{n_x^3\times n_x n_e} & 0_{n_x^3\times n_x n_e} & 0_{n_x^3\times n_e n_x^2} & 0_{n_x^3\times n_x^2 n_e} & 0_{n_x^3\times n_x^2 n_e} & R^j_{6,1} & R^j_{6,2} & R^j_{6,3} & 0_{n_x^3\times n_e^3}
\end{bmatrix}$$
$$=\left[\,\mathbf{0}\ \mathbf{R}^j\ \mathbf{0}\,\right].$$

The matrix $\mathbf{R}^j$ can then be computed element-by-element. For further details, see the article's Online Appendix.

A.9. *Third order: unconditional third and fourth moments*

The proof proceeds as for a second-order approximation except that, at third order, $\mathbf{v}_{t+1}$ also depends on $\epsilon_{t+1}\otimes\epsilon_{t+1}\otimes\epsilon_{t+1}$. Hence, unconditional third moments exist if ϵ_{t+1} has a finite ninth moment, and the unconditional fourth moment exists if ϵ_{t+1} has a finite twelfth moment.

A.10. *GIRFs: second order*

We first note that

$$\mathbf{x}^f_{t+l}\otimes\mathbf{x}^f_{t+l}=\left(\mathbf{h}^l_{\mathbf{x}}\mathbf{x}^f_t+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\eta\epsilon_{t+j}\right)\otimes\left(\mathbf{h}^l_{\mathbf{x}}\mathbf{x}^f_t+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\eta\epsilon_{t+j}\right)$$
$$=\mathbf{h}^l_{\mathbf{x}}\mathbf{x}^f_t\otimes\mathbf{h}^l_{\mathbf{x}}\mathbf{x}^f_t+\mathbf{h}^l_{\mathbf{x}}\mathbf{x}^f_t\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\eta\epsilon_{t+j}$$
$$+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\eta\epsilon_{t+j}\otimes\mathbf{h}^l_{\mathbf{x}}\mathbf{x}^f_t+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\eta\epsilon_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\eta\epsilon_{t+j}.$$

Next, let

$$\tilde{\mathbf{x}}^f_{t+l}\otimes\tilde{\mathbf{x}}^f_{t+l}=\mathbf{h}^l_{\mathbf{x}}\mathbf{x}^f_t\otimes\mathbf{h}^l_{\mathbf{x}}\mathbf{x}^f_t+\mathbf{h}^l_{\mathbf{x}}\mathbf{x}^f_t\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\eta\delta_{t+j}+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\eta\delta_{t+j}\otimes\mathbf{h}^l_{\mathbf{x}}\mathbf{x}^f_t$$
$$+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\eta\delta_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\eta\delta_{t+j},$$

where we define δ_{t+j} such that $\delta_{t+1}=\nu+(\mathbf{I}-\mathbf{S})\epsilon_{t+1}$ and $\delta_{t+j}=\epsilon_{t+j}$ for $j\neq1$. This means

$$GIRF_{\mathbf{x}^f\otimes\mathbf{x}^f}\left(l,\nu_i,\mathbf{x}^f_t\right)=\mathbb{E}\left[\mathbf{x}^f_{t+l}\otimes\mathbf{x}^f_{t+l}|\mathbf{x}^f_t,\epsilon_{i,t+1}=\nu_i\right]-\mathbb{E}\left[\mathbf{x}^f_{t+l}\otimes\mathbf{x}^f_{t+l}|\mathbf{x}^f_t\right]$$
$$=\mathbb{E}\left[\tilde{\mathbf{x}}^f_{t+l}\otimes\tilde{\mathbf{x}}^f_{t+l}|\mathbf{x}^f_t\right]-\mathbb{E}\left[\mathbf{x}^f_{t+l}\otimes\mathbf{x}^f_{t+l}|\mathbf{x}^f_t\right]$$

$$
\begin{aligned}
&= \mathbb{E}[\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\eta\delta_{t+1} + \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\eta\delta_{t+1} \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f \\
&\quad + \sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\eta\delta_{t+j} \otimes \sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\eta\delta_{t+j} \\
&\quad - \sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\eta\epsilon_{t+j} \otimes \sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\eta\epsilon_{t+j} | \mathbf{x}_t^f] \\
&= \mathbb{E}[\mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\eta\left(\nu + (\mathbf{I}-\mathbf{S})\epsilon_{t+1}\right) + \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\eta\left(\nu + (\mathbf{I}-\mathbf{S})\epsilon_{t+1}\right) \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f \\
&\quad + \left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\eta\left(\nu + (\mathbf{I}-\mathbf{S})\epsilon_{t+1}\right) + \sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\eta\epsilon_{t+j}\right) \\
&\quad \otimes \left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\eta\left(\nu + (\mathbf{I}-\mathbf{S})\epsilon_{t+1}\right) + \sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\eta\epsilon_{t+j}\right) \\
&\quad - \left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\eta\epsilon_{t+1} + \sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\eta\epsilon_{t+j}\right) \otimes \left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\eta\epsilon_{t+1} + \sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\eta\epsilon_{t+j}\right) | \mathbf{x}_t^f] \\
&= \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\eta\nu + \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\eta\nu \otimes \mathbf{h}_{\mathbf{x}}^{l}\mathbf{x}_t^f + \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\eta\nu \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\sigma\eta\nu \\
&\quad + \left(\mathbf{h}_{\mathbf{x}}^{l-1} \otimes \mathbf{h}_{\mathbf{x}}^{l-1}\right)\left(\mathbb{E}\left[\sigma\eta(\mathbf{I}-\mathbf{S})\epsilon_{t+1} \otimes \sigma\eta(\mathbf{I}-\mathbf{S})\epsilon_{t+1}\right] - \mathbb{E}\left[\sigma\eta\epsilon_{t+1} \otimes \sigma\eta\epsilon_{t+1}\right]\right).
\end{aligned}
$$

With $\mathbb{E}[\sigma\eta(\mathbf{I}-\mathbf{S})\epsilon_{t+1} \otimes \sigma\eta(\mathbf{I}-\mathbf{S})\epsilon_{t+1}] = (\sigma\eta(\mathbf{I}-\mathbf{S}) \otimes \sigma\eta(\mathbf{I}-\mathbf{S}))\,vec(\mathbf{I})$ and $\mathbb{E}\left[\sigma\eta\epsilon_{t+1} \otimes \sigma\eta\epsilon_{t+1}\right] = (\sigma\eta \otimes \sigma\eta)\,vec(\mathbf{I})$ we then obtain (22).

A.11. *Second-order accuracy of linear IRFs*

Let $\mathbf{x}_t^f = \mathbf{0}$ and suppose $\nu(i,1) = \pm 1$ and $\nu(j,1) = 0$ for $i \neq j$. These assumptions imply

$$
\begin{aligned}
\sigma\eta\nu \otimes \sigma\eta\nu + \Lambda &= \sigma\eta\nu \otimes \sigma\eta\nu + ((\sigma\eta(\mathbf{I}-\mathbf{S}) \otimes \sigma\eta(\mathbf{I}-\mathbf{S})) - (\sigma\eta \otimes \sigma\eta))\,vec(\mathbf{I}) \\
&= (\sigma\eta \otimes \sigma\eta)\{\mathbf{S} \otimes \mathbf{S} + ((\mathbf{I}-\mathbf{S}) \otimes (\mathbf{I}-\mathbf{S})) - \mathbf{I} \otimes \mathbf{I}\}\,vec(\mathbf{I}) \\
&= (\sigma\eta \otimes \sigma\eta)\{2(\mathbf{S} \otimes \mathbf{S}) - \mathbf{I} \otimes \mathbf{S} - \mathbf{S} \otimes \mathbf{I}\}\,vec(\mathbf{I})
\end{aligned}
$$

because $\nu \otimes \nu = (\mathbf{S} \otimes \mathbf{S})\,vec(\mathbf{I})$ and $\mathbf{I}_{n_e^2} = \mathbf{I} \otimes \mathbf{I}$, where $\mathbf{I}$ has dimension $n_e \times n_e$. Next, let $\mathbf{D}_i(i,i) = 1$ with all remaining elements of $\mathbf{D}_i$ equal to zero. Hence, $\mathbf{I}$ can be written as $\mathbf{I} = \sum_{j=1}^{n_\varepsilon}\mathbf{D}_j$ and $\mathbf{S} = \mathbf{D}_i$. This implies

$$
\begin{aligned}
\sigma\eta\nu \otimes \sigma\eta\nu + \Lambda &= (\sigma\eta \otimes \sigma\eta)\left\{-\sum_{\substack{j=1\\ i\neq j}}^{n_\varepsilon}\mathbf{D}_j \otimes \mathbf{D}_i - \sum_{\substack{j=1\\ i\neq j}}^{n_\varepsilon}\mathbf{D}_i \otimes \mathbf{D}_j\right\} vec\left(\sum_{k=1}^{n_\varepsilon}\mathbf{D}_k\right) \\
&= (\sigma\eta \otimes \sigma\eta)\left\{-\sum_{\substack{j=1\\ i\neq j}}^{n_\varepsilon}\sum_{k=1}^{n_\varepsilon}\left(\mathbf{D}_j \otimes \mathbf{D}_i\right) vec(\mathbf{D}_k) - \sum_{\substack{j=1\\ i\neq j}}^{n_\varepsilon}\sum_{k=1}^{n_\varepsilon}\left(\mathbf{D}_i \otimes \mathbf{D}_j\right) vec(\mathbf{D}_k)\right\} \\
&= (\sigma\eta \otimes \sigma\eta)\left\{-\sum_{\substack{j=1\\ i\neq j}}^{n_\varepsilon}\sum_{k=1}^{n_\varepsilon} vec\left(\mathbf{D}_i\mathbf{D}_k\mathbf{D}_j\right) - \sum_{\substack{j=1\\ i\neq j}}^{n_\varepsilon}\sum_{k=1}^{n_\varepsilon} vec\left(\mathbf{D}_j\mathbf{D}_k\mathbf{D}_i\right)\right\} \\
&= \mathbf{0}
\end{aligned}
$$

because $\mathbf{D}_i\mathbf{D}_k\mathbf{D}_j$ is only different from the zero matrix when $i = k = j$, but we have $i \neq j$. Thus, $GIRF_{\mathbf{x}^f \otimes \mathbf{x}^f}\left(l, \nu_i, \mathbf{x}_t^f\right) = \mathbf{0}$ and $GIRF_{\mathbf{x}^s}\left(l, \nu_i, \mathbf{x}_t^f\right) = \mathbf{0}$, which proves that GIRFs in a pruned second-order approximation reduce to the IRFs in a linearized solution when $\mathbf{x}_t^f = 0$ and ν specified as above.

A.12. *GIRFs: third order*

Deriving $GIRF_{\mathbf{x}^f\otimes\mathbf{x}^f\otimes\mathbf{x}^f}\left(j,\nu,\mathbf{x}_t^f\right)$. We first note that

$$
\begin{aligned}
&\mathbf{x}_{t+l}^f\otimes\mathbf{x}_{t+l}^f\otimes\mathbf{x}_{t+l}^f\\
&\quad=\mathbf{h}_{\mathbf{x}}^l\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^l\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^l\mathbf{x}_t^f+\mathbf{h}_{\mathbf{x}}^l\mathbf{x}_t^f\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\eta\epsilon_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^l\mathbf{x}_t^f\\
&\quad+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\eta\epsilon_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^l\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^l\mathbf{x}_t^f+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\eta\epsilon_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\eta\epsilon_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^l\mathbf{x}_t^f\\
&\quad+\mathbf{h}_{\mathbf{x}}^l\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^l\mathbf{x}_t^f\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\eta\epsilon_{t+j}+\mathbf{h}_{\mathbf{x}}^l\mathbf{x}_t^f\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\eta\epsilon_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\eta\epsilon_{t+j}\\
&\quad+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\eta\epsilon_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^l\mathbf{x}_t^f\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\eta\epsilon_{t+j}+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\eta\epsilon_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\eta\epsilon_{t+j}\otimes\\
&\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\eta\epsilon_{t+j}.
\end{aligned}
$$

Using the definition of δ_{t+j} from Appendix A.10, we have

$$
\begin{aligned}
&\tilde{\mathbf{x}}_{t+l}^f\otimes\tilde{\mathbf{x}}_{t+l}^f\otimes\tilde{\mathbf{x}}_{t+l}^f=\mathbf{h}_{\mathbf{x}}^l\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^l\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^l\mathbf{x}_t^f+\mathbf{h}_{\mathbf{x}}^l\mathbf{x}_t^f\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\eta\delta_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^l\mathbf{x}_t^f\\
&\quad+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\eta\delta_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^l\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^l\mathbf{x}_t^f\\
&\quad+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\eta\delta_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\eta\delta_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^l\mathbf{x}_t^f\\
&\quad+\mathbf{h}_{\mathbf{x}}^l\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^l\mathbf{x}_t^f\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\eta\delta_{t+j}\\
&\quad+\mathbf{h}_{\mathbf{x}}^l\mathbf{x}_t^f\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\eta\delta_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\eta\delta_{t+j}\\
&\quad+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\eta\delta_{t+j}\otimes\mathbf{h}_{\mathbf{x}}^l\mathbf{x}_t^f\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\eta\delta_{t+j}\\
&\quad+\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\eta\delta_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\eta\delta_{t+j}\otimes\sum_{j=1}^{l}\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\eta\delta_{t+j}.
\end{aligned}
$$

Simple algebra gives

$$
\begin{aligned}
&GIRF_{\mathbf{x}^f\otimes\mathbf{x}^f\otimes\mathbf{x}^f}\left(j,\nu_i,\mathbf{x}_t^f\right)=\mathbb{E}\left[\tilde{\mathbf{x}}_{t+l}^f\otimes\tilde{\mathbf{x}}_{t+l}^f\otimes\tilde{\mathbf{x}}_{t+l}^f|\mathbf{x}_t^f\right]-\mathbb{E}\left[\mathbf{x}_{t+l}^f\otimes\mathbf{x}_{t+l}^f\otimes\mathbf{x}_{t+l}^f|\mathbf{x}_t^f\right]\\
&\quad=\mathbf{h}_{\mathbf{x}}^l\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\eta\nu\otimes\mathbf{h}_{\mathbf{x}}^l\mathbf{x}_t^f\\
&\qquad+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\eta\nu\otimes\left(\left(\mathbf{h}_{\mathbf{x}}^l\otimes\mathbf{h}_{\mathbf{x}}^l\right)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\right)\\
&\qquad+\left(\left(\mathbf{h}_{\mathbf{x}}^l\otimes\mathbf{h}_{\mathbf{x}}^l\right)\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)\right)\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\eta\nu\\
&\qquad+\left(\mathbf{h}_{\mathbf{x}}^{l-1}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\right)[(\sigma\eta\nu\otimes\sigma\eta\nu)+\Lambda]\otimes\mathbf{h}_{\mathbf{x}}^l\mathbf{x}_t^f\\
&\qquad+\mathbf{h}_{\mathbf{x}}^l\mathbf{x}_t^f\otimes\left(\mathbf{h}_{\mathbf{x}}^{l-1}\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\right)[(\sigma\eta\nu\otimes\sigma\eta\nu)+\Lambda]\\
&\qquad+\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\eta\nu\otimes\mathbf{h}_{\mathbf{x}}^l\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\eta\nu\\
&\qquad+\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\eta(\mathbf{I}-\mathbf{S})\otimes\mathbf{h}_{\mathbf{x}}^l\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\eta(\mathbf{I}-\mathbf{S})-\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\eta\otimes\mathbf{h}_{\mathbf{x}}^l\mathbf{x}_t^f\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\eta\right)vec(\mathbf{I})
\end{aligned}
$$

$$
\begin{aligned}
&+\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\eta\nu\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\eta\left(\mathbf{I}-\mathbf{S}\right)\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\eta\left(\mathbf{I}-\mathbf{S}\right)\right)vec\left(\mathbf{I}\right)\\
&+\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\eta\left(\mathbf{I}-\mathbf{S}\right)\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\eta\left(\mathbf{I}-\mathbf{S}\right)\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\eta\nu\right)vec\left(\mathbf{I}\right)\\
&+\left(\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\eta\left(\mathbf{I}-\mathbf{S}\right)\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\eta\nu\right)\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\eta\left(\mathbf{I}-\mathbf{S}\right)\right)vec\left(\mathbf{I}\right)\\
&+\left\{\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\eta\left(\mathbf{I}-\mathbf{S}\right)\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\eta\left(\mathbf{I}-\mathbf{S}\right)\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\eta\left(\mathbf{I}-\mathbf{S}\right)\right\}\mathbf{m}^3\left(\epsilon_{t+1},\epsilon_{t+1},\epsilon_{t+1}\right)\\
&+\sum_{j=2}^{l}\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\eta\nu\otimes\left(\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\eta\otimes\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\eta\right)vec\left(\mathbf{I}\right)\\
&+\sum_{j=2}^{l}\left(\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\eta\otimes\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\eta\right)vec\left(\mathbf{I}\right)\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\eta\nu\\
&+\sum_{j=2}^{l}\left(\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\eta\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\eta\nu\otimes\mathbf{h}_{\mathbf{x}}^{l-j}\sigma\eta\right)vec\left(\mathbf{I}\right)\\
&-\left(\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\eta\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\eta\otimes\mathbf{h}_{\mathbf{x}}^{l-1}\sigma\eta\right)\mathbf{m}^3\left(\epsilon_{t+1},\epsilon_{t+1},\epsilon_{t+1}\right),
\end{aligned}
$$

where $\mathbf{m}^3\left(\epsilon_{t+1},\epsilon_{t+1},\epsilon_{t+1}\right)$ has dimension $n_e^3\times 1$ and contains all the third moments of ϵ_{t+1}.

Deriving $GIRF_{\mathbf{x}^f\otimes\mathbf{x}^s}\left(j,\nu,\left(\mathbf{x}_t^f,\mathbf{x}_t^s\right)\right)$. Using the law of motion for $\mathbf{x}_t^f\otimes\mathbf{x}_t^s$, we first note that

$$
\begin{aligned}
\mathbf{x}_{t+l}^f\otimes\mathbf{x}_{t+l}^s=&\left(\mathbf{h}_{\mathbf{x}}\otimes\mathbf{h}_{\mathbf{x}}\right)^l\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^s\right)+\sum_{j=0}^{l-1}\left(\mathbf{h}_{\mathbf{x}}\otimes\mathbf{h}_{\mathbf{x}}\right)^{l-1-j}\left(\mathbf{h}_{\mathbf{x}}\otimes\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\right)\left(\mathbf{x}_{t+j}^f\otimes\mathbf{x}_{t+j}^f\otimes\mathbf{x}_{t+j}^f\right)\\
&+\sum_{j=0}^{l-1}\left(\mathbf{h}_{\mathbf{x}}\otimes\mathbf{h}_{\mathbf{x}}\right)^{l-1-j}\left(\mathbf{h}_{\mathbf{x}}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\mathbf{x}_{t+j}^f\\
&+\sum_{j=0}^{l-1}\left(\mathbf{h}_{\mathbf{x}}\otimes\mathbf{h}_{\mathbf{x}}\right)^{l-1-j}\left(\sigma\eta\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\epsilon_{t+1+j}\\
&+\sum_{j=0}^{l-1}\left(\mathbf{h}_{\mathbf{x}}\otimes\mathbf{h}_{\mathbf{x}}\right)^{l-1-j}\left(\sigma\eta\otimes\mathbf{h}_{\mathbf{x}}\right)\left(\epsilon_{t+1+j}\otimes\mathbf{x}_{t+j}^s\right)\\
&+\sum_{j=0}^{l-1}\left(\mathbf{h}_{\mathbf{x}}\otimes\mathbf{h}_{\mathbf{x}}\right)^{l-1-j}\left(\sigma\eta\otimes\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\right)\left(\epsilon_{t+1+j}\otimes\mathbf{x}_{t+i}^f\otimes\mathbf{x}_{t+j}^f\right)
\end{aligned}
$$

Using the definition of δ_{t+j} from Appendix A.10, we obtain

$$
\begin{aligned}
\tilde{\mathbf{x}}_{t+l}^f\otimes\tilde{\mathbf{x}}_{t+l}^s=&\left(\mathbf{h}_{\mathbf{x}}\otimes\mathbf{h}_{\mathbf{x}}\right)^l\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^s\right)+\sum_{j=0}^{l-1}\left(\mathbf{h}_{\mathbf{x}}\otimes\mathbf{h}_{\mathbf{x}}\right)^{l-1-j}\left(\mathbf{h}_{\mathbf{x}}\otimes\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\right)\left(\tilde{\mathbf{x}}_{t+j}^f\otimes\tilde{\mathbf{x}}_{t+j}^f\otimes\tilde{\mathbf{x}}_{t+j}^f\right)\\
&+\sum_{j=0}^{l-1}\left(\mathbf{h}_{\mathbf{x}}\otimes\mathbf{h}_{\mathbf{x}}\right)^{l-1-j}\left(\mathbf{h}_{\mathbf{x}}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\tilde{\mathbf{x}}_{t+j}^f\\
&+\sum_{j=0}^{l-1}\left(\mathbf{h}_{\mathbf{x}}\otimes\mathbf{h}_{\mathbf{x}}\right)^{l-1-j}\left(\sigma\eta\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\delta_{t+1+j}\\
&+\sum_{j=0}^{l-1}\left(\mathbf{h}_{\mathbf{x}}\otimes\mathbf{h}_{\mathbf{x}}\right)^{l-1-j}\left(\sigma\eta\otimes\mathbf{h}_{\mathbf{x}}\right)\left(\delta_{t+1+j}\otimes\tilde{\mathbf{x}}_{t+j}^s\right)\\
&+\sum_{j=0}^{l-1}\left(\mathbf{h}_{\mathbf{x}}\otimes\mathbf{h}_{\mathbf{x}}\right)^{l-1-j}\left(\sigma\eta\otimes\tfrac{1}{2}\mathbf{H}_{\mathbf{xx}}\right)\left(\delta_{t+1+j}\otimes\tilde{\mathbf{x}}_{t+j}^f\otimes\tilde{\mathbf{x}}_{t+j}^f\right)
\end{aligned}
$$

Simple algebra then implies

$$GIRF_{\mathbf{x}^f\otimes\mathbf{x}^s}\left(j,\nu_i,\left(\mathbf{x}_t^f,\mathbf{x}_t^s\right)\right)=\sum_{j=1}^{l-1}\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)^{l-1-j}\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{H_{xx}}\right)GIRF_{\mathbf{x}^f\otimes\mathbf{x}^f\otimes\mathbf{x}^f}\left(j,\nu_i,\mathbf{x}_t^f\right)$$
$$+\sum_{j=1}^{l-1}\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)^{l-1-j}\left(\mathbf{h_x}\otimes\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)GIRF_{\mathbf{x}^f}\left(j,\nu_i\right)$$
$$+\left(\mathbf{h_x}\otimes\mathbf{h_x}\right)^{l-1}\left(\sigma\eta\nu\otimes\left(\mathbf{h_x}\mathbf{x}_t^s+\tfrac{1}{2}\mathbf{H_{xx}}\left(\mathbf{x}_t^f\otimes\mathbf{x}_t^f\right)+\tfrac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)\right)$$

A.13. *An alternative interpretation*

The deposit rate r_t^b only enters in equations (6), (12), and (15) of the model summary in Appendix A.14. But note that $r_t=r_t^b-\omega\times xhr_{t,L}$ and when substituted into the Taylor rule we get

$$r_t^b = (1-\rho_r)r_{ss}+\rho_r r_{t-1}^b+(1-\rho_r)\left(\beta_\pi\log\left(\frac{\pi_t}{\pi_{ss}}\right)+\beta_y\log\left(\frac{y_t}{z_t^*Y_{ss}}\right)\right)$$
$$+\omega\times xhr_{t,L}-\rho_r\omega\times xhr_{t-1,L}+(1-\rho_r)\beta_{xhr}\left(xhr_{t,L}-X_{t,L}\right).$$

Given this substitution, r_t^b only enters in equations (6) and (15) of the model summary in Appendix A.14. This implies that our model is equivalent to a standard New Keynesian model with market completeness, but with a Taylor rule for r_t^b that depends on past and current values of the excess holding period return on the long bond.

A.14. *Making the DSGE model stationary*

We eliminate all trending variables in the model by adopting the transformation $C_t\equiv\frac{c_t}{z_t^*}$, $R_t^k\equiv\Upsilon_t r_t^k$, $Q_t\equiv\Upsilon_t q_t$, $I_t\equiv\frac{i_t}{\Upsilon_t z_t^*}$, $W_t\equiv\frac{w_t}{z_t^*}$, $Y_t\equiv\frac{y_t}{z_t^*}$, $K_{t+1}\equiv\frac{k_{t+1}}{\Upsilon_t^{\frac{1}{1-\theta}}z_t}=\frac{k_{t+1}}{\Upsilon_t z_t^*}$, and $\Lambda_t\equiv\frac{\lambda_t}{m_t\left(z_t^*\right)^{-1}}$. Here q_t is the Lagrangian multiplier for the law of motion for capital and m_t for the value function in equation (27); see Rudebusch and Swanson (2012). Hence, $\mu_{\lambda,t+1}\equiv\frac{\lambda_{t+1}}{\lambda_t}=\frac{\Lambda_{t+1}}{\Lambda_t}\mu_{z^*,t+1}^{-1}\left(\mathbb{E}_t\left[V_{t+1}^{1-\phi_3}\right]\right)^{\frac{\phi_3}{1-\phi_3}}V_{t+1}^{-\phi_3}$, and the value of ψ that eliminates capital adjustment costs in the steady state is therefore given by $\psi\equiv\frac{I_{ss}}{K_{ss}}\mu_{\Upsilon,ss}\mu_{z^*,ss}$.

The transformed equilibrium conditions are summarized below. From these equilibrium conditions, it is straightforward to derive a closed-form solution for the steady state of the model.

Eq.	**The Households**
1	$V_t=\left[\frac{d_t}{1-\phi_2}\left(\left(C_t-bC_{t-1}\mu_{z^*,t}^{-1}\right)^{1-\phi_2}-1\right)+d_t\phi_0\frac{(1-h_t)^{1-\phi_1}}{1-\phi_1}\right]+\beta\left(\mathbb{E}_t\left[V_{t+1}^{1-\phi_3}\right]\right)^{\frac{1}{1-\phi_3}}$
2	$\Lambda_t=d_t\left(C_t-bC_{t-1}\mu_{z^*,t}^{-1}\right)^{-\phi_2}$ $-b\beta\mathbb{E}_t\left[\left(\frac{\left[E_t\left[V_{t+1}^{1-\phi_3}\right]\right]^{\frac{1}{1-\phi_3}}}{V_{t+1}(s)}\right)^{\phi_3}d_{t+1}\left(C_{t+1}-bC_t\mu_{z^*,t+1}^{-1}\right)^{-\phi_2}\left(\mu_{z^*,t+1}\right)^{-1}\right]$
3	$Q_t=\mathbb{E}_t\frac{\beta\mu_{\lambda,t+1}}{\mu_{\Upsilon,t+1}}[R_{t+1}^k+Q_{t+1}(1-\delta)-Q_{t+1}\frac{\kappa}{2}\left(\frac{I_{t+1}}{K_{t+1}}\mu_{\Upsilon,t+1}\mu_{z^*,t+1}-\frac{I_{ss}}{k_{ss}}\mu_{\Upsilon,ss}\mu_{z^*,ss}\right)^2$ $+Q_{t+1}\kappa\left(\frac{I_{t+1}}{K_{t+1}}\mu_{\Upsilon,t+1}\mu_{z^*,t+1}-\frac{I_{ss}}{k_{ss}}\mu_{\Upsilon,ss}\mu_{z^*,ss}\right)\frac{I_{t+1}}{K_{t+1}}\mu_{\Upsilon,t+1}\mu_{z^*,t+1}]$
4	$d_t\phi_0(1-h_t)^{-\phi_1}=\Lambda_t W_t$
5	$1=Q_t\left(1-\kappa\left(\frac{I_t}{k_t}\mu_{\Upsilon,t}\mu_{z^*,t}-\frac{I_{ss}}{K_{ss}}\mu_{\Upsilon,ss}\mu_{z^*,ss}\right)\right)$
6	$1=\mathbb{E}_t\left[\beta\mu_{\lambda t+1}\frac{\exp\{r_t^b\}}{\pi_{t+1}}\right]$

	The Firms
7	$mc_t a_t \theta \mu_{\Upsilon,t} \mu_{z,t}^{1-\theta} K_t^{\theta-1} h_t^{1-\theta} = R_t^k$
8	$mc_t (1-\theta) a_t \mu_{\Upsilon,t}^{\frac{-\theta}{1-\theta}} \mu_{z,t}^{-\theta} K_t^{\theta} h_t^{-\theta} = W_t$
9	$\frac{(\eta-1)}{\eta} X_t^2 = Y_t mc_t \tilde{p}_t^{-\eta-1} + \mathbb{E}_t\left[\alpha\beta\mu_{\lambda,t+1}\left(\frac{\tilde{p}_t}{\tilde{p}_{t+1}}\right)^{-\eta-1}\left(\frac{1}{\pi_{t+1}}\right)^{-\eta}\frac{(\eta-1)}{\eta} X_{t+1}^2 \mu_{z^*,t+1}\right]$
10	$X_t^2 = Y_t \tilde{p}_t^{-\eta} + \mathbb{E}_t\left[\alpha\beta\mu_{\lambda,t+1}\left(\frac{\tilde{p}_t}{\tilde{p}_{t+1}}\right)^{-\eta}\left(\frac{1}{\pi_{t+1}}\right)^{1-\eta} X_{t+1}^2 \mu_{z^*,t+1}\right]$
11	$1 = (1-\alpha)\tilde{p}_t^{1-\eta} + \alpha\left(\frac{1}{\pi_t}\right)^{1-\eta}$
	The Financial Intermediary
12	$r_t^b = r_t + \omega \times xhr_{t,L}$
13	$P_{t,1} = \frac{1}{\exp\{r_t\}}$
14	$P_{t,k} = \mathbb{E}_t\left[\beta\mu_{\lambda,t+1}\frac{1}{\pi_{t+1}} P_{t+1,k-1}\right]$ for $k = 2,3,...,K$
15	$xhr_{t,k} \equiv \mathbb{E}_t\left[\log P_{t+1,k-1} - \log P_{t,k}\right] - r_t$ for $k = 2,3,...,K$
	The Central Bank
16	$r_t = r_{ss}(1-\rho_r) + \rho_r r_{t-1} + (1-\rho_r)\left(\beta_\pi \log\left(\frac{\pi_t}{\pi_{ss}}\right) + \beta_y \log\left(\frac{Y_t}{Y_{ss}}\right)\right)$
	$+(1-\rho_r)\beta_{xhr}\left(xhr_{t,L} - X_{t,L}\right)$
17	$X_{t,L} = (1-\gamma) xhr_{t,L} + \gamma \mathbb{E}_t\left[X_{t+1,L}\right]$
	Other relations
18	$a_t\left(K_t \mu_{\Upsilon,t}^{\frac{-1}{1-\theta}} \mu_{z,t}^{-1}\right)^{\theta} h_t^{1-\theta} = Y_t s_{t+1}$
19	$s_{t+1} = (1-\alpha)\tilde{p}_t^{-\eta} + \alpha \pi_t^{\eta} s_t$
20	$K_{t+1} = (1-\delta) K_t \left(\mu_{\Upsilon,t}\mu_{z^*,t}\right)^{-1} + I_t$
	$-K_t\left(\mu_{\Upsilon,t}\mu_{z^*,t}\right)^{-1}\frac{\kappa}{2}\left(\frac{I_t}{K_t}\mu_{\Upsilon,t}\mu_{z^*,t} - \frac{I_{ss}}{K_{ss}}\mu_{\Upsilon,ss}\mu_{z^*,ss}\right)^2$
21	$Y_t = C_t + I_t + g_t$
22	$\mu_{z^*,t} \equiv \mu_{\Upsilon,t}^{\theta/(1-\theta)} \mu_{z,t}$
	Exogenous processes
23	$\log\left(\mu_{z,t}\right) = \log\left(\mu_{z,ss}\right)$ and $z_{t+1} \equiv z_t \mu_{z,t+1}$ (i.e. a deterministic trend)
24	$\log\left(\mu_{\Upsilon,t}\right) = \log \mu_{\Upsilon,ss}$ and $\Upsilon_{t+1} \equiv \Upsilon_t \mu_{\Upsilon,t+1}$ (i.e. a deterministic trend)
25	$\log a_{t+1} = \rho_a \log a_t + \sigma_a \epsilon_{a,t+1}$
26	$\log\left(\frac{G_{t+1}}{G_{ss}}\right) = \rho_G \log\left(\frac{G_t}{G_{ss}}\right) + \sigma_G \epsilon_{G,t+1}$
27	$\log d_{t+1} = \sigma_d \epsilon_{d,t+1}$

A.15. *An efficient perturbation approximation*

To formally present our efficient perturbation approximation, consider the decomposition $\mathbf{y}_t \equiv \left[\left(\mathbf{y}_t^{macro}\right)' \ \left(\mathbf{y}_t^{bonds}\right)'\right]$ and similarly for all derivatives of $\mathbf{g}(\mathbf{x}_t,\sigma)$. Here, $\mathbf{y}_t^{macro}$ refers to the control variables needed to solve the model without feedback effects from long-term bond prices to the real economy (when $\omega = 0$ and $\beta_{xhr} = 0$), whereas $\mathbf{y}_t^{bonds}$ denotes the remaining variables related to pricing government bonds and computing excess holding period returns. Our three-step perturbation approximation is

Step 1:Solve for $\left(\mathbf{g}_{\mathbf{x}}^{macro}, \mathbf{G}_{\mathbf{xx}}^{macro}, \mathbf{G}_{\mathbf{xxx}}^{macro}\right)$ and $(\mathbf{h}_{\mathbf{x}}, \mathbf{H}_{\mathbf{xx}}, \mathbf{H}_{\mathbf{xxx}})$ by a standard perturbation algorithm using a version of our model *without* feedback effects from government bonds to the real economy. This version of our model has only eleven control variables and eighteen equations and is solved using the `Matlab` codes of Binning (2013).

Step 2:Use the perturbation algorithm of Andreasen and Zabczyk (2015) to recursively solve for $\left(\mathbf{g}_{\mathbf{x}}^{bonds}, \mathbf{G}_{\mathbf{xx}}^{bonds}, \mathbf{G}_{\mathbf{xxx}}^{bonds}\right)$, given the derivatives obtained in Step 1.

Step 3:With the derivatives obtained in Steps 1 and 2, solve for $(\mathbf{g}_{\sigma\sigma},\mathbf{g}_{\sigma\sigma\mathbf{x}},\mathbf{g}_{\sigma\sigma\sigma})$ and $(\mathbf{h}_{\sigma\sigma},\mathbf{h}_{\sigma\sigma\mathbf{x}},\mathbf{h}_{\sigma\sigma\sigma})$ by the standard perturbation algorithm when using the full model with fifty-four control variables and sixty-one equations.

To maximize the efficiency of our perturbation algorithm, steps 2 and 3 are computed using a `FORTRAN` implementation accessible via MEX files in `Matlab`.

A.16. *Data for the application*

We use data from the Federal Reserve Bank of St. Louis covering the period 1961.Q3 to 2007.Q4, giving a total of 186 observations. The annualized growth rate in consumption is calculated from real consumption expenditures (PCECC96). The series for real private fixed investment (FPIC96) is used to calculate the growth rate in investment. Both growth rates are expressed in per capita terms based on the total population in the U.S. The ratio of government spending to output is computed as government consumption expenditures and investments divided by gross domestic production. The annual inflation rate is for consumer prices. The 3-month nominal interest rate is measured by the rate in the secondary market (TB3MS), and the 10-year nominal rate is from Gürkaynak *et al.* (2007). As in Rudebusch and Swanson (2012), observations for the 10-year interest rate from 1961.Q3 to 1971.Q3 are calculated by extrapolation of the estimated curves in Gürkaynak *et al.* (2007). All moments related to interest rates are expressed in annualized terms. Finally, we use average weekly hours of production and non-supervisory employees in manufacturing (AWHMAN) as provided by the Bureau of Labor Statistics. The series is normalized by dividing it by five times 24 hours, giving a mean level of 0.34.

A.17. *Approximate expression for excess holding period return*

First,

$$xhr_{t,L}=\mathbb{E}_t\left[\log\left(P_{t+1,L-1}\right)-\log\left\{\mathbb{E}_t\left[M_{t,t+1}\right]\mathbb{E}_t\left[P_{t+1,L-1}\right]+Cov_t\left(M_{t,t+1},P_{t+1,L-1}\right)\right\}\right]-r_t.$$

To first order,

$$\log(x_t+y_t)\approx\log(x_{ss}+y_{ss})+\frac{1}{x_{ss}+y_{ss}}(x_t-x_{ss})+\frac{1}{x_{ss}+y_{ss}}(y_t-y_{ss})$$

and let $x_t=\mathbb{E}_t\left[M_{t,t+1}\right]\mathbb{E}_t\left[P_{t+1,L-1}\right]$ and $y_t=Cov_t\left(M_{t,t+1},P_{t+1,L-1}\right)$, implying that $x_t+y_t=P_{t,L}$.

Hence,

$$\begin{aligned}
xhr_{t,L}\approx&\,\mathbb{E}_t\left[\log\left(P_{t+1,L-1}\right)\right]-\log P_{ss,L}-\tfrac{1}{P_{ss,L}}\left(\mathbb{E}_t\left[M_{t,t+1}\right]\mathbb{E}_t\left[P_{t+1,L-1}\right]-M_{ss,ss+1}P_{ss,L-1}\right)\\
&-\tfrac{1}{P_{ss,L}}Cov_t\left(M_{t,t+1},P_{t+1,L-1}\right)-r_t\\
=&\,\mathbb{E}_t\left[\log\left(P_{t+1,L-1}\right)\right]-\log P_{ss,L}-\tfrac{\mathbb{E}_t\left[P_{t+1,L-1}\right]}{P_{ss,L}}e^{-r_t-\omega\times xhr_{t,L}}+1-\tfrac{Cov_t\left(M_{t,t+1},P_{t+1,L-1}\right)}{P_{ss,L}}-r_t\\
\approx&\,\mathbb{E}_t\left[\log\left(\tfrac{P_{t+1,L-1}}{P_{ss,L}}\right)\right]-\tfrac{\mathbb{E}_t\left[P_{t+1,L-1}\right]}{P_{ss,L}}\left(e^{-r_{ss}}\left(1-r_t-\omega\times xhr_{t,L}+r_{ss}\right)\right)-\tfrac{Cov_t\left(M_{t,t+1},P_{t+1,L-1}\right)}{P_{ss,L}}\\
&+1-r_t
\end{aligned}$$

as

$$\mathbb{E}_t\left[M_{t,t+1}\right]=e^{-r_t-\omega\times xhr_{t,L}},P_{ss,L}=M_{ss,ss+1}P_{ss,L-1}$$

and

$$e^{-r_t-\omega\times xhr_{t,L}}\approx e^{-r_{ss}}\left(1-r_t-\omega\times xhr_{t,L}+r_{ss}\right).$$

Finally, using $\log\left(\frac{P_{t+1,L-1}}{P_{ss,L-1}}\right)\approx\frac{P_{t+1,L-1}}{P_{ss,L-1}}-1$, we obtain (41).

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Supplementary Data

Supplementary data are available at *Review of Economic Studies* online.

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