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A two-phase model and an application to verbal discrimination learning

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Among current mathematical learning theories that have been applied to discrete-trial two-choice experiments in which the same response (the "correct" response) is reinforced on every trial, two of the simplest and most antithetical are the all-or-none model developed by Bower (1961a)[1] and the single-operator linear model developed by Bush and Sternberg (1959).[2] In the all-or-none model with parameter values p and c, we identify an event termed "conditioning," which for any subject occurs on at most one trial; the probability that conditioning occurs on trial k given that it has not occurred on a previous trial is c. A subject's probability of making the incorrect response on trial n given that conditioning occurred on trial k is $1 - p$ if $n \leq k$ and 0 if $n > k$. According to the single-operator model with parameter values p and α, a subject's probability q_n of making an error on trial n satisfies the linear difference equation

$$q_n = \alpha q_{n-1} \qquad (n = 2, 3, \cdots),$$

which has solution

$$(1) \qquad q_n = \alpha^{n-1}(1 - p) \qquad (n = 1, 2, \cdots),$$

where $1 - p$ is the subject's initial probability of making an error.

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[1] The first all-or-none mathematical model to appear in the literature is due to Bush and Mosteller (1959), who derived many of its properties. For convenience, however, we will refer only to Bower's presentation in what follows.

The stimulus-sampling interpretation of the all-or-none model, which leads to the term "one-element model" which is often applied to it, is explained by Bower (1961a, pp. 256–257).

In the special case $p = 0$, the all-or-none model discussed in this paper is identical to the model with the same name considered by Estes (1960).

[2] The single-operator linear model is the simplest member of the family of linear-operator models studied by Bush and Mosteller (1955).

The two-phase model that will now be discussed is obtained from the two models discussed above in a very transparent way.[3] In the two-phase model with parameter values p, c, and α, we identify an event called "first learning," which for any subject occurs on at most one trial; the probability that first learning occurs on trial k given that it has not occurred on a previous trial is c. A subject's probability of making an incorrect response on trial n given that first learning occurred on trial k is $1 - p$ if $n \leq k$ and $\alpha^{n-k}(1 - p)$ if $n \geq k$. If we denote the event "first learning occurs on trial k" by $C^{(k)}$, an immediate consequence of the assumption concerning first learning is that the trial of first learning has a geometric distribution with parameter c, i.e.,

$$\text{(2)} \qquad \Pr(C^{(k)}) = c(1 - c)^{k-1} \qquad (k = 1, 2, \cdots).$$

If we denote the event "error on trial n" by $A_{2,n}$ the assumption concerning the relation between first learning and responding can be expressed by

$$\text{(3)} \qquad \Pr(A_{2,n} \mid C^{(k)}) = \begin{cases} 1 - p & \text{for } n \leq k, \\ \alpha^{n-k}(1 - p) & \text{for } n \geq k. \end{cases}$$

As a preliminary observation we note that the all-or-none and single-operator models are special cases ($\alpha = 0$ and $c = 1$, respectively) of the two-phase model. In what follows we will assume that $0 < c < 1$ and that $0 < \alpha < 1$, though we will permit p to take on any value in the closed unit interval.

1. The mean learning curve

For any positive integer n,

$$\begin{aligned} \Pr(A_{2,n}) &= \sum_{k=1}^{n} \Pr(A_{2,n} \mid C^{(k)}) \Pr(C^{(k)}) + \sum_{k=n+1}^{\infty} \Pr(A_{2,n} \mid C^{(k)}) \Pr(C^{(k)}) \\ &= \sum_{k=1}^{n} (1 - p)\alpha^{n-k} c(1 - c)^{k-1} + \sum_{k=n+1}^{\infty} (1 - p)c(1 - c)^{k-1}; \end{aligned}$$

and by Eqs. (2) and (3) it becomes

$$(1 - p)\left[\alpha^{n-1} c \sum_{k=1}^{n} \left(\frac{1 - c}{\alpha}\right)^{k-1} + c \sum_{k=n+1}^{\infty} (1 - c)^{k-1}\right].$$

Thus after performing the summations and making some elementary algebraic manipulations, we obtain[4]

$$\text{(4)} \quad \Pr(A_{2,n}) = \begin{cases} (1 - p)\left[c\,\dfrac{\alpha^n - (1 - c)^n}{\alpha - (1 - c)} + (1 - c)^n\right] & \text{if } \alpha \neq 1 - c, \\ (1 - p)\alpha^{n-1}[n(1 - \alpha) + \alpha] & \text{if } \alpha = 1 - c. \end{cases}$$

[3] The axioms for the two-phase model were suggested by Dr. Richard C. Atkinson.

[4] The summation formulas needed for the derivations in this paper are among those tabled by Cogan and Norman (1958, pp. 236–237, 240–241).

2. Mean and variance of the total number of errors

Let T be the total number of errors in an infinite sequence of trials in the two-phase model. Clearly, $T = T_u + T_c$ and thus $E(T) = E(T_u) + E(T_c)$, where T_u is the total number of errors through the trial of first learning (tfl), and T_c is the total number of errors after tfl. It is also apparent that T_u and T_c are stochastically independent, so $\text{var}(T) = \text{var}(T_u) + \text{var}(T_c)$. The stochastic structure of the trials through tfl in the two-phase model is identical with that of the trials through the trial of conditioning in the all-or-none model with corresponding parameter values. Thus the distribution of T_u is the same as the distribution of the total number of errors through the trial of conditioning in the latter model. Therefore

$$E(T_u) = \frac{1-p}{c} \tag{5}$$

and

$$\text{var}(T_u) = \frac{1-p}{c} + (1-2c)\frac{(1-p)^2}{c^2}, \tag{6}$$

since the expressions on the right-hand sides of these equations give, respectively, the mean and variance of the total number of errors through the trial of conditioning in the all-or-none model with parameter values p and c (Bower, 1961a, pp. 259, 261). Also, the probabilistic characteristics of the trials after tfl in the two-phase model are precisely those of the trials after the first trial in the single-operator model with parameter values p and α, or, equivalently, those of all trials in the single-operator model with parameter values $1 - \alpha(1-p)$ and α. Thus the distribution of T_c is identical to the distribution of the total number of errors in the latter model. Therefore

$$E(T_c) = \frac{\alpha(1-p)}{1-\alpha} \tag{7}$$

and

$$\text{var}(T_c) = \frac{\alpha(1-p)}{1-\alpha} - \frac{\alpha^2(1-p)^2}{1-\alpha^2}, \tag{8}$$

since the expressions on the right-hand sides of these equations give, respectively, the mean and variance of the total number of errors for the single-operator model with parameter values $1 - \alpha(1-p)$ and α (Tatsuoka and Mosteller, 1959, pp. 233, 236). We thus conclude that

$$E(T) = (1-p)\left(\frac{1}{c} + \frac{\alpha}{1-\alpha}\right) \tag{9}$$

and that

$$\begin{aligned} \text{var}(T) &= (1-p)\left(\frac{1}{c} + \frac{\alpha}{1-\alpha}\right) + (1-p)^2\left(\frac{1-2c}{c^2} - \frac{\alpha^2}{1-\alpha^2}\right) \\ &= E(T) + (1-p)^2\left(\frac{1-2c}{c^2} - \frac{\alpha^2}{1-\alpha^2}\right). \end{aligned} \tag{10}$$

3. Distribution of the total number of errors

The distribution of T is the convolution of the distributions of the independent random variables T_u and T_c, i.e., for all $k \geq 0$,

$$\Pr(T = k) = \sum_{m=0}^{k} \Pr(T_u = k - m)\Pr(T_c = m). \tag{11}$$

As we have previously observed, the distributions of T_u and T_c in the two-phase model with parameter values p, c, and α are identical, respectively, to the distribution of total errors in the all-or-none model with parameter values p and c and to the distribution of total errors in the single-operator model with parameter values $1 - \alpha(1 - p)$ and α. Thus (see Bower, 1961a, p. 260) we have the following expression for the distribution of T_u:

$$\Pr(T_u = k - m) = \begin{cases} bp & \text{for } k = m, \\ b\dfrac{(1 - b)^{k-m}}{1 - c} & \text{for } k > m, \end{cases} \tag{12}$$

where $b = c/1 - p(1 - c)$, and, as a simple consequence of the results obtained by Tatsuoka and Mosteller (1959, pp. 237–239),

$$\Pr(T_c = m) = [\alpha(1 - p)]^m \sum_{r=0}^{\infty} C_r(m)[\alpha(1 - p)]^r, \tag{13}$$

where

$$C_0(0) = 1 \tag{14}$$

and

$$C_r(m) = \frac{(-1)^r \alpha^{r(r-1)/2 + mr + m(m-1)/2}}{\prod_{i=1}^{m+r}(1 - \alpha^i)} \binom{r + m}{r} \tag{15}$$

for $r + m > 0$.

4. Various sequential statistics

Let x_n be the random variable that takes on the value 1 or 0 depending on whether there is an error or a correct response on trial n. Then u_j, the total number of j-tuples of errors for the two-phase model, is given by

$$u_j = \sum_{n=1}^{\infty} \prod_{i=0}^{j-1} x_{n+i}. \tag{16}$$

Clearly, $u_1 = T$. For $j \geq 2$ we have the representation

$$u_j = \sum_{i=1}^{j} u_{j,i}, \tag{17}$$

where

$u_{j,1}$ is the number of j-tuples of errors with last coordinate on or before tfl,

$u_{j,j}$ is the number of j-tuples of errors with first coordinate on or after tfl, and, for $j \geq 3$ and $1 < i < j$,

$u_{j,i}$ is the number of j-tuples of errors with $(j - i + 1)$th coordinate on tfl.

Obviously $u_{j,1}$ and $u_{j,j}$ have the same distributions as the total numbers of j-tuples of errors in the all-or-none model with parameter values p and c and in the single-operator model with parameter values p and α, respectively. Thus (see Bower, 1961a, pp. 262–63)

$$E(u_{j,1}) = \frac{(1-p)^j(1-c)^{j-1}}{c} \tag{18}$$

and (see Bush and Sternberg, 1959, p. 207)

$$E(u_{j,j}) = (1-p)^j \frac{\alpha^{j(j-1)/2}}{1-\alpha^j}. \tag{19}$$

Therefore from Eq. (17) we get

$$E(u_2) = (1-p)^2\left(\frac{1-c}{c} + \frac{\alpha}{1-\alpha^2}\right). \tag{20}$$

For $j \geq 3$ and $1 < i < j$,

$$\begin{aligned}
E(u_{j,i}) &= \sum_{k=1}^{\infty} E(u_{j,i} \mid C^{(k)}) \Pr(C^{(k)}) \\
&= \sum_{k=j-i+1}^{\infty} \Pr\left(\prod_{l=1}^{j} x_{k+l-(j-i+1)} = 1 \mid C^{(k)}\right) \Pr(C^{(k)}) \\
&= \sum_{k=j-i+1}^{\infty} \left(\prod_{l=1}^{j} \Pr(x_{k+l-(j-i+1)} = 1 \mid C^{(k)})\right) \Pr(C^{(k)}),
\end{aligned}$$

since the error indicator random variables are independent with respect to the conditional probability induced by conditioning on $C^{(k)}$. Thus using Eqs. (2) and (3), we obtain

$$\begin{aligned}
E(u_{j,i}) &= (1-p)^j \sum_{k=j-i+1}^{\infty} \left(\prod_{l=j-i+1}^{j} \alpha^{l-(j-i+1)}\right) c(1-c)^{k-1} \\
&= (1-p)^j \alpha^{(i-1)i/2}(1-c)^{j-i}
\end{aligned}$$

by elementary manipulations. Therefore, for $j \geq 3$,

$$E(u_j) = (1-p)^j\left[\frac{(1-c)^{j-1}}{c} + \sum_{i=2}^{j-1} \alpha^{i(i-1)/2}(1-c)^{(j-i)} + \frac{\alpha^{(j-1)j/2}}{1-\alpha^j}\right]. \tag{21}$$

From these results the expected total number R of runs of errors and the expected total number r_j, $j = 1, 2, \cdots$ of runs of errors of length j are easily obtained by means of the equations (see Bush, 1959, p. 216)

$$E(R) = E(u_1) - E(u_2) \tag{22}$$

and

$$E(r_j) = E(u_j) - 2E(u_{j+1}) + E(u_{j+2}). \tag{23}$$

Let A be the total number of alternations of correct responses and errors in an infinite sequence of trials. A moment's reflection shows that $A = 2R - x_1$ for all subjects who make only a finite number of errors, and, since it is easily seen that the probability is unity that a subject makes only a finite number of errors, we have

$$E(A) = 2E(R) - (1 - p). \tag{24}$$

The statistic $\mathbf{c}_l$ defined by

$$\mathbf{c}_l = \sum_{n=1}^{\infty} x_n x_{n+l} \tag{25}$$

is often of interest. Clearly, $\mathbf{c}_l = u_2$. For $1 \geq 2$ we have

$$E(\mathbf{c}_l) = (1 - p)^2 \left[\frac{(1-c)^l}{c} + (1-c)^l \sum_{j=2}^{l} \left(\frac{\alpha}{1-c} \right)^{j-1} + \frac{\alpha^l}{1-\alpha^2} \right]. \tag{26}$$

The derivation of this result is completely analogous to the derivation of Eq. (21) and we therefore omit it.

We conclude this section by obtaining an expression for $\Pr(x_n = 1, x_{n+l} = 1)$, $l \geq 1$, for the two-phase model:

$$\Pr(x_n = 1, x_{n+l} = 1) = \sum_{k=1}^{\infty} \Pr(x_n = 1, x_{n+l} = 1 \mid C^{(k)}) \Pr(C^{(k)})$$

$$= \left(\sum_{k=1}^{n} + \sum_{k=n+1}^{n+l} + \sum_{k=n+l+1}^{\infty} \right) \Pr(x_n = 1 \mid C^{(k)}) \Pr(x_{n+l} = 1 \mid C^{(k)}) \Pr(C^{(k)}).$$

By applying Eqs. (2) and (3) to each of these three sums and then performing the summations and some obvious algebraic manipulations, we obtain

$$\Pr(x_n = 1, x_{n+l} = 1) = c(1-p)^2 \left\{ \alpha^l \left[\frac{(\alpha^2)^n - (1-c)^n}{\alpha^2 - (1-c)} \right] + (1-c)^n \left[\frac{\alpha^l - (1-c)^l}{\alpha - (1-c)} \right] + \frac{(1-c)^{n+l}}{c} \right\} \tag{27}$$

for $(1 - c) \neq \alpha^2$ and $(1 - c) \neq \alpha$. Slightly simpler expressions are obtained in the cases $(1 - c) = \alpha^2$ and $(1 - c) = \alpha$.

5. Expected total number of errors following an error on trial n

Let $T^{(n)}$ be the total number of errors after trial n for the two-phase model. Then

$$E(T^{(n)} \mid x_n = 1) = \sum_{k=n+1}^{\infty} \Pr(x_k = 1 \mid x_n = 1) = \frac{1}{\Pr(x_n = 1)} \sum_{k=n+1}^{\infty} \Pr(x_k = 1, x_n = 1) \tag{28}$$

$$= \frac{c(1-p)^2}{\Pr(A_{2,n})}\left(\left\{\frac{\alpha}{[\alpha^2-(1-c)](1-\alpha)}\right\}(\alpha^2)^n + \left\{\frac{c+\alpha-1}{[\alpha-(1-c)](1-\alpha)c} - \frac{\alpha}{[\alpha^2-(1-c)](1-\alpha)} + \frac{1-c}{c^2}\right\}(1-c)^n\right)$$

for $1-c \neq \alpha^2$, $1-c \neq \alpha$ as a consequence of Eq. (27).

6. Number of errors before the first success

Let J be the number of errors before the first success. Clearly,

(29) $$\Pr(J=0) = p.$$

For $j > 0$,

$$\Pr(J=j) = \left(\sum_{k=1}^{j} + \sum_{k=j+1}^{\infty}\right)\Pr(J=j \mid C^{(k)})\Pr(C^{(k)}).$$

After applying Eqs. (2) and (3) to each of these two sums and making a few straightforward simplifications, we obtain

(30) $$\Pr(J=j) = [(1-p)(1-c)]^j \left\{p + c\sum_{m=1}^{j}(1-c)^{-m}\alpha^{(m-1)m/2}[1-\alpha^m(1-p)]\right\}.$$

The definition of the mean of a random variable, Eqs. (29) and (30), and a rather lengthy calculation involving nothing more difficult than an interchange of orders of summation yield

(31) $$E(J) = \frac{(1-p)^2}{(1-(1-c)(1-p))^2}\left\{\frac{p(1-c)}{(1-p)} + \frac{c\Phi(\alpha,1-p)}{(1-p)} - c\alpha\Phi(\alpha,(1-p)\alpha)\right\} + \frac{c(1-p)^2}{(1-(1-c)(1-p))}\left\{\frac{\Psi(\alpha,1-p)}{(1-p)} - \alpha\Psi(\alpha,(1-p)\alpha)\right\},$$

where

(32) $$\Phi(\alpha,\beta) = \sum_{\nu=0}^{\infty}\alpha^{\nu(\nu+1)/2}\beta^{\nu}$$

and

(33) $$\Psi(\alpha,\beta) = \sum_{\nu=0}^{\infty}\nu\alpha^{\nu(\nu+1)/2}\beta^{\nu}.$$

Values of Φ and Ψ for $.50 \leq \alpha \leq .99$ and $.50 \leq \beta \leq 1.00$ are tabulated in Bush and Mosteller (1955, pp. 339–343), but a table of Φ and Ψ for smaller arguments would be needed before Eq. (31) could be used, for instance, in connection with the application of the two-phase model discussed below.

7. Mean and variance of the trial of the last error

Let N be the random variable that gives the number of the trial of the last error in an infinite sequence of trials in the two-phase model, and let D be the event "no errors after tfl." Then

(34) $$E(N) = E(N \mid D)\Pr(D) + E(N \mid \sim D)(1 - \Pr(D)).$$

Clearly, $E(N \mid D) = b(1-p)/c^2$, for the expression on the right gives the expected trial number of the last error in the all-or-none model with parameter values p and c (see Bower, 1961a, p. 271).

To determine $E(N \mid \sim D)$, we denote by K the random variable that gives the number of the trial of first learning and write

$$E(N \mid \sim D) = E(K \mid \sim D) - 1 + E(N - (K-1) \mid \sim D).$$

Since K is obviously independent of what happens after tfl,

$$E(K \mid \sim D) = E(K) = 1/c.$$

Denoting by $\Pr^*$, E^*, and L the probability measure, expected value functional, and trial number of the last error random variable for the single-operator model with parameter values p and α, and denoting by D^* the event "no error after the first trial" in that model, we obtain

$$E(N - (K-1) \mid \sim D) = E^*(L \mid \sim D^*) = \frac{E^*(L) - E^*(L \mid D^*)\Pr^*(D^*)}{1 - \Pr^*(D^*)}.$$

It is evident that $\Pr^*(D^*) = \Pr(D)$, and $E^*(L \mid D^*)$ is just $1-p$, the probability of an error on trial one in the single-operator model with parameter values p and α.

Putting all of this together, we obtain

(35) $$E(N) = \frac{b(1-p)}{c^2}\Pr(D) + \left(\frac{1}{c} - 1\right)(1 - \Pr(D)) + E^*(L) - (1-p)\Pr(D)$$

or, regrouping terms,

(36) $$E(N) = \frac{1-c}{c} + E^*(L) + \Pr(D)\left[\frac{b(1-p)}{c^2} + p - \frac{1}{c}\right].$$

Bush and Sternberg (1959, p. 211) have shown that

(37) $$E^*(L) = -\sum_{j=1}^{\infty} \frac{d_j(1-p)^j}{1-\alpha^j},$$

where

(38) $$d_0 = 1$$

and

(39) $$d_j = \frac{(-1)^j \alpha^{j(j-1)/2}}{\prod_{i=1}^{j}(1-\alpha^i)}.$$

It is apparent that

$$\Pr(D) = \prod_{i=1}^{\infty} [1 - \alpha^i(1 - p)]. \tag{40}$$

An important aid in the computation of $\Pr(D)$ is the table of the infinite product $\Pi_{i=0}^{\infty}(1 - \alpha^i\beta)$ that appears in Tatsuoka and Mosteller (1959, pp. 243–246).

A derivation analogous to the one just given but using in addition the obvious independence of $\boldsymbol{K} - 1$ and $\boldsymbol{N} - (\boldsymbol{K} - 1)$ with respect to the conditional probability induced by conditioning on $\sim D$ yields the relation

$$\begin{aligned} E(\boldsymbol{N}^2) = b(1 - p)\frac{(2 - c)}{c^3}\Pr(D) + \left(\frac{2 - c}{c^2} - \frac{2}{c} + 1\right)[1 - \Pr(D)] \\ + 2\left(\frac{1 - c}{c}\right)[E^*(\boldsymbol{L}) - (1 - p)\Pr(D)] \\ + [E^*(\boldsymbol{L}^2) - (1 - p)\Pr(D)], \end{aligned} \tag{41}$$

where, as a consequence of Eq. (20) on p. 210 of Bush and Sternberg (1959),

$$\begin{aligned} E^*(\boldsymbol{L}^2) = \sum_{j=0}^{\infty} d_j(1 - p)^j\left\{1 + \frac{(1 - p)}{\alpha}\left[\frac{1 + \alpha^{j+1}}{(1 - \alpha^{j+1})^3} - 1\right]\right\} \\ - 2E^*(\boldsymbol{L}) - 1. \end{aligned} \tag{42}$$

The variance of $\boldsymbol{N}$ may be obtained from Eqs. (36) and (41) and the familiar formula $\operatorname{var}(\boldsymbol{N}) = E(\boldsymbol{N}^2) - E^2(\boldsymbol{N})$.

8. Estimation of parameters

The estimation technique to be discussed is relevant to the case in which subjects (or subject items, or, in general, those entities to which the model is applied) are assumed to be stochastically independent and identical.

In the application of the two-phase model that will be discussed below, an a priori estimate $\hat{p}$ of p will be used. In situations where such an estimate does not seem appropriate, the proportion of successes on trial 1 might be used. The latter estimator is unbiased and, for large numbers of subjects, has reasonably small variance.

Defining the quantity μ by

$$\mu = \frac{E(\boldsymbol{T})}{(1 - p)} - \frac{E(\boldsymbol{u}_2)}{(1 - p)^2}, \tag{43}$$

we obtain from Eqs. (9) and (20) by a straightforward arithmetic manipulation that

$$\mu = 1 + \frac{\alpha^2}{1 - \alpha^2} \tag{44}$$

and thus that

$$\alpha = \sqrt{\left|1 - \frac{1}{\mu}\right|}. \tag{45}$$

In the application to be reported below, the estimator $\hat{\alpha}$ of α was employed where

$$(46) \qquad \hat{\alpha} = \sqrt{\left| 1 - \frac{1}{\hat{\mu}} \right|}, \quad \hat{\mu} = \frac{\widehat{E(T)}}{(1 - \hat{p})} - \frac{\widehat{E(\boldsymbol{u}_2)}}{(1 - \hat{p})^2},$$

and $\widehat{E(T)}$ and $\widehat{E(\boldsymbol{u}_2)}$ are the sample mean numbers of errors and pairs of errors, respectively. From Eq. (9) we get

$$(47) \qquad c = \left[\frac{E(T)}{1 - p} - \frac{\alpha}{1 - \alpha} \right]^{-1},$$

and, in the application to be reported below, the estimator $\hat{c}$ of c was employed where

$$(48) \qquad \hat{c} = \left[\frac{\widehat{E(T)}}{1 - \hat{p}} - \frac{\hat{\alpha}}{1 - \hat{\alpha}} \right]^{-1}.$$

Unfortunately, except for the relative ease with which their values can be computed, $\hat{\alpha}$ and $\hat{c}$ have no discernible virtues as estimators of α and c. We note, for instance, that since $\hat{\mu}$ is zero if no subject in the experiment makes any errors, and since the latter event clearly has positive probability, it can be seen from Eq. (46) that the expected value of $\hat{\alpha}$ is infinite. Thus the important problem of estimating the parameters of the two-phase model is in need of further investigation.

9. An application of the two-phase model to a verbal discrimination learning experiment[5]

In the experiment that we will now consider, every subject was required to learn to read a "correct" digram from each of 20 3 × 5-inch cards bearing four digrams, one in each corner of the left half of the card. A subject was allowed to examine a card for as long as he wished. After he had made his choice, the right side of the card, hitherto covered, was exposed. On it was printed the "correct" digram. After two seconds the entire card was covered and the left half of another card was exposed 2 seconds later. Three different orders of presentation of the 20 cards were used and these followed each other cyclically. (The cycle was the same for all subjects, but different subjects were randomly assigned different starting points in the cycle.) Only the usual 2-second inter-card time intervened between successive presentations of the 20 cards. Learning proceeded to a criterion of two consecutive errorless runs through the 20-card collection.

All digrams on a given card had the same first letter, which was not used as a first letter on any other card. Thus we may say that a subject's task was

[5] The experiment was conducted by Dr. Gordon H. Bower. The author is indebted to him for permission to report the results here.

TABLE 1

COMPARISON OF THE TWO-PHASE MODEL WITH THE FOUR-RESPONSE VDL DATA

(Parameter Estimates: $\hat{p} = .250$, $\hat{c} = .301$, $\hat{\alpha} = .426$.)

Quantity	Data	Two-Phase Model	Quantity	Data	Two-Phase Model
$\Pr(A_{2,n})$, $n = 1$	.74	.75	$E(u_j)$, $j = 2$	1.5993	*
2	.57	.62	3	.84	.85
3	.47	.47	4	.44	.44
4	.39	.35	5	.22	.23
5	.27	.25	6	.11	.12
6	.18	.18	7	.05	.06
7	.15	.13			
8	.10	.09	$E(c_l)$, $l = 2$	1.22	1.21
9	.06	.06	3	.90	.88
10	.04	.04	4	.65	.63
11	.04	.03	5	.45	.45
12	.01	.02	6	.33	.32
13	.01	.02			
14	.01	.01			
			$\Pr(J = j)$, $j = 1$	.30	.28
$E(T)$	3.0483	*	2	.18	.21
			3	.11	.12
$\sigma(T)$	2.29	2.32	4	.06	.06
			5	.04	.03
$\Pr(T = k)$, $k = 0$	.08	.05	6	.02	.02
1	.21	.21	7	.01	.01
2	.19	.25	8	.00	.00
3	.17	.18	9	.01	.00
4	.12	.11	10	.00	.00
5	.09	.07			
6	.04	.05	$E(N)$	4.15	4.02
7	.05	.03			
8	.02	.02	$\sigma(N)$	3.20	3.10
9	.02	.01			
10	.01	.01			
11	.00	.00			

* Used to estimate parameters.

to learn to discriminate between second letters (and, as a procedural artifact, between positions on the card[6]) in the stimulus situation characterized by each first letter.[7]

There were 29 subjects (all Stanford students), and thus $29 \times 20 = 580$

[6] The position of the correct digram was randomized over the 20 cards.

[7] No digram used was a common English word.

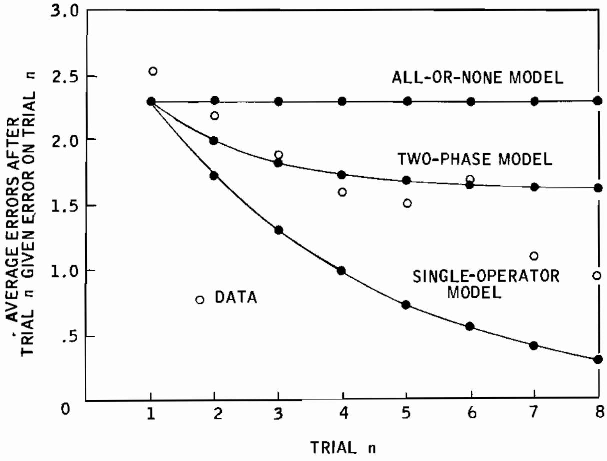

FIG. 1. Average number of errors after trial n given an error on trial n for verbal discrimination learning experiment.

subject-items. The data for one subject-item were lost. The remaining 579 subject-items were assumed to be stochastically independent and identical in the analysis below.

A comparison of some statistics of the experiment and the corresponding predictions of the two-phase model with $\hat{p} = .25$, $\hat{c} = .301$, and $\hat{\alpha} = .426$ is presented in Table 1. Since the model speaks for itself rather eloquently in the comparison presented in the table, we turn our attention to a set of statistics not included in Table 1: the average number of errors following an error on trial n. As Bower (1961a) has pointed out, these statistics differentiate quite sharply between the all-or-none model and the single-operator linear model. The former (Bower, 1961a, p. 275) predicts constancy over all n of the expected number of errors after trial n given an error on trial n at the value $[(1-p)/c](1-c)$. Since in the single-operator model responses on different trials are independent, the expected total number of errors after trial n given an error on trial n is just the expected total number of errors after trial n which turns out to be $[(1-p)/(1-\alpha)]\alpha^n$. Considered as a function of n, this is decreasing and negatively accelerated, converging to 0. The predictions of both of these models and those of the two-phase model [obtained from Eqs. (28) and (4)] are presented along with the data in Fig. 1.[8]

The first thing that should be noted from the figure is the pronounced decline with increasing n of the observed average number of errors following an error on trial n. This result contrasts sharply with that obtained in a similar experiment (Bower, 1961b) with, however, two *ccc* trigrams on each

[8] For both the all-or-none and single-operator models, p was estimated by .25. The learning parameters for the two models were then estimated using the respective theoretical expressions for the two models for the total number of errors to perfect learning (see Bower, 1961*a*, p. 259, and Tatsuoka and Mosteller, 1959, p. 233). The estimates thus obtained were $\hat{c} = .246$ and $\hat{\alpha} = .754$.

stimulus card. In that experiment the average number of errors after an error on trial n showed no systematic or pronounced deviations from the constant value predicted by the all-or-none model for $n = 1, 2, \cdots 6$ (see Bower, 1962, p. 48, esp. Fig. 4). At any rate the results presented in Fig. 1 lead us to reject the all-or-none model as an explanation of the behavior of subjects in the present experiment. Similarly, the fit to the data given by the single-operator model is quite unacceptable.

The over-all fit obtained from the two-phase model is much better than that obtained from either of the other two models. It seems to this writer that $E(T^{(n)} \mid x_n = 1)$ for the two-phase model is fairly close to the corresponding sample average for $n = 1, 2, \cdots 6$ though, especially with respect to $n = 1$ and $n = 2$, one could certainly hope to do better. As can be seen from Table 1, only 15 per cent of the subject-items in the experiment had errors on trial 7, only 10 per cent had errors on trial 8, and fewer than 10 per cent had errors on any trial thereafter. Thus the variances associated with the data points for $n = 7$ and $n = 8$ in Fig. 1 are expected to be large and even larger for $n > 8$ (we have not reported the data for these latter points for precisely this reason). It seems likely, however, that there is a difficulty with the statistic under discussion, for large n, that is more basic than any of the problems associated with making statistical inferences on the basis of relatively few observations. And this difficulty is only a particularly acute manifestation of a rather general problem associated with the kind of analysis given above. The quantities that we have (rather loosely) referred to as the predictions of this or that model are distributions and moments of random variables defined for infinite sequences of trials. The analysis given above is thus valid, in the strict sense, only under the assumption that no errors would have been made after criterion was reached, had the experiment been continued indefinitely. This assumption is, of course, completely absurd. One might hope, however, that the error introduced by making this assumption is small in some sense. Unfortunately, however, there is little hope that the latter condition holds in the present case because the list criterion used would seem to impose an insufficiently stringent criterion for "perfect learning" on the most slowly learned items in the list. To return to our discussion of the average number of errors after trial n given an error on trial n: the predictions of the two-phase model fail to take into consideration[9] the fact that the experiment has been terminated after an artificially small number of trials, and the error thus introduced is presumably pronounced enough to be noticeable when n is large and we are literally considering data only for the most slowly learned items. The effect of this error is to cause the two-phase model to predict too many errors after an error on trial n for large n. This can be seen to be the kind of "mistake" made by the two-phase model for $n = 7$ and $n = 8$ in Fig. 1.

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[9] Except for the parameter estimates—this is not well understood.

On this note we conclude our somewhat superficial analysis of this experiment.

10. Some related models

The second phase of the two-phase model as developed in the first part of this paper is a single-operator linear model. Similar models could be developed in which the second stage is some other type of incremental model, such as a two-commuting-operator linear model (see Tatsuoka and Mosteller, 1959) or some type of β model (see Luce, 1959). Another possibility for the second stage is an all-or-none model with a new (higher) guessing probability and a new learning parameter. The "two-element model" derived from stimulus-sampling considerations by Suppes and Ginsberg (1963) is of this type, with second guessing probability the specific function $\frac{1}{2} + \frac{1}{2}p$ of the first.

It would apparently be quite difficult to distinguish between the two-phase model and the various models mentioned in this section, especially when learning is relatively rapid as in the verbal discrimination learning experiment discussed above.

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