An Approach to Free-Responding on Schedules that Prescribe Reinforcement Probability as a Function of Interresponse Time

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A model is proposed for free-responding on schedules that positively reinforce an interresponse time of duration t with a probability $u(t)$ that does not depend on the experimental history of the subject prior to the previous response. The organism is characterized at any point in time by its interresponse time distribution, and the learning process takes the form of a sequence of linear transformations on this distribution. The transformation corresponding to nonreinforcement increases the probability of long interresponse times. The transformation associated with reinforcement of an interresponse time t leads to two effects: an increase in the probability of short interresponse times, and an increase in the probability of interresponse times of duration approximately t. Expressions are derived for the asymptotic interresponse time distribution $I(t)$. Predictions of the model are compared with data from random-ratio, variable-interval, and a certain DRL-like schedule.

Skinner (1938) bases his ingenious explanation of the superiority of fixed-ratio to fixed-interval response rates upon the observation that different interresponse times (IRT's) can be selectively strengthened by differential reinforcement. More recently the influence of reinforcement contingencies upon the IRT distribution has been emphasized by Anger (1956), and Logan (1960) has stressed the importance of temporal response differentiation in runway experiments that are closely related to the free-responding situations that will be considered in this paper.

To study the way the IRT distribution is controlled by the probabilities of reinforcement of different IRT's it seems natural to focus attention on the class of schedules for which the probability $u(t)$ that reinforcement will follow the IRT t is prescribed by the experimenter. That is, a function $u(t)$ is selected by the experimenter and all IRT's of duration t are reinforced with probability $u(t)$. The term *simple contingent schedule* has often been used in the literature of mathematical learning theory to

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describe analogous schedules in other learning situations, and this term will be used here.

One example of a simple contingent schedule is the *differential reinforcement of low rate* or DRL schedule in which an IRT is reinforced if and only if it exceeds a certain critical duration τ, thus

$$u(t) = \begin{cases} 0 & \text{if} \quad t \leqslant \tau \\ 1 & \text{if} \quad t > \tau. \end{cases}$$

If $u(t)$ also vanishes for $t > \tau'$ where $\tau' > \tau$ a *differential reinforcement of low rate with limited hold* or DRL LH schedule is obtained. Other simple contingent schedules are *random-ratio* (RR) where all IRT's have the same probability p of reinforcement (thus $u(t) = p$ for all $t \geqslant 0$), and Millenson's (1963) *random-interval* (RI) schedule which specifies that the time between the consumption of one reinforcement and the "setting up" of another be an exponential random variable with mean $1/\gamma$ seconds ($u(t) = 1 - \exp(-\gamma t)$ as is shown in Part B of Sect. IV below). Some other types of "variable-ratio" and "variable-interval" schedules are sufficiently similar to RR and RI, respectively, that any conclusions reached about behavior under the latter schedules will have implications for behavior under the former. Fixed-ratio and fixed-interval schedules, on the other hand, are not at all like simple contingent schedules.

To the author's knowledge the only paper in the literature that considers the entire *class* of simple contingent schedules *as such* is the very recent report on the experiments of Mallott and Cumming (1964). It is hoped that Mallott's and Cumming's paper and the present theoretical paper will call attention to the importance of this class of schedules.

It is now well established (see Kelleher, Fry, and Cook, 1959 and Sidman, 1956a, 1956b) that organisms can adapt to DRL and DRL LH contingencies in the sense of developing IRT distributions with predominant mode in the neighborhood of the cutoff point τ. Such adaptation is, however, beyond the capabilities of the simple free-responding models proposed by Bush and Mosteller (1951) and Estes (1950). These models always predict IRT frequency functions which decrease as distance from the origin increases. The same is true of the (in effect) pure birth models considered by Mueller (1950). Thus these models need be considered no further.

I. A MODEL FOR FREE-RESPONDING ON SIMPLE
CONTINGENT SCHEDULES OF POSITIVE REINFORCEMENT

The model to be discussed below is arrived at by formalizing the following rather crude conception of the dynamics of behavior under simple contingent reinforcement: The entire effect of nonreinforcement following an IRT is a general decrease in the rate of responding. Receipt of reinforcement following an IRT of duration t results

in a compromise between two effects. The first of these is a tendency for the rate of responding to increase. The second is an augmented tendency to emit IRT's of duration approximately t.

The course of acquisition is described by a sequence of transformations on the organism's probabilities of making IRT's of various durations. Clearly if different events such as "reinforcement of a 5-sec IRT" are to have the different effects ascribed to them in the preceding paragraph there must be a different transformation associated with each such event. The form of these transformations must now be determined. Let E be any event of the type under consideration, and let T_E be the associated transformation. Thus if $P(A)$ is an organism's probability of emitting an IRT in a Borel subset A of the positive half line at some point in a free-responding experiment, then $(T_E P)(A)$ is the new probability of such a response after the occurrence of E. It is natural to consider transformations of the form

$$(T_E P)(A) = g_E(P(A), A) \tag{1}$$

for some function g_E of two arguments, the first a number in the unit interval and the second a set. It is an immediate consequence of a theorem of Blau (1961, Theorem 1) that the only transformations of probability measures into probability measures having the form (1) are averages of the identity transformation and a fixed probability measure P_E, i.e., for some $0 \leqslant \theta_E \leqslant 1$ and P_E, and all P and A,

$$(T_E P)(A) = (1 - \theta_E) P(A) + \theta_E P_E(A).$$

The form of the corresponding transformation $\tilde{T}_E$ on distribution functions $F(x)$ is obtained by taking $A = (0, x)$:

$$(\tilde{T}_E F)(x) = (1 - \theta_E) F(x) + \theta_E F_E(x), \tag{2}$$

where $F_E(x) = P_E((0, x))$.

The first theorist to propose a learning model for a continuum of responses employing such transformations was Suppes (1959). Most of the differences between the development to follow and the development in Suppes' paper stem from the differences in the classes of experimental situations that are considered.

It is convenient at this point to introduce some terminology and notation. The latency t, of the first response will be referred to as the "first IRT." For $n \geqslant 2$ the time t_n between the nth response and the preceding response will be called the nth IRT. The reinforcement indicator random variable for the nth response will be denoted by k_n, i.e.,

$$k_n = \begin{cases} 1 & \text{if the } n\text{th response is reinforced} \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

The $2n$-dimensional random vector $(k_n, t_n, k_{n-1}, t_{n-1}, \cdots, k_1, t_1)$ which gives the entire history of IRT's and reinforcements through the nth for an organism will be

denoted by s_n. Finally, the conditional probability distribution of the $n + 1$st IRT given the history s_n will be denoted by $I_{n+1}(t \mid s_n)$ for $n \geqslant 0$. For $n = 0$ this symbol is to be interpreted as $I_1(t)$, the initial IRT distribution.

In view of (2) the following formalization of the conception sketched in the first paragraph of this section will be considered.

The decrease in rate of responding accompanying nonreinforcement of the nth response is represented in the model by averaging the "old" IRT distribution $I_n(t \mid s_{n-1})$ with $F_0(t)$, a distribution with very large mean, to obtain the "new" IRT distribution $I_{n+1}(t \mid s_n)$. Thus

$$I_{n+1}(t \mid s_n) = (1 - \theta_0) I_n(t \mid s_{n-1}) + \theta_0 F_0(t), \tag{4}$$

where $0 \leqslant \theta_0 < 1$. The distribution F_0 may be tentatively identified with the "operant level" IRT distribution under the prevailing experimental conditions. If responding at operant level is conceived of as approximately a Poisson process F_0 would be taken to be exponential. This choice would be unreasonable in so far as it is obviously impossible for an organism to emit arbitrarily short IRT's in a given situation, but the error thus introduced should not be important in most applications.

The dual effect of reinforcement of the nth response is represented in the model by averaging $I_n(t \mid s_{n-1})$ with $(1 - \alpha) F_1(t) + \alpha L(t, t_n)$ to obtain $I_{n+1}(t \mid s_n)$, i.e.,

$$I_{n+1}(t \mid s_n) = (1 - \theta_1) I_n(t \mid s_{n-1}) + \theta_1[(1 - \alpha) F_1(t) + \alpha L(t, t_n)], \tag{5}$$

where $0 \leqslant \theta_1 < 1$ and $0 \leqslant \alpha < 1$. Here F_1 is a distribution function with most of its mass near the origin, and its presence in the above formula represents the tendency of the organism to respond more rapidly after receipt of reinforcement. Since F_1 is concentrated near the origin it is clear that great care must be taken in postulating the behavior of F_1 very near the origin. This rules out an exponential distribution for the reason discussed above in connection with F_0. A gamma distribution with, of course, a very small mean, would be much more appropriate.

The distribution $L(\cdot, t_n)$ is assumed to be unimodal with mode at t_n, and its presence in the above formula represents the increased tendency of the organism to emit further IRT's in the neighborhood of t_n following reinforcement of an IRT of duration t_n. If it is assumed that the standard deviation of $L(\cdot, s)$ is a measure of the organism's temporal generalization in the neighborhood of s, and that Weber's law holds for this measure, then the standard deviation of $L(\cdot, s)$ is proportional to s. One family of distributions having all of these properties in addition to analytical convenience is the family

$$L(t, s) = G\left(\frac{t}{s}\right), \tag{6}$$

where G is a unimodal distribution with mode 1.

The theoretical structure described by (4) and (5) together with the stipulations that F_0 has a large mean, F_1 has a small mean, and $L(\cdot, s)$ has mode s, will be called the *linear free-responding model*. The assumptions that F_0 is exponential, F_1 is gamma, and L satisfies (6) will be in force only where indicated below.

The assumption that simple contingent schedules are being considered may be written

$$P(k_n = 1 \mid t_n = t, s_{n-1}) = u(t) \qquad \text{for all} \qquad t \geqslant 0. \tag{7}$$

The function $u(t)$ will be called the reinforcement function of the schedule.

II. GENERAL THEORY OF THE ASYMPTOTIC IRT DISTRIBUTION

A. EXISTENCE AND INTERPRETATION

The distribution $I_n(t \mid s_{n-1})$ of the nth IRT for an organism is a random variable since it depends upon the $2(n-1)$ dimensional random variable s_{n-1}. The expectation

$$I_n(t) = E[I_n(t \mid s_{n-1})] \tag{8}$$

over all possible histories s_{n-1} can be interpreted as the distribution of the nth IRT for a large group of organisms behaving according to the model with the same parameters and initial distribution $I_1(t)$. Theorem 1 shows, among other things, that under very general conditions an asymptotic IRT distribution

$$I(t) = \lim_{n \to \infty} I_n(t) \tag{9}$$

exists and does not depend on $I_1(t)$.

Theorem 1 (except for the uniformity assertion) can easily be shown to follow from the ergodic theorem in Chapt. IV, Sect. 3 of Iosifescu (1963).[2] (The latter theorem extends Theorem 2 of Iosifescu and Theodorescu (1965), which is restricted to models with finite sets of possible responses and reinforcements, to models with arbitrary response and reinforcement sets.) The statement concerning uniformity is a consequence of the first remark following the ergodic theorem of Chapt. III, Sect. 2 of Iosifescu (1963). Details of the deductions will not be presented here.

THEOREM 1. *If $\theta_0 > 0$, $\theta_1 > 0$, and there is some interval on which $F_1'(t)$ and $F_0'(t)$ are both positive, then for any positive integer m and any bounded measurable functions $v_j(t)$, $j = 1, \cdots, m$ the limit*

$$\lim_{n \to \infty} E \left[\prod_{j=1}^{m} \int_0^\infty v_j(t) \, dI_n(t \mid s_{n-1}) \right] \tag{10}$$

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[2] Vera Rich has translated Iosifescu (1963) from the Russian. A copy of the translation may be obtained for $7.50 from Addis Translations, 129 Pope Street, Menlo Park, California 94025.

exists and does not depend on the distribution I_1, of t_1. For any m and any $M < \infty$ convergence is uniform in I_1 and in functions v_j satisfying

$$\prod_{j=1}^{m} \sup_{0 < t} |v_j(t)| \leqslant M.$$

The existence and independence of I_1 of the limit (9) follows from Theorem 1 on taking $m = 1$ and

$$v_1(s) = f_t(s) = \begin{cases} 1 & s < t \\ 0 & s \geqslant t. \end{cases} \tag{11}$$

In fact the theorem can be shown to imply that the total variation of $I - I_n$ converges to 0.

Theorem 1 is extremely general, since no assumptions are made about $u(t)$ and α. It is easy to show that the hypotheses of the theorem are actually necessary to obtain the conclusion of the theorem in this generality. Suppose throughout what follows that $\alpha = 0$. If $\theta_1 = 0$ (i.e., reinforcement has no effect on the organism) then for $u(t) \equiv 1$ it is clear that $I_n(t) = I_1(t)$ for all n, so $I(t) = I_1(t)$ contradicting the assertion of Theorem 1 that the limit in (10) is independent of $I_1(t)$. Similarly if $\theta_0 = 0$ (i.e., nonreinforcement is ineffective) then for $u(t) \equiv 0$ it is apparent that $I(t) = I_1(t)$. Finally, if F_1 and F_0 are concentrated on disjoint sets S_0 and S_1, if $\theta_1 > 0$ and $\theta_0 > 0$, and if

$$u(t) = \begin{cases} 1 & \text{if} \quad t \in S_1 \\ 0 & \text{if} \quad t \in' S_1 \end{cases}$$

then it is easy to see that $I(t) = F_1(t)$ if $I_1(t)$ is concentrated on S_1, while $I(t) = F_0(t)$ if $I_1(t)$ is concentrated on S_0.

The hypothesis about the simultaneous positivity of $F_0'(t)$ and $F_1'(t)$ (and the implicit assumption that these densities exist) will be in force throughout the rest of the paper. If F_0 is exponential and F_1 is gamma, then $F_0'(t)$ and $F_1'(t)$ are positive for all $t > 0$.

As defined in (9) $I(t)$ has the interpretation of an asymptotic group average IRT distribution. However it is a consequence of Theorem 2 below that $I(t)$ is consistently estimated by the empirical IRT distributions for an individual subject. This yields an interpretation of $I(t)$ in terms of a single organism's behavior and justifies the application of the linear free-responding model to data from individual subjects. Some applications of this sort are presented in Sect. IV of this paper.

THEOREM 2.   *Under the hypotheses of Theorem 1, for any bounded function f and any $r \geqslant 1$*

$$\lim_{n \to \infty} E\left[\left(\frac{\sum_{j=r}^{r+n-1} f(t_j)}{n} - \int_0^\infty f(t)\, dI(t) \right)^2 \right] = 0. \tag{12}$$

If this theorem is applied to the functions $f_t(\cdot)$ given by (11), then

$$S_{r,n}(t) = \sum_{j=r}^{r+n-1} \frac{f_t(t_j)}{n}$$

is just the empirical IRT distribution for an individual subject for the n trials starting at trial r, $\int_0^\infty f_t(s)\, dI(s) = I(t)$, and the conclusion

$$\lim_{n\to\infty} E[(S_{r,n}(t) - I(t))^2] = 0$$

is the consistency property mentioned above.

Theorem 2 is a law of large numbers for the random variables $f(t_1), f(t_2), \cdots$ where f is bounded. This theorem is an immediate consequence of statement (i) in Theorem 3. Much more refined information about the asymptotic distribution of the sums

$$\sum_{j=r}^{r+n-1} f(t_j)$$

is given by statement (ii) of Theorem 3, which is a central limit theorem for $\{f(t_j)\}_{j=1}^\infty$. Theorem 3 is a consequence of Theorem 1 in Chapt. IV, Sect. 4 of Iosifescu (1963). (The latter theorem extends Theorem 3 of Iosifescu and Theodorescu (1965) to arbitrary response and reinforcement sets.)

THEOREM 3. *Under the hypotheses of Theorem 1, for every bounded function f and every reinforcement function u there exists a non-negative constant $\sigma_{u,f}$ which does not depend on I_1 such that*

(i) *for every $r \geqslant 1$*

$$\lim_{n\to\infty} \frac{1}{n} E\left[\left(\sum_{j=r}^{r+n-1} f(t_j) - n \int_0^\infty f(t)\, dI(t)\right)^2\right] = \sigma_{u,f}^2,$$

and

(ii) *if $\sigma_{u,f}^2 \neq 0$ then*

$$\lim_{n\to\infty} P\left(\frac{\sum\limits_{j=r}^{r+n-1} f(t_j) - n \int_0^\infty f(t)\, dI(t)}{\sqrt{n}\, \sigma_{u,f}} < y\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^y \exp\left(-\frac{x^2}{2}\right) dx. \qquad (13)$$

B. AN INTEGRAL EQUATION FOR $I(t)$

The investigation of the form of $I(t)$ begins with the following recursive expression for its precursors $I_n(t)$.

THEOREM 4.   *For $n \geqslant 1$,*

$$I_{n+1}(t) = (1 - \bar{\theta}_n) I_n(t) + \theta_0(1 - p_n) F_0(t) + (1 - \alpha) \theta_1 p_n F_1(t)$$

$$+ \alpha\theta_1 \int_0^\infty L(t, s) u(s) dI_n(s) + E_n(t)$$

where

$$p_n = P(k_n = 1) = \int_0^\infty u(t) dI_n(t), \quad \bar{\theta}_n = E[\theta_{k_n}] = \theta_0(1 - p_n) + \theta_1 p_n,$$

and

$$E_n(t) = (\theta_0 - \theta_1) \operatorname{cov} \left(\int_0^\infty u(s) dI_n(s \mid s_{n-1}), I_n(t \mid s_{n-1}) \right).$$

Before presenting the proof it is appropriate to point out that the term $E_n(t)$ and all quantities that will derive from it are to be regarded as residual terms. It will be shown that these terms are negligible in many cases of interest. As a start, note that for any random-ratio schedule $u(t) \equiv p$ for some constant p so

$$\int_0^\infty u(s) dI_n(s \mid s_{n-1}) = p$$

has zero variance and $E_n(t)$ vanishes.

PROOF.   Let

$$u_1(t) = u(t), \quad u_0(t) = 1 - u(t),$$

$$K_1(t, s) = (1 - \alpha) F_1(t) + \alpha L(t, s), \quad \text{and} \quad K_0(t, s) = F_0(t).$$

The function $u_k(t)$ gives the probability that $k_n = k$ when $t_n = t$. Introduction of the K_k's permits the compression of (4) and (5) into the single equation

$$I_{n+1}(t \mid s_n) = (1 - \theta_{k_n}) I_n(t \mid s_{n-1}) + \theta_{k_n} K_{k_n}(t, t_n).$$

We have

$$I_{n+1}(t) = P(t_{n+1} < t) = E[P(t_{n+1} < t \mid s_n)]$$

$$= E[I_{n+1}(t \mid s_n)] = E[(1 - \theta_{k_n}) I_n(t \mid s_{n-1}) + \theta_{k_n} K_{k_n}(t, t_n)]$$

$$= E[E[(1 - \theta_{k_n}) I_n(t \mid s_{n-1}) + \theta_{k_n} K_{k_n}(t, t_n) \mid s_{n-1}]]$$

$$= E[E[(1 - \theta_{k_n}) \mid s_{n-1}] I_n(t \mid s_{n-1}) + E[\theta_{k_n} K_{k_n}(t, t_n) \mid s_{n-1}]]$$

$$= E \left[\sum_{k=0}^1 (1 - \theta_k) \int_0^\infty u_k(s) dI_n(s \mid s_{n-1})) I_n(t \mid s_{n-1}) \right]$$

$$+ E \left[\sum_{k=0}^1 \theta_k \int_0^\infty K_k(t, s) u_k(s) dI_n(s \mid s_{n-1}) \right]$$

$$= E\left[\sum_{k=0}^{1} (1 - \theta_k) \int_0^\infty u_k(s) \, dI_n(s \mid s_{n-1})\right] E[I_n(t \mid s_{n-1})]$$

$$+ \text{cov}\left[\sum_{k=0}^{1} (1 - \theta_k) \int_0^\infty u_k(s) \, dI_n(s \mid s_{n-1}), I_n(t \mid s_{n-1})\right]$$

$$+ \sum_{k=0}^{1} \theta_k \int_0^\infty K_k(t, s) \, u_k(s) \, dE[I_n(s \mid s_{n-1})].$$

However

$$\sum_{k=0}^{1} (1 - \theta_k) \, u_k(s) = (1 - \theta_0)(1 - u(s)) + (1 - \theta_1) \, u(s)$$

$$= (1 - \theta_0) + (\theta_0 - \theta_1) \, u(s).$$

Thus

$$\text{cov}\left[\sum_{k=0}^{1} (1 - \theta_k) \int_0^\infty u_k(s) \, dI_n(s \mid s_{n-1}), I_n(t \mid s_{n-1})\right]$$

$$= \text{cov}\left[(1 - \theta_0) + (\theta_0 - \theta_1) \int_0^\infty u(s) \, dI_n(s \mid s_{n-1}), I_n(t \mid s_{n-1})\right]$$

$$= (\theta_0 - \theta_1) \, \text{cov}\left[\int_0^\infty u(s) \, dI_n(s \mid s_{n-1}), I_n(t \mid s_{n-1})\right].$$

Therefore we obtain

$$I_{n+1}(t) = \left(1 - \sum_{k=0}^{1} \theta_k \int_0^\infty u_k(s) \, dI_n(s)\right) I_n(t) + \sum_{k=0}^{1} \theta_k \int_0^\infty K_k(t, s) u_k(s) \, dI_n(s) + E_n(t),$$

which reduces to (14) upon substitution for the $K_k(t, s)$ in terms of $F_0(t)$, $F_1(t)$, and $L(t, s)$.

$$\text{Q.E.D.}$$

Letting n approach infinity in Theorem 4 an integral equation for the asymptotic IRT distribution $I(t)$ is obtained.

COROLLARY.   *If $\theta_0 > 0$ and $\theta_1 > 0$ then*

$$I(t) = \frac{\theta_0(1 - p)}{\bar{\theta}} F_0(t) + (1 - \alpha) \frac{\theta_1 p}{\bar{\theta}} F_1(t) + \alpha \frac{\theta_1}{\bar{\theta}} \int_0^\infty L(t, s) \, u(s) \, dI(s) + \frac{1}{\bar{\theta}} E(t), \quad (15)$$

where

$$p = \lim_{n \to \infty} p_n = \int_0^\infty u(s) \, dI(s), \quad (16)$$

$$\bar{\theta} = \lim_{n \to \infty} \bar{\theta}_n = \theta_0(1 - p) + \theta_1 p,$$

and

$$E(t) = \lim_{n \to \infty} E_n(t). \quad (17)$$

C. A Series Expansion of $I(t)$

If α is not too large the integral equation (15) can be solved by iteration, a standard technique in integral equation theory. The solution takes the form of an infinite series which is called a Neumann series. Though the presentation that follows is self-contained the reader desiring additional background in integral equation theory may wish to consult standard sources such as Courant and Hilbert (1953) and Tricomi (1957).

THEOREM 5.   *If $\theta_0 > 0$, $\theta_1 > 0$, and $\alpha < \bar{\theta}/\theta_1 \sup u$ then*

$$I(t) = \frac{\theta_0(1 - p)}{\bar{\theta}} F_0(t) + (1 - \alpha) \frac{\theta_1 p}{\bar{\theta}} F_1(t)$$

$$+ \sum_{k=1}^{\infty} \left(\alpha \frac{\theta_1}{\bar{\theta}} \right)^k \int_0^\infty W^{(k)}(t, s) \, d \left[\frac{\theta_0(1 - p)}{\bar{\theta}} F_0(s) + (1 - \alpha) \frac{\theta_1 p}{\bar{\theta}} F_1(s) \right] + \mathscr{E}(t), \quad (18)$$

where

$$\mathscr{E}(t) = \frac{1}{\bar{\theta}} \int_0^\infty R(t, s) \, dE(s), \tag{19}$$

$$R(t, s) = \sum_{k=0}^{\infty} \left(\alpha \frac{\theta_1}{\bar{\theta}} \right)^k W^{(k)}(t, s), \tag{20}$$

$$W^{(0)}(t, s) = \begin{cases} 0 & \text{if} \quad t \leqslant s \\ 1 & \text{if} \quad t > s, \end{cases} \tag{21}$$

$$W^{(1)}(t, s) = L(t, s) \, u(s), \tag{22}$$

and

$$W^{(m+1)}(t, s) = \int_0^\infty W^{(m)}(t, z) \, d_z W^{(1)}(z, s) \qquad for \qquad m \geqslant 1. \tag{23}$$

PROOF.   To facilitate our work with Eq. 15 we introduce some notation:

$$G(t) = \frac{\theta_0(1 - p)}{\bar{\theta}} F_0(t) + (1 - \alpha) \frac{\theta_1 p}{\bar{\theta}} F_1(t) + \frac{1}{\bar{\theta}} E(t) \qquad for \qquad t \geqslant 0 \tag{24}$$

and

$$b = \alpha \frac{\theta_1}{\bar{\theta}}. \tag{25}$$

In these terms (15) becomes

$$I(t) = G(t) + b \int_0^\infty W^{(1)}(t, s) \, dI(s). \tag{26}$$

This is a Fredholm equation of the second kind. We now iterate this equation.
Suppose that for some $n \geqslant 0$

$$I(t) = G(t) + \sum_{k=1}^{n} b^k \int_0^\infty W^{(k)}(t, s) \, dG(s) + b^{n+1} \int_0^\infty W^{(n+1)}(t, s) \, dI(s). \qquad (27)$$

Then substituting expression (26) for $I(s)$ into the right side of (27) we obtain

$$I(t) = G(t) + \sum_{k=1}^{n+1} b^k \int_0^\infty W^{(k)}(t, s) \, dG(s) + b^{n+2} \int_0^\infty W^{(n+1)}(t, z) \, d_z \left(\int_0^\infty W^{(1)}(z, s) \, dI(s) \right)$$

$$= G(t) + \sum_{k=1}^{n+1} b^k \int_0^\infty W^{(k)}(t, s) \, dG(s) + b^{n+2} \int_0^\infty \left(\int_0^\infty W^{(n+1)}(t, z) \, d_z W^{(1)}(z, s) \right) dI(s)$$

$$= G(t) + \sum_{k=1}^{n+1} b^k \int_0^\infty W^{(k)}(t, s) \, dG(s) + b^{n+2} \int_0^\infty W^{(n+2)}(t, s) \, dI(s).$$

Since for $n = 0$ (27) reduces to (26), we conclude by induction that (27) is valid for all $n \geqslant 0$.
  Now

$$0 \leqslant W^{(1)}(t, s) = L(t, s) \, u(s) \leqslant u(s).$$

Suppose $n \geqslant 1$ and

$$0 \leqslant W^{(n)}(t, s) \leqslant (\sup_{0 \leqslant t} u(t))^{n-1} \, u(s).$$

Then

$$0 \leqslant W^{(n+1)}(t, s) = \left(\int_0^\infty W^{(n)}(t, z) \, d_z L(z, s) \right) u(s)$$

$$\leqslant (\sup u)^{n-1} \left(\int_0^\infty u(z) \, d_z L(z, s) \right) u(s) \leqslant (\sup u)^n \, u(s).$$

Thus, for all $n \geqslant 1$,

$$0 \leqslant W^{(n)}(t, s) \leqslant (\sup u)^{n-1} \, u(s). \qquad (28)$$

Therefore,

$$0 \leqslant \int_0^\infty W^{(n+1)}(t, s) \, dI(s) \leqslant (\sup u)^n \int_0^\infty u(s) \, dI(s) = (\sup u)^n \, p.$$

It follows that

$$\left| I(t) - G(t) - \sum_{k=1}^{n} b^k \int_0^\infty W^{(k)}(t, s) \, dG(s) \right| \leqslant bp(b \sup u)^n = \alpha \frac{\theta_1 p}{\bar{\theta}} \left(\alpha \frac{\theta_1}{\bar{\theta}} \sup u \right)^n.$$

The term on the right converges to 0 as $n \to \infty$ if

$$\alpha \frac{\theta_1}{\bar{\theta}} \sup u < 1 \qquad \text{or} \qquad \alpha < \frac{\bar{\theta}}{\theta_1 \sup u}.$$

The theorem follows upon rewriting in our previous notation and observing that

$$E(t) = \int_0^\infty W^{(0)}(t, s) \, dE(s),$$

since $E(0) = 0$.   Q.E.D.

Since the number $p = \int_0^\infty u(s) \, dI(s)$ that determines the upper limit $d = \bar{\theta}/\theta_1 \sup u$ of the interval of convergence producing α's specified by Theorem 5 is still in general unknown, the question arises: Is the range of convergence producing α's specified by the theorem adequate for applications of the model? If $\theta_0 \geqslant \theta_1$ then $d \geqslant 1$ and there is no problem. If $\theta_0 < \theta_1$ then at the very least the interval

$$\Delta = \left\{ \alpha : \alpha < \frac{\theta_0}{\theta_1 \sup u} \right\}$$

is always allowed. But in Secs. III and IV of this paper attention will be focused on the case of small θ_0/θ_1, and in that case Δ would often be too restrictive. However rewriting d as

$$d = \int_0^\infty w(s) \, dI(s) + \frac{\theta_0}{\theta_1} \frac{(1 - p)}{\sup u},$$

where $w(s) = u(s)/\sup u$ may be described as the relative reinforcement availability function, it is easily seen that, whenever d and, a fortiori, $\int_0^\infty w(s) \, dI(s)$ are small, the organism asymptotically is not doing much responding in the region of high reinforcement availability. This means that α, which measures the organism's capacity to adapt to his environment, is small. The conclusion is that if α is chosen sensibly (from a psychological point of view) the Neumann series expansion of $I(t)$ given by Theorem 5 will probably be available.

In order to determine p approximately and estimate $\mathscr{E}(t)$ additional assumptions must be made. One approach is suggested by the observation that on a random-ratio schedule $\mathscr{E}(t)$ vanishes (since $E(t)$ does), and p, the asymptotic reinforcement pro-

bability which appears in the expression for $I(t)$ in Theorem 5, is simply the constant value of $u(t)$. This suggests that when $\sup_{0<t} u(t) - \inf_{0<t} u(t)$ is small $\mathscr{E}(t)$ will be small and p can be conveniently approximated. The following theorem validates this intuition.

THEOREM 6. *If $\theta_0 > 0$, $\theta_1 > 0$, and $\alpha < \bar{\theta}/\theta_1 \sup u$ then*

$$| \mathscr{E}(t) | \leqslant \frac{1}{4} \left(\frac{| \theta_1 - \theta_0 |}{\bar{\theta}} \right) \left(\frac{1}{1 - \alpha(\theta_1/\bar{\theta}) \sup u} \right) (\sup u - \inf u) \qquad (29)$$

and

$$| p - p^* | \leqslant \tfrac{1}{2} (\sup u - \inf u),$$

where

$$p^* = \inf u + \tfrac{1}{2} (\sup u - \inf u).$$

PROOF. The second part of the theorem is trivial:

$$| p - p^* | = \left| \int_0^\infty (u(t) - p^*) \, dI(t) \right| \leqslant \sup_t | u(t) - p^* | \int_0^\infty dI(t) \leqslant \frac{1}{2} (\sup u - \inf u).$$

To prove the first part we note that

$$\mathscr{E}(t) = \frac{1}{\bar{\theta}} \lim_{n \to \infty} \int_0^\infty R(t, s) \, dE_n(s)$$

$$= \frac{(\theta_0 - \theta_1)}{\bar{\theta}} \lim_{n \to \infty} \text{cov} \left[\int_0^\infty u(s) \, dI_n(s \mid s_{n-1}), \int_0^\infty R(t, s) \, dI_n(s \mid s_{n-1}) \right]. \qquad (30)$$

But

$$\left| \text{cov} \left[\int_0^\infty u(s) \, dI_n(s \mid s_{n-1}), \int_0^\infty R(t, s) \, dI_n(s \mid s_{n-1}) \right] \right|$$

$$\leqslant \sigma \left[\int_0^\infty u(s) \, dI_n(s \mid s_{n-1}) \right] \sigma \left[\int_0^\infty R(t, s) \, dI_n(s \mid s_{n-1}) \right]. \qquad (31)$$

Also for any random variable Y with $M_1 \leqslant Y \leqslant M_2$ for some finite constants M_1 and M_2

$$\sigma[Y] \leqslant E^{1/2}[(Y - \tfrac{1}{2}(M_1 + M_2))^2] \leqslant \tfrac{1}{2}(M_2 - M_1),$$

since

$$| Y - \tfrac{1}{2}(M_1 + M_2) | \leqslant \tfrac{1}{2}(M_2 - M_1).$$

Thus

$$\sigma \left[\int_0^\infty u(s) \, dI_n(s \mid s_{n-1}) \right] \leqslant \frac{1}{2} (\sup u - \inf u)$$

and   (32)

$$\sigma\left[\int_0^\infty R(t,s)\, dI_n(s \mid s_{n-1})\right] \leqslant \frac{1}{2}\left(1 - \alpha\frac{\theta_1}{\bar{\theta}}\sup u\right)^{-1}$$

(the latter in view of

$$0 \leqslant R(t,s) \leqslant \sum_{k=0}^\infty \left(\alpha\frac{\theta_1}{\bar{\theta}}\right)^k (\sup u)^k.$$

The proof is completed by combining (30), (31), and (32).   Q.E.D.

It is clear, however, that even in the random-ratio case where p is known and $\mathscr{E}(t)$ vanishes some approximations will have to be introduced to obtain conceptually and computationally tractable formulas from (18).

III. APPROXIMATIONS TO $I(t)$ WHEN α AND θ_0/θ_1 ARE SMALL

One way to obtain simple and easily interpertable formulas from (18) is to assume that α and $\lambda = \theta_0/\theta_1$ are small. In this section the first order Taylor expansions of $I(t)$ and its moments in these variables will be obtained. In Sect. IV these formulas will be compared with data obtained under three different schedules of reinforcement. The fact that the correspondence of theory and data is respectable is taken as an indication of the usefulness of the small α and λ assumptions.

Further evidence for the small λ assumption is provided by the estimates of λ from random-ratio and variable-interval rate of responding data obtained in Parts A and B of Sect. IV. These estimates suggest that for these data the appropriate value of λ is of the order of magnitude of 10^{-4}. Intuitively, small λ is a reflection of the sensitivity of response rate to small changes in rate r of reinforcement in the neighborhood of $r = 0$. This effect is apparent in Catania and Reynold's (1963) variable-interval data. By glancing ahead at Table 2 the reader can see that most of the variation in mean interresponse time (M) occurs in the upper lines of the table, corresponding to the largest values of mean interreinforcement time $(1/\gamma)$. The effect is obscured in Brandauer's (1958) random-ratio data (Table 1) by the fact that his birds were not run at lower p values. The range of variation in M in Table 1 is comparable to that in the lower part of Table 2. From the vantage point of the linear free-responding model sensitivity of response rate to small changes in r when r is very small must mean that the effect of reinforcement is so much more powerful than the effect of nonreinforcement that only occasional reinforcement is necessary to drive response rate into the vicinity of the highest rate attainable under the given experimental conditions.

It is instructive to consider the random-ratio case first, since it is by far the simplest. If $p = 0$ it is easy to see that (18) reduces to $I(t) = F_0(t)$, so assume henceforth that $p > 0$. The formula (18) can then be rewritten

$$I(t) = \lambda \left(\frac{1-p}{p}\right) g_p(\lambda) \sum_{k=0}^{\infty} (\alpha g_p(\lambda))^k \int_0^{\infty} L^{(k)}(t, s) \, dF_0(s)$$

$$+ (1 - \alpha) g_p(\lambda) \sum_{k=0}^{\infty} (\alpha g_p(\lambda))^k \int_0^{\infty} L^{(k)}(t, s) \, dF_1(s)$$

or

$$I(t) = \lambda \left(\frac{1-p}{p}\right) \sum_{k=0}^{\infty} \alpha^k g_p^{k+1}(\lambda) \int_0^{\infty} L^{(k)}(t, s) \, dF_0(s) + \sum_{k=0}^{\infty} \alpha^k g_p^{k+1}(\lambda) \int_0^{\infty} L^{(k)}(t, s) \, dF_1(s)$$

$$- \sum_{k=0}^{\infty} \alpha^{k+1} g_p^{k+1}(\lambda) \int_0^{\infty} L^{(k)}(t, s) \, dF_1(s),$$

where

$$g_p(\lambda) = \frac{1}{1 + \lambda(1 - p)/p},$$

and $L^{(k)}$ is defined in a manner analogous to $W^{(k)}$ (see (21), (22), and (23)) but with $L^{(1)}(t, s) = L(t, s)$. By substituting the power series representations

$$g_p^n(\lambda) = \sum_{m=0}^{\infty} \binom{n-1+m}{n-1} \left(-\left(\frac{1-p}{p}\right) \lambda\right)^m$$

and regrouping terms, the formula

$$I(t) = F_1(t) + \alpha \left(\int_0^{\infty} L^{(1)}(t, s) \, dF_1(s) - F_1(t)\right) + \lambda \left(\frac{1-p}{p}\right) (F_0(t) - F_1(t))$$

$$+ \sum_{k+m \geqslant 2} \binom{k+m}{k} (-1)^m \left(\frac{1-p}{p}\right)^m \left[\int_0^{\infty} L^{(k)}(t, s) \, dF_1(s)\right.$$

$$\left. - \frac{m}{k+m} \int_0^{\infty} L^{(k)}(t, s) \, dF_0(s) - \frac{k}{k+m} \int_0^{\infty} L^{(k-1)}(t, s) \, dF_1(s)\right] \alpha^k \lambda^m \qquad (33)$$

is obtained.

Such a formula makes up in power what it lacks in simplicity. From it approxima-
tions to $I(t)$ of any order in α and λ can be obtained as well as an explicit expression
for the error in any such approximation. Unfortunately, the distributions $I(t)$ for
other schedules are not as well understood.

The first order approximation to $I(t)$ for small α and λ and general reinforcement
function u is given by Theorem 7. The requirement that

$$\int_0^\infty u(t)\, dF_i(t) = \mu_i > 0, \qquad i = 0, 1$$

corresponds to the assumption that $p > 0$ in the preceeding paragraphs.

THEOREM 7.   *If $\mu_0 > 0$ and $\mu_1 > 0$ then*

$$I(t) = (1 - \alpha - \beta)\, F_1(t) + \beta F_0(t) + \alpha H(t) + O(\alpha^2 + \lambda^2) \tag{34}$$

for $\lambda > 0$ and $\alpha \geqslant 0$, where

$$\beta = \frac{1 - \mu_1}{\mu_1}\, \lambda \tag{35}$$

and

$$H(t) = \frac{1}{\mu_1} \int_0^\infty L(t, s)\, u(s)\, dF_1(s). \tag{36}$$

Since formulas like (34) have not appeared very frequently in the psychological
literature, a few words about the interpretation of this formula are in order. By
definition the formula means that there are constants $B < \infty$ and $b > 0$ such that

$$|\, I(t) - (1 - \alpha - \beta)\, F_1(t) - \beta F_0(t) - \alpha H(t)\,| \leqslant B(\alpha^2 + \lambda^2)$$

or

$$\frac{|\, I(t) - (1 - \alpha - \beta)\, F_1(t) - \beta F_0(t) - \alpha H(t)\,|}{(\alpha^2 + \lambda^2)^{1/2}} \leqslant B(\alpha^2 + \lambda^2)^{1/2} \tag{37}$$

for all $\lambda > 0$, $\alpha \geqslant 0$ and $\alpha^2 + \lambda^2 \leqslant b$. Thus the error in approximating $I(t)$ by the
linear function

$$L_0(\alpha, \lambda) = (1 - \alpha - \beta)\, F_1(t) + \beta F_0(t) + \alpha H(t) \tag{38}$$

of α and λ not only goes to 0 as α and λ approach 0, but is actually an arbitrarily small
proportion of $(\alpha^2 + \lambda^2)^{1/2}$ if α and λ are sufficiently small. If another linear function
$L(\alpha, \lambda)$ of α and λ had this property, then

$$\lim_{\alpha, \lambda \to 0} \frac{|\, L(\alpha, \lambda) - L_0(\alpha, \lambda)\,|}{(\alpha^2 + \lambda^2)^{1/2}} = 0,$$

which, however, implies that $L(\alpha, \lambda) = L_0(\alpha, \lambda)$. Thus $L_0(\alpha, \lambda)$ is the best linear approximation to $I(t)$ in the sense that it is the only linear approximation for which the error goes to zero faster than $(\alpha^2 + \lambda^2)^{1/2}$ as α and λ approach 0.

Of course to say that $L_0(\alpha, \lambda)$ becomes an arbitrarily good approximation to $I(t)$ as α and λ approach 0 does not imply that for particular "small" values of α and λ (e.g., $\alpha = .1, \lambda = .0001$) the approximation is good, but it certainly suggests that this is the case. This suggestion will be followed up in Sect. IV, where L_0 and other first order approximations will be compared with experimental data. To the extent that the results of this analysis are encouraging, several paths toward greater refinement present themselves for future research:

(a)   Monte Carlo computations could be made to give information about the characteristics of the approximations for particular specifications of the functions and parameters of the model and of the reinforcement function u in which the researcher has particular interest.

(b)   Formulas for second and higher order approximations to $I(t)$ and its moments could be sought.

(c)   Finally, and perhaps most fruitfully, attempts could be made to determine experimentally what variables control the magnitude of the contribution of temporal discrimination to total performance and the relative effectiveness of reinforcement and nonreinforcement. Not only would the results of such experiments be of great intrinsic interest, but they would permit an experimenter to obtain especially small values of α and λ, for which the first order approximations developed here are especially appropriate, at will.

PROOF OF THEOREM 7.   Since all of the asymptotic quantities with which we will work below are independent of $I_1(t)$, we may assume, without loss of generality, that $I_1(t) = F_0(t)$.

Let sequences $\{a_n(s_{n-1})\}$ and $\{b_n(s_{n-1})\}$ be defined recursively as follows: for $n = 1$,

$$a_n(s_{n-1}) = a_1 = 0 \quad \text{and} \quad b_n(s_{n-1}) = b_1 = 1, \tag{39}$$

while for $n \geq 1$

$$a_{n+1}(s_n) = (1 - \theta_{k_n}) a_n(s_{n-1}) + k_n \theta_{k_n}$$

and

$$b_{n+1}(s_n) = (1 - \theta_{k_n}) b_n(s_{n-1}) + (1 - k_n) \theta_{k_n}. \tag{40}$$

It is easily shown by induction that

$$I_n(t \mid s_{n-1}) = b_n(s_{n-1}) F_0(t) + a_n(s_{n-1})(1 - \alpha) F_1(t) + \alpha K_n(t \mid s_{n-1}) \tag{41}$$

for $n \geqslant 1$, where

$$K_n(t \mid s_{n-1}) = \sum_{v=1}^{n-1} \Big(\prod_{j=v+1}^{n-1} (1 - \theta_{k_j}) \Big) \theta_{k_v} k_v L(t, t_v). \tag{42}$$

Also from (40) it follows that, for all $n \geqslant 1$

$$a_{n+1}(s_n) + b_{n+1}(s_n) = (1 - \theta_{k_n}) (a_n(s_{n-1}) + b_n(s_{n-1})) + \theta_{k_n}.$$

But by (39) $a_1 + b_1 = 1$. Thus by induction it follows that

$$a_n(s_{n-1}) + b_n(s_{n-1}) = 1$$

for all $n \geqslant 1$. Therefore, (41) can be rewritten

$$I_n(t \mid s_{n-1}) = b_n(s_{n-1}) F_0(t) + (1 - b_n(s_{n-1})) (1 - \alpha) F_1(t) + \alpha K_n(t \mid s_{n-1}). \tag{43}$$

Also, it is easily verified that the solution to the recursion (40) for $a_n(s_{n-1})$ satisfying the initial condition (39) is

$$1 - b_n(s_{n-1}) = a_n(s_{n-1}) = \sum_{v=1}^{n-1} \Big(\prod_{j=v+1}^{n-1} (1 - \theta_{k_j}) \Big) \theta_{k_v} k_v . \tag{44}$$

The formulas (43) and (44) will be the basis of much of our later work.

For any bounded functions f we obtain from (43)

$$\int_0^\infty f(t) \, dI_n(t \mid s_{n-1}) = b_n(s_{n-1}) \int_0^\infty f(t) \, dF_0(t) + (1 - b_n(s_{n-1})) (1 - \alpha) \int_0^\infty f(t) \, dF_1(t)$$

$$+ \alpha \sum_{v=1}^{n-1} \Big(\prod_{j=v+1}^{n-1} (1 - \theta_{k_j}) \Big) \theta_{k_v} k_v \int_0^\infty f(t) \, d_t L(t, t_v). \tag{45}$$

Applying this equality to $u(t)$ and noting that the last term is bounded above and below by $\alpha(1 - b_n(s_{n-1})) \sup f$ and $\alpha(1 - b_n(s_{n-1})) \inf f$, respectively, we obtain

$$w_*(b_n(s_{n-1})) = b_n(s_{n-1}) \mu_0 + (1 - b_n(s_{n-1})) ((1 - \alpha) \mu_1 + \alpha \inf u)$$

$$\leqslant \int_0^\infty u(t) \, dI_n(t \mid s_{n-1}) \leqslant b_n(s_{n-1}) \mu_0 + (1 - b_n(s_{n-1})) ((1 - \alpha) \mu_1 + \alpha \sup u)$$

$$= w^*(b_n(s_{n-1})). \tag{46}$$

Taking expectations on both sides we find that

$$E[b_n(s_{n-1})]\, \mu_0 + (1 - E[b_n(s_{n-1})])\, ((1 - \alpha)\mu_1 + \alpha \inf u) \leqslant p_n$$

$$\leqslant E[b_n(s_{n-1})]\, \mu_0 + (1 - E[b_n(s_{n-1})])\, ((1 - \alpha)\, \mu_1 + \alpha \sup u)$$

or

$$E[b_n(s_{n-1})]\, (\mu_0 - \mu_1) - \alpha(1 - E[b_n(s_{n-1})])\, (\mu_1 - \inf u) \leqslant p_n - \mu_1$$

$$\leqslant E[b_n(s_{n-1})]\, (\mu_0 - \mu_1) + \alpha(1 - E[b_n(s_{n-1})])\, (\sup u - \mu_1).$$

From this and $\inf u \leqslant \mu_i \leqslant \sup u$, $i = 1, 2$, it follows that

$$\mid p_n - \mu_1 \mid \leqslant (\sup u - \inf u)\, ((1 - \alpha)\, E[b_n(s_{n-1})] + \alpha),$$

so that, taking upper limits on both sides, the important relation

$$\mid p - \mu_1 \mid \leqslant (\sup u - \inf u)\, ((1 - \alpha)\, \overline{\lim_{n \to \infty}}\, E[b_n(s_{n-1})] + \alpha) \tag{47}$$

is obtained.

By means of (44), (45) can be rewritten in the form

$$\int_0^\infty f(t)\, dI_n(t \mid s_{n-1}) = \int_0^\infty f(t)\, dF_1(t) + b_n(s_{n-1}) \left(\int_0^\infty f(t)\, dF_0(t) - \int_0^\infty f(t)\, dF_1(t) \right)$$

$$+ \alpha \sum_{v=1}^{n-1} \left(\prod_{j=v+1}^{n-1} (1 - \theta_{k_j}) \right) \theta_{k_v} k_v \left[\int_0^\infty f(t)\, d_t L(t, t_v) - \int_0^\infty f(t)\, dF_1(t) \right]. \tag{48}$$

But the third term on the right is bounded in absolute value by $\alpha(\sup f - \inf f)$. Using this fact to obtain an estimate of the second term on the right in (49), which equation is obtained from (48) by Minkowski's inequality,

$$\sigma \left[\int_0^\infty f(t)\, dI_n(t \mid s_{n-1}) \right]$$

$$\leqslant \left| \int_0^\infty f(t)\, dF_0(t) - \int_0^\infty f(t)\, dF_1(t) \right| \sigma[b_n(s_{n-1})]$$

$$+ \sigma \left[\alpha \sum_{v=1}^{n-1} \left(\prod_{j=v+1}^{n-1} (1 - \theta_{k_j}) \right) \theta_{k_v} k_v \left[\int_0^\infty f(t)\, d_t L(t, t_v) - \int_0^\infty f(t)\, dF_1(t) \right] \right] \tag{49}$$

we find that

$$\sigma \left[\int_0^\infty f(t)\, dI_n(t \mid s_{n-1}) \right] \leqslant (\sup f - \inf f)\, (\sigma[b_n(s_{n-1})] + \alpha).$$

Taking upper limits on both sides leads to the conclusion that

$$\lim_{n\to\infty} \sigma\left[\int_0 f(t)\, dI_n(t \mid s_{n-1})\right] \leqslant (\sup f - \inf f)\, (\overline{\lim_{n\to\infty}}\, \sigma[b_n(s_{n-1})] + \alpha) \qquad (50)$$

for any bounded function $f(t)$. If (50) is applied twice, once to $f(t) = u(t)$ and once to $f(s) = R(t, s)$, and the results are combined with (30) and (31), the important formula

$$|\,\mathscr{E}(t)\,| \leqslant \frac{|\,1 - \lambda\,|}{p + (1 - p)\,\lambda}\, \frac{(\sup u - \inf u)}{1 - \alpha \sup u/(p + (1 - p)\,\lambda)}\, (\overline{\lim_{n\to\infty}}\, \sigma[b_n(s_{n-1})] + \alpha)^2 \qquad (51)$$

is obtained.

To make further progress we must have estimates of the upper limits of the moments of the stochastic process $\{b_n(s_{n-1})\}$ that appear on the right in (47) and (51). Referring back to (40) we see that this process satisfies

$$P(b_{n+1}(s_n) = (1 - \theta_0)\, b_n(s_{n-1}) + \theta_0 \mid s_{n-1}) = P(k_n = 0 \mid s_{n-1}) = 1 - \int_0^\infty u(t)\, dI_n(t \mid s_{n-1}) \qquad (52)$$

and

$$P(b_{n+1}(s_n) = (1 - \theta_1)\, b_n(s_{n-1}) \mid s_{n-1}) = P(k_n = 1 \mid s_{n-1}) = \int_0^\infty u(t)\, dI_n(t \mid s_{n-1}). \qquad (53)$$

As λ approaches 0 the "up" operator becomes weak in comparison with the "down" operator, so it is to be expected that the random variables $b_n(s_{n-1})$ for n large will concentrate at 0 as λ approaches 0. We will now study the rate at which this concentration occurs.

We write

$$\begin{aligned}
E[b_{n+1}^2(s_n)] &= E[(b_n(s_{n-1}) + (b_{n+1}(s_n) - b_n(s_{n-1}))^2] \\
&= E[b_n{}^2(s_{n-1})] + 2E[b_n(s_{n-1})\,(b_{n+1}(s_n) - b_n(s_{n-1}))] \\
&\quad + E[(b_{n+1}(s_n) - b_n(s_{n-1}))^2] \\
&= E[b_n{}^2(s_{n-1})] + 2E[b_n(s_{n-1})\, E[b_{n+1}(s_n) - b_n(s_{n-1}) \mid s_{n-1}]] \\
&\quad + E[E[(b_{n+1}(s_n) - b_n(s_{n-1}))^2 \mid s_{n-1}]]. \qquad (54)
\end{aligned}$$

But

$$E[(b_{n+1}(s_n) - b_n(s_{n-1}))^k \mid s_{n-1}]$$
$$= [\theta_0(1 - b_n(s_{n-1}))]^k\, (1 - P(k_n = 1 \mid s_{n-1})) + [-\theta_1 b_n(s_{n-1})]^k\, P(k_n = 1 \mid s_{n-1}). \qquad (55)$$

Thus, using the bounds for $P(k_n = 1 \mid s_{n-1})$ given by (46), we obtain from (54) and (55)

$$
\begin{aligned}
E[b_{n+1}^2(s_n)] \leqslant\ & E[b_n{}^2(s_{n-1})] \\
& + 2E[b_n(s_{n-1})\,(\theta_0(1 - b_n(s_{n-1}))\,(1 - w_*(b_n(s_{n-1}))) - \theta_1 b_n(s_{n-1})\,w_*(b_n(s_{n-1}))] \\
& + E[\theta_0{}^2(1 - b_n(s_{n-1}))^2\,(1 - w_*(b_n(s_{n-1}))) + \theta_1{}^2 b_n{}^2(s_{n-1})\,w^*(b_n(s_{n-1}))] \\
=\ & E[b_n{}^2(s_{n-1})\,(1 - 2\theta_1 w_*(b_n(s_{n-1})) + \theta_1{}^2 w^*(b_n(s_{n-1})))] \\
& + 2E[b_n(s_{n-1})\,\theta_0(1 - b_n(s_{n-1}))\,(1 - w_*(b_n(s_{n-1})))] \\
& + E[\theta_0{}^2(1 - b_n(s_{n-1}))^2\,(1 - w_*(b_n(s_{n-1})))].
\end{aligned}
$$

But

$$
\begin{aligned}
\theta_1[2w_*(b_n(s_{n-1})) &- \theta_1 w^*(b_n(s_{n-1}))] \\
= \theta_1[b_n(s_{n-1})\,\mu_0(2 - \theta_1) &+ (1 - b_n(s_{n-1}))\,((1 - \alpha)\,\mu_1(2 - \theta_1) + \alpha(2 \inf u - \theta_1 \sup u)) \\
\geqslant \theta_1[b_n(s_{n-1})\,\mu_0 &+ (1 - b_n(s_{n-1}))\,\tfrac{1}{2}\mu_1]
\end{aligned}
$$

$$
\geqslant \theta_1 \min\left(\mu_0,\,\frac{\mu_1}{2}\right)
$$

for all $\alpha \leqslant \tfrac{1}{4}\mu$, $\sup u$, and

$$
\inf u \leqslant w^*(b_n(s_{n-1})) \leqslant \sup u.
$$

Therefore,

$$
\begin{aligned}
E[b_{n+1}^2(s_n)] \leqslant\ & \left(1 - \theta_1 \min\left(\mu_0,\,\frac{\mu_1}{2}\right)\right) E[b_n{}^2(s_{n-1})] \\
& + 2(1 - \inf u)\,\theta_0 E[b_n(s_{n-1})] + (1 - \inf u)\,\theta_0{}^2 \\
\leqslant\ & \left(1 - \theta_1 \min\left(\mu_0,\,\frac{\mu_1}{2}\right)\right) E[b_n{}^2(s_{n-1})] \\
& + 2(1 - \inf u)\,\theta_0 E^{1/2}[b_n{}^2(s_{n-1})] + (1 - \inf u)\,\theta_0{}^2.
\end{aligned}
$$

Taking upper limits on both sides, we obtain

$$
\begin{aligned}
\varlimsup_{n\to\infty} E[b_n{}^2(s_{n-1})] \leqslant\ & \left(1 - \theta_1 \min\left(\mu_0,\,\frac{\mu_1}{2}\right)\right) \varlimsup_{n\to\infty} E[b_n{}^2(s_{n-1})] \\
& + 2(1 - \inf u)\,\theta_0(\varlimsup_{n\to\infty} E[b_n{}^2(s_{n-1})])^{1/2} + (1 - \inf u)\,\theta_0{}^2
\end{aligned}
$$

or

$$
\begin{aligned}
\theta_1 \min\left(\mu_0,\,\frac{\mu_1}{2}\right) & \varlimsup_{n\to\infty} E[b_n{}^2(s_{n-1})] \\
& \leqslant 2(1 - \inf u)\,\theta_0(\varlimsup_{n\to\infty} E[b_n{}^2(s_{n-1})])^{1/2} + (1 - \inf u)\,\theta_0{}^2.
\end{aligned}
$$

Division by $\lambda^2 \theta_1 \min(\mu_0, \mu_1/2) > 0$ yields

$$\frac{1}{\lambda^2} \varlimsup_{n \to \infty} E[b_n{}^2(s_{n-1})] \leqslant 2 \left(\frac{1 - \inf u}{\min(\mu_0, \mu_1/2)}\right) \left(\frac{1}{\lambda^2} \varlimsup_{n \to \infty} E[b_n{}^2(s_{n-1})]\right)^{1/2} + \left(\frac{1 - \inf u}{\min(\mu_0, \mu_1/2)}\right),$$

which implies that

$$\frac{1}{\lambda^2} \varlimsup_{n \to \infty} E[b_n{}^2(s_{n-1})] \leqslant K^2, \tag{56}$$

where K is the unique positive root of the quadratic equation

$$x^2 = 2 \left(\frac{1 - \inf u}{\min(\mu_0, \mu_1/2)}\right) x + \left(\frac{1 - \inf u}{\min(\mu_0, \mu_1/2)}\right). \tag{57}$$

It follows that

$$\varlimsup_{n \to \infty} \sigma[b_n(s_{n-1})] = O(\lambda) \tag{58}$$

and

$$\varlimsup_{n \to \infty} E[b_n(s_{n-1})] = O(\lambda) \tag{59}$$

uniformly in α for α sufficiently small.

Combining (59) with (47) we obtain

$$p = \mu_1 + O((\alpha^2 + \lambda^2)^{1/2}). \tag{60}$$

This in turn implies that

$$\frac{|1 - \lambda|}{p + (1 - p)\lambda} \frac{(\sup u - \inf u)}{1 - \alpha \sup u/(p + (1 - p)\lambda)} \to \frac{1}{\mu_1}(\sup u - \inf u)$$

as α and λ approach 0. Consequently the term on the left is bounded when $(\alpha^2 + \lambda^2)^{1/2}$ is sufficiently small. Therefore, (58) and (51) yield

$$\mathscr{E}(t) = O(\alpha^2 + \lambda^2). \tag{61}$$

We now consider the other terms in the formula (18). Clearly

$$\sum_{k=2}^{\infty} \left(\alpha \frac{\theta_1}{\bar\theta}\right)^k \int_0^\infty W^{(k)}(t, s) \, d \left(\frac{\theta_0(1 - p)}{\bar\theta} F_0(s) + (1 - \alpha) \frac{\theta_1 p}{\bar\theta} F_1(s)\right) = O(\alpha^2)$$

uniformly in λ when λ is sufficiently small. Also,

$$\frac{\theta_1 p}{\bar\theta} = \frac{1}{1 + \lambda\left(\dfrac{1 - p}{p}\right)} = 1 - \lambda\left(\frac{1 - p}{p}\right) + O(\lambda^2)$$

$$= 1 - \lambda\left(\frac{1 - \mu_1}{\mu_1}\right) + O(\alpha^2 + \lambda^2).$$

Therefore,

$$I(t) = \left(\lambda \left(\frac{1-\mu_1}{\mu_1}\right) + O(\alpha^2 + \lambda^2)\right) F_0(t) + (1-\alpha)\left(1 - \lambda\frac{(1-\mu_1)}{\mu_1} + O(\alpha^2 + \lambda^2)\right)F_1(t)$$

$$+ \alpha(1 + O((\alpha^2 + \lambda^2)^{1/2}))\left(\frac{1}{\mu_1} + O((\alpha^2 + \lambda^2)^{1/2})\right).$$

$$\left(O((\alpha^2 + \lambda^2)^{1/2}) + (1-\alpha)(1 + O((\alpha^2 + \lambda^2)^{1/2}))\int_0^\infty W^{(1)}(t, s)\, dF_1(s)\right) + O(\alpha^2 + \lambda^2)$$

$$= \lambda\left(\frac{1-\mu_1}{\mu_1}\right) F_0(t) + \left(1 - \alpha - \lambda\frac{(1-\mu_1)}{\mu_1}\right)F_1(t)$$

$$+ \alpha\frac{1}{\mu_1}\int_0^\infty W^{(1)}(t, s)\, dF_1(s) + O(\alpha^2 + \lambda^2).$$

Q.E.D.

The formula (34) shows that, for small α and λ, $I(t)$ is approximately a weighted average of the fast responding distribution F_1, the slow responding distribution F_0, and the distribution H. To get some feeling for H, consider its density $h = H'$, writing it as

$$h(t) = \frac{\displaystyle\int_0^\infty \ell(t, s)\, u(s)\, f_1(s)\, ds}{\displaystyle\int_0^\infty u(s)\, f_1(s)\, ds},$$

where

$$\ell(t, s) = \frac{\partial}{\partial t} L(t, s) \qquad \text{and} \qquad f_1 = F_1'.$$

For fixed $s, \ell(\cdot, s)$ has been assumed to be a density with most of its mass around s. It follows that, if $\int_0^\infty \ell(t, s)\, ds < \infty$ for some t, then

$$\frac{\ell(t, \cdot)}{\displaystyle\int_0^\infty \ell(t, s)\, ds}$$

is a density with most of its mass around t. Thus if $u(s)\, f_1(s)$ is temperate (i.e., has small variation) in the vicinity of t, it follows that

$$h(t) = \frac{\displaystyle\int_0^\infty \ell(t, s)\, ds}{\displaystyle\int_0^\infty u(s)\, f_1(s)\, ds} \quad \frac{\displaystyle\int_0^\infty u(s)\, f_1(s)\, \ell(t, s)\, ds}{\displaystyle\int_0^\infty \ell(t, s)\, ds}$$

$$\doteq \frac{\displaystyle\int_0^\infty \ell(t, s)\, ds}{\displaystyle\int_0^\infty u(s)\, f_1(s)\, ds}\, u(t)\, f_1(t).$$

Under the assumption (6)

$$\ell(t, s) = G'\left(\frac{t}{s}\right)\frac{1}{s},$$

so that

$$\int_0^\infty \ell(t, s)\, ds = \int_0^\infty G'\left(\frac{t}{s}\right)\frac{ds}{s} = \int_0^\infty G'(z)\frac{dz}{z} \doteq 1,$$

since G' has most of its mass around 1. Thus the extremely crude approximation

$$h(t) \doteq \frac{u(t)\, f_1(t)}{\displaystyle\int_0^\infty u(s)\, f_1(s)\, ds} \tag{62}$$

is obtained. From this it can be seen that large values of $u(t)$ representing regions of high reinforcement availability are reflected in large values of $h(t)$. Thus the measure dH has a tendency to concentrate in regions of high reinforcement availability. It may be described as an adaptive component of the asymptotic IRT measure

$$dI \doteq (1 - \alpha - \beta)\, dF_1 + \beta\, dF_0 + \alpha\, dH.$$

In the case $L(t, s) = G(t/s)$, where

$$G^{(m)} = \int_0^\infty x^m\, dG(x) < \infty,$$

the mth moment of $L(\cdot, s)$ is given by

$$\int_0^\infty t^m\, d_t L(t, s) = G^{(m)} s^m.$$

Thus

$$\int_0^\infty t^m\, dH(t) = G^{(m)}\, \frac{\displaystyle\int_0^\infty s^m u(s)\, dF_1(s)}{\displaystyle\int_0^\infty u(s)\, dF_1(s)}.$$

In view of this Theorem 7 suggests the following proposition.

THEOREM 8. *If*

$$\mu_0 > 0, \qquad \mu_1 > 0, \qquad L(t, s) = G\left(\frac{t}{s}\right),$$

$$G^{(m)} = \int_0^\infty x^m\, dG(x) < \infty, \qquad and \qquad M_i^{(m)} = \int_0^\infty x^m\, dF_i(x) < \infty, \qquad i = 1, 2$$

then

$$M^{(m)} = (1 - \alpha - \beta)\, M_1^{(m)} + \beta M_0^{(m)} + \alpha G^{(m)} N^{(m)} + O(\alpha^2 + \lambda^2), \qquad (63)$$

where

$$M^{(m)} = \int_0^\infty t^m \, dI(t) \qquad and \qquad N^{(m)} = \frac{\displaystyle\int_0^\infty s^m u(s) \, dF_1(s)}{\displaystyle\int_0^\infty u(s) \, dF_1(s)}.$$

In the case where $u(t)$ vanishes for t sufficiently large, Theorem 8 can be proved by a straightforward modification of the proof of Theorem 7, which will not be presented here. Given an arbitrary schedule $u(t)$, consider approximating schedules $u_\Gamma(t)$ which vanish for t sufficiently large, defined by

$$u_\Gamma(t) = \begin{cases} u(t) & \text{if} \quad t \leqslant \Gamma \\ 0 & \text{if} \quad t > \Gamma. \end{cases}$$

Just as it is clear that the performance obtained from an organism under u_Γ will converge to that obtained under u as Γ approaches infinity, it is apparent that formulas obtained in the linear free-responding model for u_Γ will go over into the corresponding formulas for u in the limit as Γ approaches infinity. Thus (63) holds for u, and Theorem 8 may be regarded as proved.

Theorem 8 and the assumption $G^{(1)} \doteq 1$ yield the following approximation to the asymptotic mean IRT valid for small α and λ:

$$M \doteq (1 - \alpha - \beta)\, M_1 + \beta M_0 + \alpha N \qquad (64)$$

where all upper 1's have been suppressed. This approximation will be used in the first two parts of Sect. IV.

IV. APPLICATIONS

Specialization of (64) to random ratio schedules leads directly to Eq. 65 below, which expresses asymptotic mean IRT (M) as a linear function of the mean ratio $1/p$ of the schedule. When (64) is applied to random interval schedules with small γ the formula (67), which expresses M as a linear function of the mean interreinforcement interval $1/\gamma$, is obtained. Thus in both cases the linear free-responding model predicts that M is approximately a linear function of a natural measure of infrequency of reinforcement. Data relevant to these predictions are discussed in Parts A and B below.

In Part C, as an illustration of the use of formula (34), a case is considered in which a bimodal asymptotic IRT distribution is predicted.

A. Random-Ratio Schedules of Reinforcement

The random-ratio schedule of reinforcement with reinforcement probability p is the simple contingent schedule with $u(t) \equiv p$. Clearly $N = M_1$ for this schedule so the small α, small λ, $\int_0^\infty t \, dG(t) \doteq 1$ formula (64) for the asymptotic mean IRT reduces in the present case to

$$M \doteq (1 - \beta) M_1 + \beta M_0 = \left(1 - \frac{1 - p}{p} \lambda\right) M_1 + \frac{1 - p}{p} \lambda M_0$$

$$= [M_1 - (M_0 - M_1) \lambda] + \frac{1}{p} [\lambda(M_0 - M_1)].$$

Denoting $\lambda(M_0 - M_1)$ by μ and $M_1 - (M_0 - M_1)\lambda$ by η yields

$$M \doteq \eta + \frac{1}{p} \mu \tag{65}$$

as the function giving the mean asymptotic IRT for various values of p, the probability that any given response will be reinforced.

Obviously M decreases as p increases—the organism is supposed to speed up as he is reinforced more frequently. This was not observed in Brandauer's (1958) experiment for moderate and large values of p. Mean IRT's for all three of his birds decreased substantially as p was *decreased* from 1 to .1 and decreased a bit further when p was further decreased from .1 to .02. This might be attributable to a disruptive effect of reinforcement upon performance of this sort—reinforcement breaks the momentum of behavior, so to speak—and this affects response rate noticeably when reinforcement occurs frequently. However, such complications are not taken account of by the linear free-responding model, so the model will only be applied to the extreme lower range of p values. Sufficient data from this lower range are available from Pigeons 14 and 15 in Brandauer's experiment.[3] These data are presented in Table 1 along with corresponding predictions of the model. The parameter estimates $\hat\mu$ and $\hat\eta$ given in this table have been obtained for each bird by the method of least squares. On the whole it is felt that Table 1 shows fair agreement between the data and the predicted linear relation (65).

[3] See Brandauer (1958, Table 2, p. 25). For Pigeon 17, the mean IRT declined slightly when p was reduced from .02 to .01 and only increased when p was dropped further to .005. No p values below .005 were tested for this bird. In order to check the reliability of the rate data, some rates were redetermined some time after the first determination. The agreement between the two rates thus obtained and reported in Brandauer's Table 2 was good. The mean IRT's reported in Table 1 above are all first determinations.

TABLE 1

OBSERVED (O) AND PREDICTED[a] (P) MEAN IRT'S IN SECONDS AS A
FUNCTION OF $1/p$ FOR BRANDAUER'S EXPERIMENT

	Bird			
$1/p$	14		15	
	O	P	O	P
599	[b]		.30	.33
400	.46	.48	.34	.29
200	.46	.40	.23	.24
100	.37	.37	.22	.22
50	.31	.35	.21	.21

[a] From Eq. 65 with $\hat{\eta}$, $10^3 \hat{\mu}$ equal .328, .377 and .203, .211 for Birds 14 and 15, respectively.
[b] Bird 14 was not run at $1/p = 599$.

Estimates of M_1 obtained from the formula $\hat{M}_1 = \hat{\eta} + \hat{\mu}$, are .328 and .203 for Birds 14 and 15, respectively. Assuming that for each bird 10 sec $< M_0 - M_1 < 100$ sec the following bounds for $\lambda = \theta_0/\theta_1$ are obtained by substituting $\hat{\mu}$ for μ in the equation $\lambda = \mu/(M_0 - M_1)$:

$$\text{Bird 14:} \quad .377 \times 10^{-5} < \lambda < .377 \times 10^{-4}$$

$$\text{Bird 15:} \quad .211 \times 10^{-5} < \lambda < .211 \times 10^{-4}.$$

B. RANDOM-INTERVAL SCHEDULES OF REINFORCEMENT

Millenson (1963) described a reinforcement programming apparatus such that the nth response reinforced is the first response which occurs T_n seconds or more after the $n - 1$st reinforcement, where $\{T_n\}$ is a sequence of independent random variables with a common exponential distribution $F_T(t) = 1 - e^{-\gamma t}$. Millenson called this a random-interval schedule, and this term will be used here. It will now be shown that the random-interval schedule is a simple contingent schedule with reinforcement function $u(t) = 1 - e^{-\gamma t}$.

Implicit in the definition of this schedule is the assumption that

$$P(k_1 = 0 \mid t_1 = t) = P(T_1 > t) = e^{-\gamma t}.$$

Suppose now that $n > 1$ and that s_{n-1} is a history with $k_{n-1} = 0, \cdots, k_{n-l} = 0$,

$$k_{n-l-1} = 1, \qquad \sum_{j=1}^{n-l-1} k_j = m \qquad \text{and} \qquad t_{n-1} = t^*_{n-1}, \cdots, t_{n-k} = t^*_{n-k}.$$

Then

$$P(k_n = 0 \mid t_n = t \,\&\, s_{n-1}) = P(T_{m+1} > t + t^*_{n-1} + \cdots + t^*_{n-k} \mid T_{m+1}$$

$$> t^*_{n-1} + \cdots + t^*_{n-k})$$

$$= P(T_{m+1} > t) = e^{-\gamma t}.$$

This proves the assertion.

The small α, small λ, $\int_0^\infty t\,dG(t) \doteq 1$ formula (64) for the asymptotic mean IRT specializes in this case to

$$M \doteq (1 - \alpha - \beta)\, M_1 + \beta M_0 + \frac{\alpha \int_0^\infty s(1 - e^{-\gamma s})\,dF_1(s)}{\int_0^\infty (1 - e^{-\gamma s})\,dF_1(s)},$$

where

$$\beta = \frac{1 - \mu_1}{\mu_1}\, \lambda \qquad \text{and} \qquad \mu_1 = \int_0^\infty (1 - e^{-\gamma s})\,dF_1(s)$$

or

$$M \doteq [M_1 - (M_0 - M_1)\,\lambda] + \left[\frac{\int_0^\infty s(1 - e^{-\gamma s})\,dF_1(s)}{\int_0^\infty (1 - e^{-\gamma s})\,dF_1(s)} - M_1 \right] \alpha$$

$$+ \frac{1}{\int_0^\infty (1 - e^{-\gamma s})\,dF_1(s)}\,(M_0 - M_1)\,\lambda. \qquad (66)$$

Interest will center below around the case in which γ, the rate of reinforcement, is very small. Thus a small γ approximation to the above approximation will be developed.

By Taylor's theorem $1 - e^{-\gamma s} = \gamma s + \epsilon(\gamma, s)\, \gamma^2 s^2$, where $|\epsilon(\gamma, s)| \leqslant \frac{1}{2}$. Therefore,

$$\int_0^\infty (1 - e^{-\gamma s})\,dF_1(s) = \gamma \int_0^\infty s\,dF_1(s) + \delta(\gamma)\,\gamma^2 \int_0^\infty s^2\,dF_1(s),$$

where

$$|\delta(\gamma)| \leqslant \frac{1}{2},$$

and

$$\int_0^\infty s(1 - e^{-\gamma s})\, dF_1(s) = \gamma \int_0^\infty s^2\, dF_1(s) + \nu(\gamma)\, \gamma^2 \int_0^\infty s^3\, dF_1(s),$$

where $|\nu(\gamma)| \leqslant \frac{1}{2}$. Thus for small γ

$$\frac{\displaystyle\int_0^\infty s(1 - e^{-\gamma s})\, dF_1(s)}{\displaystyle\int_0^\infty (1 - e^{-\gamma s})\, dF_1(s)} \doteq \frac{\displaystyle\int_0^\infty s^2\, dF_1(s)}{\displaystyle\int_0^\infty s\, dF_1(s)},$$

so that the second term in the expression (66) for M is approximately

$$\left[\frac{\displaystyle\int_0^\infty s^2\, dF_1(s) - \left(\int_0^\infty s\, dF_1(s)\right)^2}{\displaystyle\int_0^\infty s\, dF_1(s)}\right] \alpha.$$

Assuming, to get some idea of the order of magnitude of the term in brackets, that F_1 is the famma distribution

$$F_1(t) = \int_0^t \frac{a}{\Gamma(r)} (ax)^{r-1} e^{-ax}\, dx,$$

where $a > 0, r > 1$, it follows that this term is equal to

$$\frac{r/a^2}{r/a} = \frac{1}{a} < \frac{r}{a} = M_1.$$

But it has been assumed that M_1 is small, and that α is small. Thus αM_1 is very small, and the quantity

$$\left[\frac{\displaystyle\int_0^\infty s^2\, dF_1(s) - \left(\int_0^\infty s\, dF_1(s)\right)^2}{\displaystyle\int_0^\infty s\, dF_1(s)}\right] \alpha$$

which αM_1 dominates will thus be neglected in all further calculations.

Next note that

$$\left[\int_0^\infty (1 - e^{-\gamma s})\, dF_1(s)\right]^{-1} = \left[\gamma \int_0^\infty s\, dF_1(s) + \delta(\gamma)\, \gamma^2 \int_0^\infty s^2\, dF_1(s)\right]^{-1} \sim \left[\gamma \int_0^\infty s\, dF_1(s)\right]^{-1}$$

as $\gamma \to 0$. It follows that for small γ the following approximation to the asymptotic mean IRT holds:

$$M \doteq v + w \frac{1}{\gamma}, \qquad (67)$$

where

$$v = M_1 - (M_0 - M_1)\lambda \quad \text{and} \quad w = \frac{1}{M_1}(M_0 - M_1)\lambda$$

so that

$$M_1 = \frac{v}{1-w} \quad \text{and} \quad (M_0 - M_1)\lambda = \frac{wv}{1-w}. \qquad (68)$$

There do not seem to be any data in the literature on rates of responding on random-interval schedules as a function of γ. However data on rates of responding on similar "variable-interval" schedules have recently been reported by Catania and Reynolds (1963). In their schedule of food reinforcement the sequence $\{T_n\}$ was deterministic and periodic with period 15. The phase of this sequence was varied from experimental session to experimental session. The values of the T_n were the first 15 non-negative integer multiples of a constant number τ of seconds. Different rates of reinforcement were obtained by varying τ. A cycle of T_n/τ values was 14, 8, 11, 6, 5, 9, 2, 13, 7, 1, 12, 4, 10, 0, 3. Catania and Reynold's pigeons would not have been expected to have behaved much differently had $\{T_n\}$ been a sequence of independent random

TABLE 2

OBSERVED (O) AND PREDICTED[a] (P) MEAN IRT's IN SECONDS AS A
FUNCTION OF $1/\gamma$ FOR CATANIA AND REYNOLD's EXPERIMENT

$1/\gamma$ (sec)	Bird											
	118		121		129		278		279		281	
	O	P	O	P	O	P	O	P	O	P	O	P
427	[b]		[b]		1.71	1.86	1.46	1.38	2.07	2.08	1.36	1.36
323	1.43	1.41	[b]		[b]		[b]		[b]		[b]	
216	[b]		[b]		1.71	1.35	.90	1.06	1.54	1.48	[b]	
108	.70	.77	3.00	3.24	.94	1.09	.84	.89	1.08	1.17	[b]	
45.5	.60	.59	2.86	2.21	.97	.94	.80	.80	.97	.99	.96	.96
23.5	.55	.53	1.97	1.85	.96	.89	.82	.76	.98	.92	.95	.93
12	.51	.49	1.13	1.66	.71	.86	.80	.75	.88	.89	.90	.92

[a] From Eq. 67 with $\hat{v}$, $10^2\,\hat{w}$ equal .456, .294; 1.460, 1.648; .833, .240; .728, .152; 856, .287; .907, .107 for Birds 118, 121, 129, 278, 279, and 281, respectively.
[b] These bird-schedule combinations were not run.

variables all uniformly distributed on $[0, 14\tau]$; thus the asserted "similarity" to Millenson's schedule. Going one step further it will be assumed that the difference between uniform and exponential distributions with the same mean $7\tau = 1/\gamma$ is of little importance as far as the rate of responding problem is concerned, and the mean IRT formula (67) will thus be applied to Catania and Reynold's birds. Comparsions of theory and data for the six pigeons appear in Table 2.[4] The least squares estimates $\hat{v}$ and $\hat{w}$ given in this table were obtained separately for each bird. Clearly the data in Table 2 conform rather well to the predicted linear relation (67).

The quantity M_1 was estimated by means of the first formula in (68). The values .458, 1.485, .835, .729, .858, and .908 were obtained for Birds 118, 121, 129, 278, 279, and 281, respectively. The quantity $(M_0 - M_1)\lambda$ was estimated by the second formula in (68), and the bounds for λ given below were obtained from these estimates by supposing that $10\text{ sec} < M_0 - M_1 < 100\text{ sec}$:

$$\text{Bird } 118: \quad .135 \times 10^{-4} < \lambda < .135 \times 10^{-3}$$

$$\text{Bird } 121: \quad .245 \times 10^{-3} < \lambda < .245 \times 10^{-2}$$

$$\text{Bird } 129: \quad .200 \times 10^{-4} < \lambda < .200 \times 10^{-3}$$

$$\text{Bird } 278: \quad .111 \times 10^{-4} < \lambda < .111 \times 10^{-3}$$

$$\text{Bird } 279: \quad .246 \times 10^{-4} < \lambda < .246 \times 10^{-3}$$

$$\text{Bird } 281: \quad .971 \times 10^{-5} < \lambda < .971 \times 10^{-4}.$$

C. Schedules that Differentially Reinforce Low Rates of Responding

Qualitatively speaking it is clear that an organism behaving according to the linear free-responding model can adapt at least fairly well to DRL contingencies. However, for the model to predict the extremes of temporal discrimination often observed on DRL schedules moderate or large values of α will plainly be required. Thus the approximation to $I(t)$ given by the first three terms on the right in (34) is in some measure inappropriate. Since part of the second order term in (34) is contributed by $\mathscr{E}(t)$, Theorem 6 suggests that this residue will be made smaller if DRL-like schedules are considered for which $\sup u - \inf u$ is less than 1, the value of this difference associated with standard DRL.

Norman (1964) has collected data from a pigeon pecking a key to obtain a 3-sec presentation of mixed grains on the simple contingent schedules with reinforcement functions

$$u_p(t) = \begin{cases} 0 & t \leqslant 6\text{ sec} \\ p & t > 6\text{ sec,} \end{cases}$$

[4] Six of the 31 data points in this table are averages over two measurements widely spaced in time.

where $p = 1$, .7, and .4.[5] The bird was maintained at a weight between 75 and 80% of its ad libitum weight. The lengths of the experimental sessions, which usually occurred daily, were adjusted so that the bulk of the bird's daily ration of food was received as reinforcement for key pecking. The rest of this ration was given soon after the experimental session. The histogram in Fig. 1 presents the IRT densities for 2-sec blocks from 3 sessions with $p = .4$. IRT"s after a reinforced response were timed from the end of food presentation. The total number of IRT's represented is 502. Previously the bird had had 15 sessions with $p = 1$, 7 sessions with $p = .7$, and 4 sessions with $p = .4$, in that order. Figure 1 also presents the density obtained from (34) by disregarding the residual term on the right, differentiating what remains, and taking

$$f_1(t) = 4te^{-2t}, \qquad f_0(t) = \frac{1}{15} e^{-(1/15)t},$$

$$\ell(t, s) = \frac{\partial}{\partial t} L(t, s) = \begin{cases} \dfrac{3}{s} & \text{if} \quad \dfrac{5}{6} \leqslant \dfrac{t}{s} \leqslant \dfrac{7}{6} \\ 0 & \text{otherwise,} \end{cases}$$

$$\beta = \left(\frac{1}{\mu_1} - 1\right) \lambda = \frac{1}{4}, \qquad \text{and} \qquad \alpha = \frac{1}{2}.$$
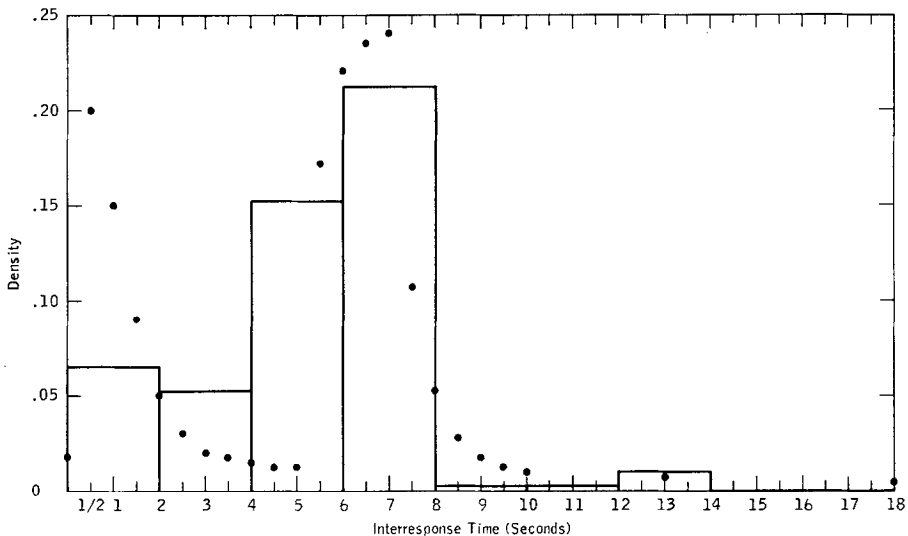

FIG. 1. Observed and predicted IRT densities for a pigeon for 3 consecutive daily sessions on the simple contingent schedule with reinforcement function $u.4$. Heights of blocks with solid borders give halved proportions of observed IRT"s in the corresponding 2-sec intervals, so that the areas within the blocks give the proportions proper. Dots give values of the asymptotic IRT density predicted by the linear free-responding model.

[5] I wish to thank Dr. Gordon Bower for the use of the laboratory in which this experiment was conducted. Similar schedules have been considered by Mallott and Cumming (1964).

These choices were made by trial and error and, particularly in the case of $\ell(t, s)$, on the basis of ease of computation. The density $h(t)$ corresponding to the above choice of $f_1(t)$ and $\ell(t, s)$ is easily shown to be

$$h(t) = \frac{6}{13} \begin{cases} e^{-12((1/7)t-1)} - e^{-12((1/5)t-1)} & \text{if} & 7 \leqslant t \\ 1 - e^{-12((1/5)t-1)} & \text{if} & 5 \leqslant t \leqslant 7 \\ 0 & \text{if} & t \leqslant 5. \end{cases}$$

The value of λ corresponding to the above choice of β is approximately $.8 \times 10^{-5}$.

The results presented in Fig. 1 are regarded as encouraging. Both the empirical and predicted densities are bimodal. Though the quantitative correspondence of observation and theory is not impressively close, a really good fit was not to be expected because of the rough and ready manner in which the theoretical density was obtained.

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