*M. Frank Norman*[1]

# Mathematical Learning Theory

I. **Introduction.** The purpose of mathematical learning theory is to provide simple quantitative descriptions of those processes that are basic to behavior modification in organisms. The objective of providing simple descriptions of basic learning processes determines the flavor of the field: learning models, as we will see, are, by design, simple gadgets for doing simple things.

Most of the experiments to which learning models have been applied consist of a sequence of *trials* on each of which a configuration of stimuli is presented to the subject. The subject then makes one of several *responses*, and this is followed by an *outcome*, perhaps a reward or punishment, determined by the experimenter. If an outcome does not decrease the probability of a certain response on subsequent trials, it is said to *reinforce* that response. The sequence of responses of a subject in such an experiment is conceived to be a realization of a discrete parameter stochastic process. Mathematical learning theorists try to predict the statistics of this process. Most learning models are intrinsically stochastic.

I will now give some examples of learning experiments and models that have been applied to them.

[1] The author's work in this area is supported by the National Science Foundation under grant GP-7335.

A. *Two-choice simple learning experiments.* A *simple learning experiment* is one in which the stimulus situation is the same on every trial. If the stimulus situation varies from trial to trial, we have *discrimination learning.* Consider now a simple learning experiment in which the same two responses, $A_1$ and $A_2$, are available to the subject on each trial. Let $A_{i,n}$ be the event "$A_i$ on trial $n$", $n = 1, 2, \cdots$; let $X_n$ be the indicator random variable for the event $A_{1,n}$ (i.e. $X_n = 1$ or 0 depending on whether or not $A_{1,n}$); and let $p_n$ be the probability of $A_{1,n}$ for an individual subject. Suppose that either response can be reinforced on any trial, regardless of what response is made on that trial. Let $O_{ij}$ be the outcome that follows $A_i$ and reinforces $A_j$, and let $O_{ij,n}$ be the event "outcome $O_{ij}$ occurred on trial $n$". Suppose that, on any trial, $O_{ij,n}$ follows $A_{i,n}$ with probability $\pi_{ij}$ fixed throughout the experiment.

1. *Examples.*

a. *Probability learning.* A human subject is seated before a panel to which are affixed two lamps numbered 1 and 2. The subject's task is to predict on each trial which of the two lamps will flash. After he has made his prediction one of the lights flashes. Under suitable conditions the flashing of a light will reinforce, in some degree, the prediction of that light, regardless of what light was predicted on that trial. Here $A_i$ is the prediction of light $i$ and $O_{ij}$ is the consequent flashing of light $j$. Monetary payoffs are sometimes used. Experiments of this kind are called *probability learning experiments.*

The special case of *noncontingent reinforcement,* in which the probability of reinforcing $A_j$ does not depend on the response made ($\pi_{1j} = \pi_{2j} = \pi_j$), has been studied most intensively experimentally. It is often found that the asymptotic probability of $A_{1,n}$ is approximately equal to the probability $\pi_1$ that $A_1$ will be reinforced (Estes [7]). This probability matching is somewhat surprising since the frequency of correct predictions is obviously maximized by always choosing $A_1$ if $\pi_1 > 1/2$ and always choosing $A_2$ if $\pi_1 < 1/2$.

b. *T-maze learning.* On each trial, an animal, say a rat, is placed in the bottom of a $T$-shaped alley. Eventually he enters the top of the maze and goes to the end of the left ($A_1$) or right ($A_2$) arm. There he may receive food, which reinforces the response just made, or he may simply be detained, which, in some cases, reinforces the other response. This is called a *noncorrection procedure* to distinguish it from *correction procedures* in which, after failing to find food in one arm, the rat is allowed to go to the end on the other arm where he receives food. This difference has profound behavioral consequences. Correction procedures sometimes lead to probability matching under noncontingent reinforcement, while noncorrection invariably leads to overwhelming preference for the more favorable side (see Norman [19]).

c. *Avoidance and punishment.* Suppose that, within a specified time after the beginning of a trial, a dog may ($A_1$) or may not ($A_2$) jump over a barrier, and that shock may or may not follow. We consider experiments in which the avoidance of shock reinforces the response that preceded it, while shock reinforces the other response. When the animal is shocked if and only if he fails to jump ($\pi_{11} = \pi_{21} = 1$), he usually learns to avoid shock altogether. Thus this paradigm is called *avoidance conditioning.* If, on the other hand, he is shocked only when he jumps, i.e. punished for jumping, it is clear that he will learn not to jump. For a comparative evaluation of a number of learning models for an avoidance conditioning experiment, see Bush and Mosteller [4].

2. *Models.* The following models have been proposed to account for data obtained in two-choice simple learning experiments.

i. *Stimulus-sampling theory.* (See Atkinson and Estes [1]). The subject's experimental environment is represented by $N$ "stimulus elements," each of which is "conditioned" to $A_1$ or $A_2$. On each trial the subject "samples" $s$ of these and makes $A_1$ with probability $k/s$ if $k$ elements of the sample are conditioned to $A_1$. If $O_{ij}$ occurs, conditioning is "effective" with probability $c_{ij}$, in which case everything in the sample becomes conditioned to the reinforced response $A_j$. If conditioning is not effective, then nothing in the sample changes its state of conditioning. (Symmetries in the experimental situation may, of course, reduce the number of distinct $c_{ij}$ that need be considered. More generally, theoretical or empirical considerations may suggest the relative or absolute orders of magnitude of these parameters. Similar comments apply to all the models discussed below.) Variants of this model may be distinguished on the basis of the composition of the sample. For definiteness we will restrict our attention to the *fixed sample size model* in which $s$ is constant and any sample of size $s$ has the same probability. However, the methods to be discussed are equally applicable to other stimulus-sampling models.

In this model we identify $p_n$, the probability of $A_{1,n}$, with the proportion of the $N$ stimulus elements that are conditioned to $A_1$ at beginning of trial $n$. Given $p_n$, the probability that the sample contains $j$ elements conditioned to $A_2$, $A_1$ occurs and is reinforced, and conditioning is effective, so that $\Delta p_n = p_{n+1} - p_n = j/N$, is

$$\phi(p_n) = \frac{\binom{Np_n}{s-j}\binom{N(1-p_n)}{j}}{\binom{N}{s}}\left(1 - \frac{j}{s}\right)\pi_{11}c_{11}.$$

The entire conditional probability distribution of $\Delta p_n$ given $p_n$ is determined similarly (see Norman [18, Appendix]).

ii. *The four-operator linear model.* (See Sternberg [24].) Here it is assumed that if $A_{i,n}$ and $O_{i1,n}$ then the increment $\Delta p_n$ is a fixed proportion $\theta_{i1}$ of the maximum possible increment $1 - p_n$, while if $A_{i,n}$ and $O_{i2,n}$ then the decrement $\Delta p_n$ is a proportion $\theta_{i2}$ of the maximum possible decrement $-p_n$, $0 \leqq \theta_{ij} \leqq 1$. Thus $p_n$ satisfies the stochastic difference equation

$$\begin{aligned}\Delta p_n &= \theta_{11}(1-p_n) && \text{if } A_{1,n}O_{11,n},\\ &= \theta_{21}(1-p_n) && \text{if } A_{2,n}O_{21,n},\\ &= -\theta_{12}p_n && \text{if } A_{1,n}O_{12,n},\\ &= -\theta_{22}p_n && \text{if } A_{2,n}O_{22,n},\end{aligned}$$

where, denoting $P(A_{i,n}O_{ij,n} | p_n = p)$ by $\phi_{ij}(p)$,

$$\phi_{11}(p) = p\pi_{11},$$
$$\phi_{12}(p) = p\pi_{12},$$
$$\phi_{21}(p) = (1-p)\pi_{21},$$

and

$$\phi_{22}(p) = (1-p)\pi_{22}.$$

iii. *The beta model.* (See Sternberg [24].) Let $v_n$, the "strength" of the response $A_1$ on trial $n$, be related to $p_n$ by the equation $p_n = v_n/(1 + v_n)$, or, equivalently $v_n = p_n/(1 - p_n)$. We suppose that to every outcome $O_{ij}$ there is a constant $\beta_{ij}$ such that $v_{n+1} = \beta_{ij}v_n$ if $A_{i,n}O_{ij,n}$. At the level of $p_n$ we have the transition equation

$$p_{n+1} = \frac{\beta_{ij}p_n}{(1-p_n) + \beta_{ij}p_n} \quad \text{if } A_{i,n}O_{ij,n}.$$

The quantities $\phi_{ij}(p) = P(A_{i,n}O_{ij,n} | p_n = p)$ are the same as in the four-operator linear model. Since $O_{ij}$ reinforces $A_j$, we must have $\beta_{11} \geqq 1$, $\beta_{21} \geqq 1$, $\beta_{12} \leqq 1$, and $\beta_{22} \leqq 1$.



B. *A discrimination learning experiment.* Consider a $T$-maze with two lights $S_1$ and $S_2$ beside each other at the choice point. On each trial one or the other is lit, and food is available on the left ($A_1$) or right ($A_2$) depending on which. $S_{1,n}$ and $S_{2,n}$ each occur on a random 50% of the trials. Clearly, most rats will eventually learn to make the appropriate response on every trial of such an experiment.

The model that will now be presented is Bush's linearization of a theory proposed by L. B. Wyckoff, Jr. (see Bush [3, Section 1.3]). We will refer to it as the *Wyckoff model.* Let $u_n$ be the rat's probability of noting which light is lit on trial $n$ (event $T_n$). Let $x_{1,n}$ and $x_{2,n}$ be, respectively, the probabilities of getting food (event $F_n$) on $S_1$ and $S_2$ trials if the rat attends to the lights. The probability of getting food is clearly 1/2 if he does not attend. If the rat attends to the lights on an $S_i$ trial, it is assumed that either a rewarded $A_i$ response or a nonrewarded alternative response increases the probability $x_i$ of making the $A_i$ response on subsequent trials of this sort. These are the only circumstances in which $x_i$ changes. The probability $u$ of attending to the lights is assumed to increase if attention is followed by reward or nonattention is followed by nonreward, and to decrease in complementary cases. Finally, it is assumed that all of the increments and decrements are effected by linear functions. These assumptions are summarized in Table 1,

TABLE 1

| Event | Probability | $\Delta u$ | $\Delta x_1$ | $\Delta x_2$ |
|---|---|---|---|---|
| $S_1TF$ | $(1/2)ux_1$ | + | + | 0 |
| $S_1T\tilde{F}$ | $(1/2)u(1-x_1)$ | − | + | 0 |
| $S_2TF$ | $(1/2)ux_2$ | + | 0 | + |
| $S_2T\tilde{F}$ | $(1/2)u(1-x_2)$ | − | 0 | + |
| $\tilde{T}F$ | $(1-u)/2$ | − | 0 | 0 |
| $\tilde{T}\tilde{F}$ | $(1-u)/2$ | + | 0 | 0 |

in which + (−, 0) indicates a transformation of the form $\Delta y = \theta(1-y)$ ($\Delta y = -\theta y$, $\Delta y = 0$) where $\theta > 0$. The $\theta$'s associated with different events may be different.

C. *An experiment with a continuum of responses.* The subject is seated before a large disc. On each trial a spot of light appears somewhere on the rim $R$. The subject's task is to predict where.

Let $Y_n$ be the subject's $n$th prediction, and let $Z_n$ be the position of the reinforcement spot. It is assumed that the probability distribution $\pi(Y_n, \cdot)$ of $Z_n$ depends at most on $Y_n$.

Let $\mu_n$ be the probability distribution of $Y_n$ for a single subject. In the model proposed by Suppes [25], $\mu_n$ is a random measure satisfying the stochastic difference equation

$$\mu_{n+1} = (1 - \theta)\mu_n + \theta\tau(Z_n, \cdot),$$

where $\theta$ is a constant, and, for every $z$, $\tau(z, \cdot)$ is a probability distribution on $R$ with mode at $z$. Thus the occurrence of a reinforcement at $z$ moves the subject in the direction of a response distribution centered at $z$ as, intuitively, it should.

D. *A general theoretical framework.* All of these examples have the following structure. The behavior of the subject on trial $n$ is determined by his *state* $S_n$ at the beginning of the trial. $S_n$ is a random variable taking on values in a *state space* $S$. On trial $n$ an *event* $E_n$ occurs that results in a change of state. The random variable $E_n$ takes on values in an *event space* $E$. That is, to each event $e$ there corresponds a function $f_e$ from $S$ into $S$ such that

$$S_{n+1} = f_{E_n}(S_n), \quad n = 1, 2, \cdots.$$

The function $f_e$ will be called the *operator for the event* $e$ or simply an event operator. The probability distribution $\phi\,.\,(S_n)$ of $E_n$ depends only on the state $S_n$ at the beginning of the trial. Henceforth by a *learning model* I will mean a structure $(S, E, f, \phi)$.

In the stimulus-sampling model described above, $S_n = p_n$, $S = \{k/N : k = 0, 1, \cdots, N\}$, and each specification of number of elements in the trial sample conditioned to $A_2$, response, outcome, and effectiveness or ineffectiveness of conditioning determines an event. The operators $f_e$ are translations when restricted to $\{s : \phi_e(s) > 0\}$. Outside of this set the value of $f_e$ is inconsequential. In the four-operator linear model, $S_n = p_n$, $S = [0, 1]$, $E$ is the set of possible response-outcome pairs, and $f_{ij}(p) = (1 - \theta_{ij})p + \theta_{ij}\delta_{1j}$. For the beta model we may take $S_n = p_n$, $v_n$, or $\ln v_n$; $S$, $f$ and $\phi$ vary accordingly. As in the linear model, $E$ is the set of response-outcome pairs. In the Wyckoff model, it is natural to take $S_n = (u_n, x_{1,n}, x_{2,n})$ and $S$ the unit cube. The events, their conditional probabilities, and their corresponding event operators are given, respectively, by the first, second, and last three columns of Table 1. Finally, in Suppes' model, we take $S_n = \mu_n$ and $E_n = Z_n$, so that $E$ is the rim $R$ of the disc and $S$ is the set of Borel probability measures on $R$.



For any learning model within the framework presented above, it is clear that both the stochastic processes $\{S_n\}$ and $\{W_n\}$, where $W_n = (S_n, E_n)$, are Markov with stationary transition probabilities. This observation is basic to much of what follows.

## II. Compact Markov processes and distance diminishing models.

A. *Models with finite state and event spaces.* Learning models like the stimulus-sampling models for which both $S$ and $E$ have only a finite number of elements will be called *finite state models*. In this case the stochastic processes $\{S_n\}$ and $\{W_n\}$ $(W_n = (S_n, E_n))$ are finite Markov chains. The general theory of such chains is exceedingly well developed (see Chung [5]) and provides a powerful tool for analyzing these models. Whereas the fact that $\{S_n\}$ is a Markov chain has been used extensively by learning theorists, they have seldom applied Markov chain theory to the process $\{W_n\}$.

As an example of what is available, suppose that the chain $\{W_n\}$ has only one recurrent class and that this class is aperiodic. (For this it suffices that the same be true for $\{S_n\}$.) Let $\psi$ be an arbitrary real valued function on $S \times E$. Then, whatever the initial value $s$ of $S_1$, the limits

$$\lim_{n\to\infty} E_s[\psi(W_n)] = \mu$$

and

$$\lim_{n\to\infty} nE_s\left[\left(\frac{1}{n}\sum_{j=1}^{n}\psi(W_j) - \mu\right)^2\right] = \sigma^2$$

exist and do not depend on $s$. The average $(1/n)\sum_{j=1}^{n}\psi(W_j)$ converges with probability 1 to $\mu$ and, if $\sigma^2 > 0$, is asymptotically normally distributed with mean $\mu$ and variance $\sigma^2/n$.

In the stimulus-sampling model, the event that occurs on trial $n$ includes a description of the subject's response on that trial, hence $X_n = \psi(W_n)$ for some $\psi$, where $X_n$ is the indicator random variable of the event $A_{1,n}$. Thus the above theorems give us precise information about the asymptotic behavior of the proportion of $A_1$ responses over the first $n$ trials for a single subject. Similar information about the frequency of compound events such as $A_{1,n}O_{11,n}A_{1,n+1} \cup A_{2,n}O_{21,n}A_{1,n+1}$ (reinforcement of $A_1$ followed by $A_1$ on the next trial) can be obtained by considering the corresponding compound finite Markov chain, in this case $\{(W_n, W_{n+1})\}$.

B. *Distance diminishing models*. The four-operator linear model, the Wyckoff model, and, more generally, the class of distance diminishing models to be defined shortly, can be treated by a Markov process theory completely analogous to the theory of finite Markov chains. Later in this section, I will introduce this theory; In Section C, I will outline it; and in Section D, I will indicate how it can be applied to learning models.

Suppose that $d$ is a metric on the state space $S$. For mappings $\psi$ and $g$ of $S$ into the complex numbers and into $S$, respectively, their maximum "difference quotients" $m(\psi)$ and $\mu(g)$ are defined by

$$m(\psi) = \sup_{s \neq s'} \frac{|\psi(s) - \psi(s')|}{d(s,s')}$$

and

$$\mu(g) = \sup_{s \neq s'} \frac{d(g(s),g(s'))}{d(s,s')}.$$

If $m(\psi) < \infty$, then $\psi$ is said to satisfy a Lipschitz condition. The mapping $g$ is *distance diminishing* if $\mu(g) \leqq 1$ and *strictly distance diminishing* if $\mu(g) < 1$. A learning model is distance diminishing (with respect to $d$) if $(S,d)$ is compact; the event space $E$ is a finite set; for each event $e$, $\phi_e(\cdot)$ satisfies a Lipschitz condition and $f_e(\cdot)$ is distance diminishing; and, finally, for any state $s$ there is a positive integer $k$ and there are $k$ events $e_1, \cdots, e_k$ such that this sequence of events succeeds $s$ with positive probability, and the composite mapping of $S$ corresponding to it is strictly distance diminishing:

$$\phi_{e_1 \cdots e_k}(s) > 0 \quad \text{and} \quad \mu(f_{e_1 \cdots e_k}) < 1,$$

where

$$\phi_{e_1 \cdots e_k}(s) = P_s(E_j = e_j,\ j = 1, \cdots, k)$$

and

$$f_{e_1 \cdots e_k} = f_{e_k} \circ f_{e_{k-1}} \circ \cdots \circ f_{e_1}.$$

The four-operator linear model is distance diminishing if and only if for every $i$ there is a $j_i$ such that $\theta_{ij_i} > 0$ and $\pi_{ij_i} > 0$ (Norman [18, Lemma 3.2]). The Wyckoff model can be shown to be distance diminishing under the restrictions on its parameters given when it was introduced. In both cases the metric considered is Euclidean.

Let $K(\cdot,\cdot)$ be the transition function for a Markov process $\{S_n\}_{n=1}^{\infty}$ in a compact metric space $(S,d)$, i.e.



$$K(s,A) = P(S_{n+1} \in A \mid S_n = s).$$

It is assumed that

(i) $K(s,\cdot)$ is a Borel probability measure for each $s \in S$, and

(ii) $K(\cdot,A)$ is Borel measurable for each Borel set $A$.

Let $U$ be the linear operator on bounded measurable complex valued functions on $S$ defined by

$$U\psi(s) = \int K(s,dt)\psi(t) = E[\psi(S_{n+1}) \mid S_n = s].$$

Let $C(S)$ be the Banach space of continuous complex valued functions on $S$ under the norm

$$|\psi| = \max_{s \in S} |\psi(s)|,$$

and let $CL$ be the subspace of functions that satisfy a Lipschitz condition. $CL$ is a Banach space with respect to the norm

$$\|\psi\| = m(\psi) + |\psi|.$$

I have shown (Norman [18, Proof of Theorem 5.1]) that, if the Markov process $\{S_n\}$ is associated with a distance diminishing learning model, then the following conditions are satisfied:

(iii) $U$ maps $CL$ into $CL$ and is bounded with respect to the norm $\|\cdot\|$, i.e.

$$\|U\| = \sup_{\psi \in CL;\, \psi \neq 0} \frac{\|U\psi\|}{\|\psi\|} < \infty,$$

and

(iv) there is a positive integer $k$ and there are two real numbers $r < 1$ and $R$ such that

$$m(U^k\psi) \leqq rm(\psi) + R|\psi| \quad \text{for all } \psi \in CL.$$

Any Markov process in a compact metric space that satisfies (i)—(iv) will be called a *compact Markov process*.

Let $\delta$ be any metric on the event space $E$ of a distance diminishing model $((S,d),E,f,\phi)$ (e.g. $\delta(e,e') = 0$ or $1$ depending on whether or not $e = e'$). Since $E$ is finite, $(E,\delta)$ is compact. It follows that $S \times E$ is compact with respect to the metric $D((s,e),(s',e')) = d(s,s') + \delta(e,e')$. Using the fact that the process $\{S_n\}$ for a distance diminishing model is compact, it can be shown that the process $\{W_n\}$ $(W_n = (S_n, E_n))$ is also a compact Markov process with respect to the metric $D$. Thus compact Markov processes stand in the same relation to distance diminishing models that finite Markov chains do to finite state models.

C. *Compact Markov processes.* If $\{S_n\}$ is any finite Markov chain and $d$ is any metric on its state space $S$, then it is clear that $C(S)$ and $CL$ consist of all complex valued functions on $S$. For any such function $\psi$, $m(\psi) \leqq \nu|\psi|$ where $\nu = 2/\min_{s\neq s'} d(s,s')$, so

$$m(U\psi) \leqq \nu|U\psi| \leqq \nu|\psi|,$$

and (iv) is satisfied. Thus any finite Markov chain is a compact Markov process. On the other hand, it is easy to show that, if $\{S_n\}$ is a compact Markov process, then $U$ maps $C(S)$ into $C(S)$, and, for any $\psi \in C(S)$, the sequence $\{U^n\psi\}$ is equicontinuous. An extensive theory of Markov processes having this property is emerging (Jamison [11], [12]; Rosenblatt [22], [23]), and this theory has been helpful in the work discussed below. Thus compact Markov processes stand between finite Markov chains and processes for which the sequence $\{U^n\psi\}$ is equicontinuous. Their theory is actually much closer to that of the former than to that of the latter.

The following basic theorem is a specialization of a uniform ergodic theorem of Ionescu Tulcea and Marinescu ([9, Section 9]) along lines suggested by these authors.

THEOREM 1. *Let the operator $U$ correspond to a compact Markov process. Then*

(a) *there are at most a finite number of eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_p$ of $U$ for which $|\lambda_i| = 1$;*

(b) *for all positive integers $n$*

$$U^n = \sum_{i=1}^{p} \lambda_i^n U_i + V^n,$$

*where $V$ and $U_i$ are linear operators on $CL$, bounded with respect to $\|\cdot\|$;*

(c) $U_i^2 = U_i$, $U_i U_j = 0$ *for* $i \neq j$, $U_i V = VU_i = 0$;

(d) *if* $D(\lambda_i) = \{\psi \in CL: U\psi = \lambda_i\psi\}$ *then* $D(\lambda_i) = U_i(CL)$ *is finite dimensional,* $i = 1, \cdots, p$; *and*

(e) *for some* $M < \infty$ *and* $h > 0$,

$$\|V^n\| \leqq M/(1+h)^n$$

*for all positive integers $n$.*

From the fact that the norms $\|\cdot\|$ and $|\cdot|$ are equivalent on the



finite dimensional linear space $U_i(CL)$, it follows that the condition (iv) is necessary for (a)—(e). It is also immediate from (a)—(e) that, for $n$ sufficiently large, the strongly completely continuous operator $\sum_{i=1}^{p} \lambda_i^n U_i$ on $CL$ satisfies

$$\left\| U^n - \sum_{i=1}^{p} \lambda_i^n U_i \right\| < 1.$$

Thus the operator $U$ on $CL$ is quasi-strongly completely continuous (see Yosida and Kakutani [27]).

If the higher transition kernels $K^{(n)}$ are defined in the usual way:

$$K^{(0)}(s,A) = 1 \quad \text{if } s \in A,$$
$$\qquad\quad = 0 \quad \text{if } s \in' A,$$

and

$$K^{(n+1)}(s,A) = \int K(s,dt)K^{(n)}(t,A);$$

then

$$K^{(n)}(s,A) = P_s(S_{n+1} \in A)$$

and

$$U^n\psi(s) = \int K^{(n)}(s,dt)\psi(t) = E_s[\psi(S_{n+1})].$$

In view of the latter equations, Theorem 1 gives us a firm grip on the asymptotic behavior of the quantities $K^{(n)}(s,A)$ and $E_s[\psi(S_{n+1})]$, $\psi \in CL$.

Clearly 1 is an eigenvalue of $U$ and all of the nonzero complex constants are corresponding eigenfunctions. Without loss of generality we suppose that $\lambda_1 = 1$ and $\lambda_i \neq 1$ for $i \neq 1$. Theorem 1 implies that $|\overline{U}_n\psi - U_1\psi| \to 0$ as $n \to \infty$ for any $\psi \in C(S)$, where $\overline{U}_n = (1/n)\sum_{j=0}^{n-1} U^j$ and $U_1$ has been extended (uniquely) to a bounded linear operator on $C(S)$. It follows, in turn, that $U_1$ has the representation

$$U_1\psi(s) = \int K^\infty(s,dt)\psi(t)$$

where $K^\infty$ satisfies (i) and (ii), and that $\overline{K}_n(s,\cdot)$ converges weakly to $K^\infty(s,\cdot)$ for every $s \in S$, where $\overline{K}_n = (1/n)\sum_{j=0}^{n-1} K^{(j)}$. In fact, the convergence is, in a certain sense, uniform over $s$ (see Norman

[18, Definition 2.1, and the proof of Lemma 5.3]). If 1 is the only eigenvalue of modulus 1 of $U$, the Cesaro averaging in the above statements is unnecessary.

For any $s \in S$, $K^\infty(s, \cdot)$ is a *stationary probability distribution* in the sense that $S_{n+1}$ has this distribution if $S_n$ has. Let $M(S)$ be the space of complex valued Borel measures on $S$, and let $T$ be the operator on $M(S)$ defined by

$$T\nu(A) = \int \nu(ds) K(s, A).$$

Then $T$ is the adjoint of the operator $U$ on continuous functions, and, more intuitively, $T$ takes the distribution of $S_n$ into that of $S_{n+1}$. Thus the stationary probability distributions are those that are fixed under $T$.

A Borel set $B$ is *stochastically closed* if it is nonempty and if $K(s, B) = 1$ for all $s \in B$. $B$ is an *ergodic kernel* if it is stochastically and topologically closed and if it has no proper subset with these properties. It is easy to show that distinct ergodic kernels are disjoint. Whatever the initial state of the process, it is attracted to its ergodic kernels as the following theorem shows.

THEOREM 2. *A compact Markov process $\{S_n\}$ has $l$ ergodic kernels, where $l$ is the dimension of $D(1)$. Denote these $E_1, \cdots, E_l$, and let*

$$\gamma_i(s) = P_s\left(\lim_{n\to\infty} d(S_n, E_i) = 0\right).$$

*Then $\sum_{i=1}^{l} \gamma_i(s) \equiv 1$ and $\{\gamma_1, \cdots, \gamma_l\}$ is a basis for $D(1)$. There is a unique stationary probability measure $\mu_i$ with support $E_i$, and $\{\mu_1, \cdots, \mu_l\}$ is a basis for $\{\mu \in M(S) : T\mu = \mu\}$. Finally*

$$K^\infty(s, B) = \sum_{i=1}^{l} \gamma_i(s)\mu_i(B).$$

It follows that a process has a unique stationary probability distribution if and only if there is only one ergodic kernel. In this case we denote this distribution $K^\infty$. Thus $U_1\psi(s) \equiv \int \psi(t) K^\infty(dt)$.

Any subprocess of a compact Markov process obtained by restricting the process to a stochastically and topologically closed subset of the state space is a compact Markov process. Just as in the theory of finite Markov chains, the subprocesses on the ergodic kernels may have a cyclic character. The periods of these cycles determine the eigenvalues of modulus 1 of the original process.



THEOREM 3. *A complex number of modulus 1 is an eigenvalue of a compact Markov process if and only if it is an eigenvalue of the subprocess on some ergodic kernel. For any ergodic kernel $E$ there is a positive integer $k$ (called the period of $E$) such that the eigenvalues of modulus 1 of the corresponding subprocess are* $\exp[2\pi ij/k]$, *$j = 1, \cdots, k$. There are $k$ topologically closed pairwise disjoint sets $E^1, \cdots, E^k$ with union $E$, such that $K(s, E^{j+1}) = 1$ for all $s \in E^j$, $j = 1, \cdots, k$ $(E^{k+1} = E^1)$.*

If $U$ has no eigenvalues of modulus 1 other than 1, we say that the process is *aperiodic*.

Let us now consider the asymptotic behavior of the sums $(1/n)\sum_{j=1}^{n} \psi(S_j)$ for a compact Markov process $\{S_n\}$ and a real valued continuous function $\psi$. The following strong law of large numbers is obtained by combining a theorem of Jamison [12, Theorem 3.2] with Theorem 2 above: Let $M_i$ be the indicator random variable for the event "$\lim_{n\to\infty} d(S_n, E_i) = 0$". Then for any initial state $s$ the probability is 1 that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \psi(S_j) = \sum_{i=1}^{l} M_i \int \psi(t)\mu_i(dt)$$

for all continuous real valued functions $\psi$. Thus, if there is a unique stationary probability distribution $K^\infty$,

$$\frac{1}{n}\sum_{j=1}^{n} \psi(S_j) \to \int \psi(t) K^\infty(dt) \quad \text{as } n \to \infty.$$

Suppose now that $\psi$ is real valued and satisfies a Lipschitz condition, and that the process is aperiodic and has only one ergodic kernel. Assume, without loss of generality, that $\int \psi(t)K^\infty(dt) = 0$. Then $R(m - n) = E[\psi(S_m)\psi(S_n)]$ converges geometrically to 0 as $|m - n| \to \infty$, so the series $\sigma^2 = \sum_{j=-\infty}^{\infty} R(j)$ converges absolutely. (In these expressions and those that follow, expectations without subscripts are taken with respect to the stationary process that arises when $S_1$ has distribution $K^\infty$.) Moreover, for any $s \in S$

$$E_s\left[\left(\frac{1}{\sqrt{n}}\sum_{j=1}^{n} \psi(S_j)\right)^2\right] = \sigma^2 + O\left(\frac{1}{n}\right).$$

If $\sigma^2 > 0$, then $(1/n)\sum_{j=1}^{n} \psi(S_j)$ is asymptotically normally distributed with mean 0 and variance $\sigma^2/n$, for any initial state. As in the case of the analogous central limit theorem for Markov processes satisfying Doeblin's condition (Doob [6, Theorem 7.5]), this can be proved by reduction to the case of sums of independent random variables. Since $\int \psi(t) K^{\infty}(dt) = 0$, the series

$$\bar{\psi} = \sum_{j=0}^{\infty} U^j \psi$$

converges absolutely in $CL$. From the representation

$$\sigma^2 = E[(\bar{\psi}(S_{n+1}) - U\bar{\psi}(S_n))^2],$$

which follows from the series representation given above, and the fact that $\bar{\psi} - U\bar{\psi} = \psi$, it is clear that if $\sigma^2 = 0$, then $\psi(S_n) = \bar{\psi}(S_n) - \bar{\psi}(S_{n+1})$ with probability 1 for the stationary process. Using this condition, Norma Graham and I have shown that if $\psi$ has only two values, say $a$ and $b$ $(a \neq b)$ and if both have positive probability asymptotically $(K^{\infty}(\psi^{-1}\{a\}) > 0$ and $K^{\infty}(\psi^{-1}\{b\}) > 0)$, then $\sigma^2 > 0$.

Since $R(j)$ converges geometrically to zero as $|j| \to \infty$, the process $\{\psi(S_n)\}$ ($\{S_n\}$ stationary) has a spectral density function

$$f(t) = \frac{1}{2\pi} \sum_{j=-\infty}^{\infty} e^{-itj} R(j).$$

The quantity $\sigma^2$ is $2\pi f(0)$. Let

$$p_{a,b,c} = E[\psi(S_n)\psi(S_{n+a})\psi(S_{n+b})\psi(S_{n+c})] - R(a)R(c-b) - R(b)R(c-a) - R(c)R(b-a).$$

Under our assumptions ($\psi$ real valued, $\psi \in CL$, $\{S_n\}$ aperiodic with only one ergodic kernel) it can be shown that

$$\sum_{a,b,c=-\infty}^{\infty} |p_{a,b,c}| < \infty.$$

Thus the spectral estimation techniques for processes whose covariances decrease exponentially described by Parzen [21] are applicable. Briefly, there are estimators of *exponential type* for which various measures of the error of estimation are of the order of magnitude of $(\ln n)/n$, where $n$ is the number of successive observations of which the estimator is a function.

One final fact: just as the compound process $\{(S_n, S_{n+1}, \cdots, S_{n+k})\}_{n=1}^{\infty}$ is a finite Markov chain if $\{S_n\}$ is, processes compounded out of any compact Markov process are also compact Markov processes. Furthermore, if $\{S_n\}$ is aperiodic or has only one ergodic kernel the same will be true of $\{(S_n, S_{n+1}, \cdots, S_{n+k})\}_{n=1}^{\infty}$.



D. *Application of compact Markov process theory to distance diminishing models.* I stated earlier that the sequence $\{W_n\}$ of state-events pairs from a distance diminishing learning model is a compact Markov process. I now add that if the sequence $\{S_n\}$ of states is aperiodic or has only one ergodic kernel then the same is true of $\{W_n\}$. These facts are important since they permit us to establish regularities of $\{W_n\}$ by considering the simpler process $\{S_n\}$.

Returning to our examples: I have shown that if a four-operator linear model has no absorbing states (stochastically closed unit sets), then it is distance diminishing and $\{p_n\}$ has only one ergodic kernel $E_1$. If $\theta_{ij} = 1$ and $\pi_{ij} = 1$ for $i \neq j$ then $E_1 = \{0, 1\}$ and $E_1$ has period 2. Otherwise the process is aperiodic. If, on the other hand, either 0 or 1 is absorbing for a distance diminishing four-operator linear model then the process is aperiodic and these absorbing states are the ergodic kernels (Norman [18, Theorems 3.1, 3.2]). For the Wyckoff model it can be shown that, for any state $s$, the distance from the support $T_n(s)$ of the distribution $K^{(n)}(s, \cdot)$ of $S_{n+1}$ to the absorbing state $(1,1,1)$ converges to 0 as $n \to \infty$. From this it follows that $E_1 = \{(1,1,1)\}$ is the only ergodic kernel. Clearly the process is aperiodic.

For a critical discussion of earlier mathematical work on distance diminishing models see Norman [18, Section 4]. That paper also contains an alternative treatment of the event process $\{E_n\}$, via the theory of random systems with complete connections (see Iosifescu [10]).

## III. Some special results for the four-operator linear model.

A. *No absorbing barriers.* Consider first the special case where all $\theta_{ij}$'s are equal to some constant $\theta > 0$ and reinforcement is noncontingent $(\pi_{1j} = \pi_{2j} = \pi_j)$ with $0 < \pi_1 < 1$. The difference equation for a subject's probability $p_n$ of making response $A_1$ on trial $n$ reduces in this case to

$$\Delta p_n = \theta(k_n - p_n),$$

where $k_n$ is 1 if $A_1$ is reinforced on trial $n$ and 0 if $A_2$ is reinforced on trial $n$. The random variables $\{k_n\}$ are mutually independent with $P(k_n = 1) = \pi_1$ and $P(k_n = 0) = 1 - \pi_1$. The solution to this equation is

$$p_{n+1} = (1-\theta)^n p_1 + \theta \sum_{m=1}^{n} (1-\theta)^{n-m} k_m.$$

We are going to consider what happens to $p_n$ as $n$ becomes large and $\theta$ becomes small. To this end let $\{\theta_n\}$ be a sequence in $(0,1)$ converging to 0. Assuming $\operatorname{var}(p_1) = 0$,

$$p_{n+1} - \xi_{n+1} = \sum_{m=1}^{n} X_{n,m}$$

where $\xi_{n+1} = E_{p_1}[p_{n+1}]$ and

$$X_{n,m} = \theta_n(1-\theta_n)^{n-m}(k_m - \pi_1).$$

Clearly, $X_{n,1}, \cdots, X_{n,n}$ are independent and $E[X_{n,m}] = 0$. Furthermore, a routine computation shows that the triangular array $\{X_{n,m}\}$ of random variables satisfies the Lindeberg condition

$$\lim_{n\to\infty} \frac{1}{\sigma_n^2} \sum_{m=1}^{n} \int_{|x| \geqq \lambda\sigma_n} x^2 dG_{n,m}(x) = 0$$

for all $\lambda > 0$, where $\sigma_n^2 = \operatorname{var}_{p_1}(p_{n+1})$ and $G_{n,m}$ is the probability distribution of $X_{n,m}$. It follows that $(p_n - \xi_n)/\sigma_n$ is asymptotically normally distributed with mean 0 and variance 1. Thus

$$\lim_{n\to\infty;\theta\to 0} F_{n,\theta,p_1}(x) = \Phi(x)$$

for all $x$, where

$$F_{n,\theta,p_1}(x) = P_{p_1}((p_n - E_{p_1}[p_n])/\operatorname{var}_{p_1}^{1/2}[p_n] < x),$$

$\Phi$ is the standard normal cumulative distribution function, and the limit is taken as $n \to \infty$ and $\theta \to 0$ simultaneously. *After long experience in a situation in which learning occurs slowly, $p_n$ is approximately normally distributed according to this model.*

Now

$$J_n(x) = \lim_{\theta\to 0} F_{n,\theta,p_1}(x) \quad \text{and} \quad H_\theta(x) = \lim_{n\to\infty} F_{n,\theta,p_1}(x)$$

both exist. As $\theta \to 0$, $(p_n - \xi_n)/\sigma_n$ converges to the normalized binomial random variable

$$\sum_{m=1}^{n-1} \frac{k_m - \pi_1}{[(n-1)\pi_1(1-\pi_1)]^{1/2}},$$

so $J_n$ is a normalized binomial distribution. The existence of the limit of $F_{n,\theta,p_1}$ as $n \to \infty$ and its independence of $p_1$ follow from the fact that the process $\{p_n\}$ has no absorbing states and is aperiodic.



The asymptotic normality of $F_{n,\theta,p_1}$ as $n \to \infty$ and $\theta \to 0$ simultaneously then implies that

$$\lim_{n\to\infty} J_n(x) = \Phi(x),$$

and

$$\lim_{\theta\to 0} H_\theta(x) = \Phi(x).$$

The former result is, of course, nothing but DeMoivre's central limit theorem. Since $\sigma_n^2 \to \pi_1(1-\pi_1)\theta/(2-\theta)$ and $\xi_n \to \pi_1$ as $n \to \infty$, the latter result can be rewritten

$$\lim_{\theta\to 0}\lim_{n\to\infty} P_{p_1}\left(\frac{p_n - \pi_1}{[\pi_1(1-\pi_1)\theta/2]^{1/2}} < x\right) = \Phi(x).$$

I will now show how the latter result can be generalized to four-operator linear models for which the process $\{p_n\}$ is aperiodic and has no absorbing states. The method that I will describe leads to an asymptotic expansion of the characteristic function of the distribution

$$\lim_{n\to\infty} P_{p_1}\left(\frac{p_n - \rho}{\theta^{1/2}} < x\right)$$

in powers of $\theta^{1/2}$, where all $\theta_{ij}$ are assumed to be proportional to $\theta$, and $\rho$ does not depend on $\theta$.

Consider then a family $\{p_n^{(\theta)}\}_{n=1}^{\infty}$ $(0 < \theta < 1)$ of Markov processes in the unit interval satisfying stochastic difference equations of the form

$$\Delta p_n^{(\theta)} = \theta u_j(p_n^{(\theta)}) \quad \text{with prob. } \phi_j(p_n^{(\theta)}),$$

$1 \leqq j \leqq N$, where $-p \leqq u_j(p) \leqq 1-p$ and $\sum_j \phi_j(p) = 1$ for $0 \leqq p \leqq 1$, and $u_j$ and $\phi_j$ are infinitely differentiable on $[0,1]$. Let

$$h_m(p) = \sum_j u_j^m(p)\phi_j(p)$$

$$= E[(\Delta p_n^{(\theta)}/\theta)^m \mid p_n^{(\theta)} = p],$$

and suppose that $h_1(0) > 0$, $h_1(1) < 0$, and that there is a unique $\rho \in (0,1)$ such that $h_1(\rho) = 0$. Necessarily $h_1'(\rho) \leqq 0$ and $h_2(\rho) \geqq 0$. We assume that $h_1'(\rho) < 0$ and $h_2(\rho) > 0$. Finally we suppose that, for every $0 < \theta < 1$, $F_\theta$ is a stationary cumulative distribution function for the Markov process $\{p_n^{(\theta)}\}$.

For the four-operator linear model, for instance, let $b_{ij}$ $(1 \leqq i, j \leqq 2)$ be a constant in $[0,1]$ and let $\theta_{ij} = \theta b_{ij}$. In terms of this convenient double index notation we then have

$$u_{i1}(p) = b_{i1}(1-p), \qquad u_{i2}(p) = -b_{i2}p,$$

and

$$\phi_{1j}(p) = p\pi_{1j}, \qquad \phi_{2j}(p) = (1-p)\pi_{2j}.$$

The assumption that the process $\{p_n^{(\theta)}\}$ is aperiodic with no absorbing states is equivalent to the restrictions $b_{ij} > 0$ and $\pi_{ij} > 0$ if $i \neq j$. For such a model the distribution of $p_n^{(\theta)}$ converges as $n \to \infty$ to a stationary distribution $F_\theta$ that is, in fact, independent of $p_1$. That $h_1(p)$ has a unique zero $\rho$ in $(0,1)$ and that $h_1'(\rho) < 0$ follow from the fact that $h_1$ is quadratic with $h_1(0) > 0$ and $h_1(1) < 0$.

We return now to the general case and henceforth omit $\theta$ superscripts. Let $W(\cdot, p)$ be the conditional characteristic function of the normalized increment $\Delta p_n/\theta$ given $p_n$:

$$W(s,p) = E[\exp(is\Delta p_n/\theta) \mid p_n = p].$$

Notice that $W$ is independent of $\theta$. Let

$$W(s,p) = \sum_{j,k=0}^{\infty} a_{j,k}(is)^j(p-\rho)^k \tag{1}$$

be the formal Taylor expansion of $W$ in powers of $i \cdot s$ and $p - \rho$. Then

$$\sum_{k=0}^{\infty} a_{j,k}(p-\rho)^k = \frac{1}{j!}E\left[\left(\frac{\Delta p_n}{\theta}\right)^j \middle| p_n = p\right].$$

These equations give

$$a_{0,k} = \delta_{0,k}, \quad a_{1,0} = h_1(\rho) = 0, \quad a_{1,1} = h_1'(\rho) < 0,$$

and

$$a_{2l,0} = h_{2l}(\rho)/(2l)! > 0, \qquad l = 0, 1, 2, \cdots.$$

For any distribution of $p_1$ we have

$$E[\exp(it(p_{n+1}-\rho)/\theta^{1/2})] = E[W(\theta^{1/2}t, p_n)\exp(it(p_n-\rho)/\theta^{1/2})].$$



Letting $p_1$ have distribution $F_\theta$ and introducing $G_\theta(x) = F_\theta(\theta^{1/2}x + \rho)$, we obtain

$$\int_{-\infty}^{\infty} e^{itx}dG_\theta(x) = \int_{-\infty}^{\infty} e^{itx}W(\theta^{1/2}t, \rho + \theta^{1/2}x)\,dG_\theta(x). \tag{2}$$

Let

$$g(t,\theta) = \int_{-\infty}^{\infty} e^{itx}dG_\theta(x)$$

be the characteristic function of $G_\theta$. The result toward which we are working is that $g$ has an asymptotic expansion of the form

$$g(t,\theta) = \sum_{j=0}^{\infty} g_j(t)\theta^{j/2} \tag{3}$$

in powers of $\theta^{1/2}$. The heuristic argument that follows will suggest the form of the $g_j$. That (3) (and (5)) actually gives an asymptotic expansion can then be established by a supplementary argument that will not be presented here.

Substituting (1) into (2) and using

$$\int_{-\infty}^{\infty} x^k e^{itx}dG_\theta(x) = \frac{1}{i^k}\frac{\partial^k}{\partial t^k}g(t,\theta)$$

we obtain

$$g(t,\theta) = \sum_{j,k=0}^{\infty} a_{j,k}\theta^{(j+k)/2}(it)^j \frac{1}{i^k}\frac{\partial^k}{\partial t^k}g(t,\theta). \tag{4}$$

Formal differentiation of (3) $k$ times with respect to $t$ yields

$$\frac{\partial^k}{\partial t^k}g(t,\theta) = \sum_{l=0}^{\infty} g_l^{(k)}(t)\theta^{l/2}. \tag{5}$$

Substitution of (3) on the left and (5) on the right in (4), and equation of coefficients of like powers of $\theta^{1/2}$ on the two sides of the resulting equation leads to the system

$$g_m(t) = \sum_{\substack{j+k+l=m \\ j,k,l \geqq 0}} a_{j,k}(it)^j(-i)^k g_l^{(k)}(t), \qquad m \geqq 0,$$

of difference-differential equations for $g_m(t)$. In terms of the quantities

$$Q_m(t) = \sum_{\substack{j+k+l=m+2 \\ 1 \leq j; l \leq m-1}} \frac{a_{j,k}}{-a_{1,1}} i^{j+k}(-1)^k t^{j-1} g_l^{(k)}(t),$$

$m \geqq 1$, and

$$\sigma^2 = a_{2,0}/-a_{1,1} = h_2(\rho)/-2h_1'(\rho),$$

this can be rewritten

$$g_0'(t) + \sigma^2 t g_0(t) = 0,$$
$$g_m'(t) + \sigma^2 t g_m(t) = Q_m(t), \qquad m \geqq 1.$$

Since

$$1 = g(0,\theta) = \sum_{m=0}^{\infty} g_m(0)\theta^{m/2},$$

the initial conditions $g_m(0) = \delta_{0,m}$ are plausible. Assuming them we obtain

$$g_0(t) = \exp[-\sigma^2 t^2/2]$$

and

$$g_m(t) = \exp\left[\frac{-\sigma^2 t^2}{2}\right] \int_0^t \exp\left[\frac{\sigma^2 s^2}{2}\right] Q_m(s)\, ds$$

for $m \geqq 1$. Since $Q_m$ depends only on $g_0, \cdots, g_{m-1}$, these equations permit the successive calculation of $g_1, g_2, \cdots$. It can also be shown inductively from them that $g_m$ is of the form $g_m(t) = P_m(t)\exp(-\sigma^2 t^2/2)$ where $P_m$ is a polynomial, $m \geqq 1$.

In particular $g(t,\theta) \to \exp(-\sigma^2 t^2/2)$ as $\theta \to 0$, so $G_\theta(x) \to \Phi(x/\sigma)$ as $\theta \to 0$. The asymptotic expansion derived above has the same relation to this central limit theorem that the Edgeworth expansion has to the standard central limit theorem for sums of independent random variables.

A proof that $G_\theta(x) \to \Phi(x/\sigma)$ as $\theta \to 0$ in a slightly more general setting and some additional examples are given by Norman and Graham [**20**].

B. *One absorbing barrier.* If $A_1$ is reinforced regardless of what response occurs ($\pi_{11} = \pi_{21} = 1$) (e.g. $A_1$ is avoidance in an avoidance conditioning experiment), then the probability $q_n$ of $A_{2,n}$ (an error or failure to avoid on trial $n$) satisfies



$$q_{n+1} = \alpha_1 q_n \quad \text{if } A_{1,n},$$
$$= \alpha_2 q_n \quad \text{if } A_{2,n},$$

where $\alpha_1 = 1 - \theta_{11}$ and $\alpha_2 = 1 - \theta_{21}$. Clearly,

$$q_n \leqq q_1 \max{}^{n-1}(\alpha_1, \alpha_2),$$

so, if $\alpha_1, \alpha_2 < 1$, $q_n \to 0$ with probability 1, i.e. the animal learns. To test the model in a given experimental situation, one wants to to know such things as the predicted distribution of the total number of errors, the mean and variance of the trial of last error, the mean number of runs of errors of length $j$, the probability of an error on trial $n$, etc. All of these quantities depend on the initial error probability $q_1$. Explicit series expansions of all of these quantities in powers of $q_1$ have been worked out by a number of investigators using a variety of methods (see Bush [**2**], Tatsuoka and Mosteller [**26**], and Sternberg [**24**, Equation 80, p. 85]). For example, Bush reduced the computation of many of these quantities to that of the quantities

$$S_k(q_1) = \sum_{n=1}^{\infty} E_{q_1}[q_n^k]$$

and obtained the expansion

$$S_k(q_1) = \frac{q_1^k}{1-\alpha_1^k} + \sum_{i=k+1}^{\infty} \frac{q_1^i}{1-\alpha_1^i} \prod_{j=k}^{i-1} \frac{\alpha_2^j - \alpha_1^j}{1-\alpha_1^j}$$

by an interesting direct computation (see Bush [**2**, Equation 9, p. 219]).

C. *Two absorbing barriers.* The four-operator linear model will have both 0 and 1 absorbing if and only if $\pi_{12} = 0$ or $\theta_{12} = 0$, and $\pi_{21} = 0$ or $\theta_{21} = 0$. This means that the only response that is ever effectively reinforced is the one just made. Generalizing slightly, we consider an experimental situation in which $A_i$ has $N_i$ possible outcomes (e.g. various amounts of food) all of which reinforce $A_i$ and the $j$th one of which has probability $\pi_{ij}$. In the corresponding linear model, $p_n$ satisfies

$$\Delta p_n = \theta_{1j}(1-p_n) \quad \text{with prob. } p_n\pi_{1j},$$
$$= -\theta_{2k}p_n \quad \text{with prob. } (1-p_n)\pi_{2k}$$

where $1 \leqq j \leqq N_1$, $1 \leqq k \leqq N_2$, $0 \leqq \theta$, $\pi_{il} \leqq 1$, and $\sum_j \pi_{1j} = \sum_k \pi_{2k} = 1$. We assume further that, for some $j$, $\theta_{1j} > 0$ and $\pi_{1j} > 0$ and, for some $k$, $\theta_{2k} > 0$ and $\pi_{2k} > 0$. The model is then distance diminishing, and the absorbing states $\{0\}$ and $\{1\}$ are the only ergodic kernels.

In contrast to the one absorbing barrier linear model discussed above, few explicit expressions for properties of two absorbing barrier linear models are available at this time. For example, the absorption probability

$$\gamma(p) = P_p\left(\lim_{n\to\infty} p_n = 1\right)$$

is basic to the theory and application of the models. It can be shown that $\gamma$ can be extended to a function meromorphic in the entire complex plane (see Norman [19, Section 5] for the four-operator case). However, except for the few relatively uninteresting cases when $\gamma$ is a polynomial, no one has yet been able to compute the parameters of any of the standard function theoretic representations of $\gamma$.

Some information is available. Let $b_i = \sum_l \theta_{il}\pi_{il}$, $r = b_2/b_1$, $t = \min\{\theta_{il} : \theta_{il} > 0\}$, $T = \max\{\theta_{il}\}$, and

$$V(y) = (e^y - 1)/y.$$

The function $V$ is obviously strictly increasing, hence has an inverse $V^{-1}$. The quantity $r$ is a measure of the efficacy of reinforcement of $A_2$ relative to reinforcement of $A_1$. Thus $r > 1$ means that $A_2$ is the "better" response. The following theorem treats only this case. The comparable result for the case $r < 1$ is obtained by reversing the roles of $A_1$ and $A_2$.

THEOREM 1. *Suppose $r > 1$ and define $x$ and $x^*$ by $x = -V^{-1}(1/r)/t$ and $x^* = V^{-1}(r)/T$. Then*

$$\frac{e^{xp}-1}{e^x-1} \leqq \gamma(p) \leqq \frac{e^{x^*p}-1}{e^{x^*}-1}$$

*for all $0 \leqq p \leqq 1$.*

This theorem generalizes Theorem 4 of Norman [19]. A slight modification of the proof of the latter theorem will yield a proof of Theorem 1.

It is easy to show that $\gamma(p) = p$ if $r = 1$. Hence we expect $\gamma(p) \doteq p$ if $r \doteq 1$ and $\gamma(p) \leqq p$ when $r > 1$. If $r - 1$ is "large" $\gamma(p)$ is the probability of being absorbed on a very unfavorable response. Thus, we expect $\gamma(p) \doteq 0$. The following theorem on the behavior of $\gamma$ as the $\pi$'s and $\theta$'s vary follows from Theorem 1 and shows that departures of $r$ from 1 should be measured relative to the $\theta$'s.



THEOREM 2. *Suppose $r > 1$.*

(a) *If $(r-1)/T \to \infty$, then $\gamma(p) \to 0$ for all $0 < p < 1$. If $\gamma(p) \to 0$ for some $0 < p < 1$, then $(r-1)/t \to \infty$.*

(b) *If $(r-1)/t \to 0$, then $\gamma(p) \to p$ for all $0 < p < 1$. Furthermore*

$$\limsup \frac{p-\gamma(p)}{p(1-p)(r-1)/t} \leqq 1 \quad \text{and} \quad \liminf \frac{p-\gamma(p)}{p(1-p)(r-1)/T} \geqq 1$$

*for all $0 < p < 1$. If $\gamma(p) \to p$ for some $0 < p < 1$, then $(r-1)/T \to 0$ and*

$$\liminf \frac{p-\gamma(p)}{p(1-p)(r-1)/T} \geqq 1$$

*for all $0 < p < 1$.*

In applying a model like this one to experimental data, one might want to predict the mean number $\chi(p)$ of times a response configuration of the type

$$A_{i_1,n+1}A_{i_2,n+2}\cdots A_{i_k,n+k}$$

occurs, when $p_1 = p$. If all the responses in the configuration are the same, then this number will be infinite if the subject is absorbed on that response. Assume then that both responses occur in the configuration. In that case the $k$th degree (real) polynomial

$$g(p) = P_p(A_{i_1,1}\cdots A_{i_k,k})$$

vanishes on the ergodic kernels $\{0\}$ and $\{1\}$, so $\|U^n g\| \to 0$ geometrically as $n \to \infty$. Thus the series $\sum_{n=0}^{\infty} U^n g$ converges to a $CL$ function, and from

$$U^n g(p) = P_p(A_{i_1,n+1}\cdots A_{i_k,n+k})$$

it is clear that this limit is $\chi(p)$. Thus $\chi$ satisfies the functional equation

$$(*) \qquad \chi = g + U\chi$$

and the boundary conditions $\chi(0) = \chi(1) = 0$. If $\Omega$ is any $CL$

solution of $(*)$, then $\Delta = \chi - \Omega$ is a $CL$ solution of the homogeneous equation $U\Delta = \Delta$. It then follows from Theorem 2 of §II that $\Delta = \Delta(1)\gamma + \Delta(0)(1-\gamma)$, i.e.

$$\chi(p) = \Omega(p) - \Omega(1)\gamma(p) - \Omega(0)(1-\gamma(p)).$$

The next theorem gives a sufficient condition for the existence of polynomial solutions of $(*)$ when $g$ is a polynomial that vanishes on the absorbing states as well as a closely related necessary and sufficient condition for $\gamma$ to be a polynomial. The usefulness of this theorem is enhanced by the fact that, if a polynomial solution of $(*)$ exists, it is very easy to calculate it explicitly. Let

$$\delta_n = \sum_{k=1}^{N_2} (1-\theta_{2k})^n \pi_{2k} - \sum_{j=1}^{N_1} (1-\theta_{1j})^n \pi_{1j}.$$

THEOREM 3. (a) *If $\delta_n \neq 0$ for all $n \geq 1$, then $\gamma$ is not a polynomial. Let $g$ be any polynomial of degree at least 2, such that $g(0) = g(1) = 0$. Then there is a unique polynomial $\psi$ such that $\psi = g + U\psi$ and $\psi(0) = 0$. The degree of $\psi$ is one less than that of $g$.*

(b) *If $\delta_1 = 0$ then $\gamma(p) \equiv p$. Suppose $n \geq 1$, $\delta_j \neq 0$ for $1 \leq j \leq n$ and $\delta_{n+1} = 0$. Then $\gamma$ is a polynomial of degree $n+1$. If $g$ is a polynomial of degree at least 2 and at most $n+1$ such that $g(0) = g(1) = 0$, there is a unique polynomial $\psi$ such that (degree $\psi$) $\leq n$, $\psi = g + U\psi$, and $\psi(0) = 0$. The degree of $\psi$ is one less than that of $g$.*

## IV. Learning models with noncompact state spaces.

### A. *The beta model.*

1. *Recurrence criteria.* At the level of the logarithm $w_n$ of the strength $v_n$ of the $A_1$ response on trial $n$ the stochastic difference equation for the four-operator beta model is

$$\Delta w_n = \ln \beta_{ij} \quad \text{if } A_{i,n} O_{ij,n},$$

where $\beta_{i1} \geq 1$ and $\beta_{i2} \leq 1$. In terms of $w_n$, the probability of $A_{1,n}$ is

$$p_n = e^{w_n}/(1+e^{w_n}),$$

and, as usual, $P(O_{ij,n} | A_{i,n}) = \pi_{ij}$.

To begin to appreciate the magnitude of the difference in the asymptotic behavior of the linear and beta models, consider the special case of noncontingent reinforcement under the additional assumption that outcomes $O_{11}$ and $O_{21}$ have exactly the same effect on $A_1$ response probability that $O_{12}$ and $O_{22}$ have on $A_2$ response probability ($\beta_{11} = \beta_{21} = \beta > 1$, $\beta_{12} = \beta_{22} = 1/\beta$). In this case the



above difference equation reduces to

$$\begin{aligned} \Delta w_n &= \ln\beta \quad \text{if } A_{1,n}O_{11,n} \text{ or } A_{2,n}O_{21,n} \\ &= -\ln\beta \quad \text{if } A_{1,n}O_{12,n} \text{ or } A_{2,n}O_{22,n}, \end{aligned}$$

so that $w_n$ is just $w_1$ plus the sum of the $n-1$ independently and identically distributed random variables $\Delta w_1, \cdots, \Delta w_{n-1}$ with mean

$$E[\Delta w_n | w_n] = E[\Delta w_n] = 2(\pi_1 - 1/2)\ln\beta.$$

Since $\ln\beta > 0$ it follows from the strong law of large numbers that $\lim_{n\to\infty} w_n = \infty$ with probability 1 if $E[\Delta w_n] > 0$ (i.e. if $\pi_1 > 1/2$) while $\lim_{n\to\infty} w_n = -\infty$ if $E[\Delta w_n] < 0$ ($\pi_1 < 1/2$). This means that the asymptotic probability of making the more frequently reinforced response (e.g. predicting the light that flashes most often) is 1. By way of comparison,

$$P_{p_1}\left(\liminf_{n\to\infty} p_n = 0\right) = 1,$$

$$P_{p_1}\left(\limsup_{n\to\infty} p_n = 1\right) = 1,$$

and

$$\lim_{n\to\infty} E_{p_1}[p_n] = \pi_1$$

for any $0 < p_1$, $\pi_1 < 1$ in the comparable linear model. Thus the process $\{p_n\}$ for $0 < p_1 < 1$ and $\pi_1 \neq 0$, 1/2, or 1 is recurrent in the linear model but not in the beta model.

In the general case, where the distribution of $\Delta w_n$ depends on $w_n$, the qualitative asymptotic behavior of $\{w_n\}$ and $\{p_n\}$ is still determined by the conditional mean $E[\Delta w_n | w_n = w]$, or, more precisely, by its limits at $\pm\infty$. Let

$$\mu_+ = \lim_{w\to\infty} E[\Delta w_n | w_n = w] = \pi_{11}\ln\beta_{11} + \pi_{12}\ln\beta_{12}$$

and

$$\mu_- = \lim_{w\to-\infty} E[\Delta w_n | w_n = w] = \pi_{21}\ln\beta_{21} + \pi_{22}\ln\beta_{22}.$$

THEOREM 1. *For the four-operator beta model with $\beta_{i1} > 1$ and $\beta_{i2} < 1$, $i = 1, 2$, and for all $0 < p_1 < 1$:*

(a) *If $\mu_+ < 0$ and $\mu_- > 0$, then*

(1) $$P_{p_1}\left(\liminf_{n\to\infty} p_n = 0\right) = 1$$

*and*

(2) $$P_{p_1}\left(\limsup_{n\to\infty} p_n = 1\right) = 1.$$

(b) *If* $\mu_+ < 0$ $(> 0)$ *and* $\mu_- < 0$ $(> 0)$, *then*

(3) $$P_{p_1}\left(\lim_{n\to\infty} p_n = 0\ (1)\right) = 1.$$

(c) *If* $\mu_+ > 0$ *and* $\mu_- < 0$, *then*

(4) $$P_{p_1}\left(\lim_{n\to\infty} p_n = 1\right) = \delta(p_1)$$

*and*

(5) $$P_{p_1}\left(\lim_{n\to\infty} p_n = 0\right) = 1 - \delta(p_1)$$

*for some* $0 < \delta(p_1) < 1$.

This theorem is due to Lamperti and Suppes [15].

In comparison, for a distance diminishing four-operator linear model, (1) and (2) hold if neither 0 nor 1 is absorbing, (3) holds if 0 (1) is absorbing and 1 (0) is not, while (4) and (5) (with $0 < \delta(p_1) < 1$) hold if both 0 and 1 are absorbing. These results follow from the theory presented in §II above.

2. *Other results.* Far fewer analytic formulas for predictions are available for beta models than for linear models. The best results to date are those of L. Kanal. He has shown [13] that, for the two-operator model,

$$\begin{aligned}\Delta w_n &= \ln \beta_{12} \quad \text{if } A_{1,n},\\ &= \ln \beta_{22} \quad \text{if } A_{2,n}\end{aligned}$$

($\pi_{12} = \pi_{22} = 1$, $\beta_{12} = \beta_{22} < 1$), many functions of interest in testing the model empirically are solutions of the functional equation $\chi = g + U\chi$ that vanish at $w = -\infty$ ($v = 0$, $p = 0$). His general solution of this equation is closely related to the standard expression $\sum_{n=0}^{\infty} U^n g$. Concerning the model

$$\begin{aligned}\Delta w_n &= \quad \ln \beta \quad \text{if } A_{1,n},\\ &= -\ln \beta \quad \text{if } A_{2,n}\end{aligned}$$

($\pi_{11} = \pi_{22} = 1$, $\beta_{11} = 1/\beta_{22} = \beta$), he has shown (Kanal [14]) that the probability $\delta(w_1)$ that $w_n \to \infty$ ($p_n \to 1$) as $n \to \infty$ is given by

$$\delta(w_1) = \frac{\sum_{k=0}^{\infty} \exp\left\{-\frac{1}{2b}\left[w_1 - \left(k + \frac{1}{2}\right) b\right]^2\right\}}{\sum_{k=-\infty}^{\infty} \exp\left\{-\frac{1}{2b}\left[w_1 - \left(k + \frac{1}{2}\right) b\right]^2\right\}},$$

where $b = \ln \beta$.



3. *Commutativity.* A central feature conceptually of the beta model is the fact that the operators

$$p \to F(p, \beta) = \beta p / ((1 - p) + \beta p)$$

corresponding to different $\beta$'s (different events) commute. This is obvious at the level of $v = p/(1 - p)$ or $w = \ln v$ where the comparable transformations are $v \to \beta v$ and $w \to w + \ln \beta$. Luce [16] has studied abstract one parameter families of functions $F(p, \beta)$ corresponding intuitively to learning operators, with special emphasis on the commutative case. His article contains a discussion (see his Section 6, p. 396) of the extent to which beta model results like those discussed above generalize to models generated by other commutative families of operators.

B. *Models with distance diminishing event operators.* Generalizing Suppes' continuous linear model (see IC above) we consider the following set-up: (i) The state space $(S, d)$ is a bounded metric space (that is, $d(s, s') \leq \nu$ for some $\nu < \infty$ and all $s$, $s' \in S$). (ii) The event space $(E, \Gamma)$ is a measurable space. As before, the state $S_{n+1}$ on trial $n + 1$ is related to the state $S_n$ and event $E_n$ on trial $n$ by $S_{n+1} = f_{E_n}(S_n)$, $n \geqq 1$. We impose the very strong condition that the event operators be uniformly strictly distance diminishing: (iii) there is some $r < 1$ such that $\mu(f_e) \leqq r$ for all $e \in E$. Also (iv) $f_.(s)$ is a measurable transformation for every $s \in S$. The conditional probability distribution of $E_n$ given $S_n = s$ is $\phi_.(s)$. Thus (v) $\phi_.(s)$ is a probability measure on $\Gamma$ for each $s \in S$. It is assumed that the functions $\phi_G(\cdot)$ for $G \in \Gamma$ satisfy a uniform Lipschitz condition (vi) $m(\phi_G) \leqq \omega$ for some $\omega < \infty$ and all $G \in \Gamma$. Our final condition is (vii) There is a nonnegative integer $j$, a positive real number $\lambda$, and a probability measure $\zeta$ on $\Gamma$ such that

$$\phi_G(f_{e_1 \cdots e_j}(s)) \geqq \lambda \zeta(G)$$

for all $e_1, \cdots, e_j \in E$, $s \in S$, and $G \in \Gamma$. (For $j = 0$ read $\phi_G(s) \geqq \lambda\zeta(G)$ for all $s$ and $G$.) This condition is quite restrictive. Theorem 2 below shows that, under these hypotheses, the Markov process $\{S_n\}$ behaves like a compact Markov process with only one, aperiodic, ergodic kernel.

In what follows, $C(S)$ is the set of bounded continuous real valued functions on $S$, $|\psi| = \sup_{s\in S}|\psi(s)|$, $CL$ is the subset of $C(S)$ whose elements satisfy a Lipschitz condition, and $\|\psi\| = m(\psi) + |\psi|$ for $\psi \in CL$. The linear operator $U$ is defined by

$$U\psi(s) = E[\psi(S_{n+1}) \mid S_n = s] = \int \psi(f_e(s))\,\phi_{de}(s).$$

It is easy to show that, under the above assumptions, $U$ is a bounded mapping of $C(S)$ into $C(S)$ ($|U| = 1$) and of $CL$ into $CL$ ($\|U\| \leqq 2\omega + 1$).

THEOREM 2. *Under conditions* (i)—(vii) *there are constants* $\rho$ *and* $\eta$ *such that, for every* $\psi \in CL$, *there is a constant function* $U^\infty\psi$ *for which*

$$|U^{n+j}\psi - U^\infty\psi| \leqq \eta\rho^{n^{1/2}}\|\psi\|$$

*for all* $n \geqq 1$. *If* $k$ *is* 1 *when* $\omega\nu r/(1-r) \leqq 1/8$ *and otherwise is the least integer greater than or equal to*

$$\frac{\ln[(1-r)/8\omega\nu]}{\ln r},$$

*and* $h = 1 - \lambda^k/4$, *then we can take*

$$\rho = \max(r, h) \quad \text{and} \quad \eta = \max\left(\frac{\nu}{r(1-h)}, \frac{4\omega\nu}{r(1-r)(1-h)} + \frac{2}{h}\right).$$

The case $j = 0$ is a consequence of a theorem of Ionescu Tulcea [8, Theorem 1], and the other cases are consequences of this one.

It follows from results of Iosifescu [10, Chapter 3, Section 3] that, under (i)—(vii),

$$\lim_{n\to\infty} E_s[f(E_n)] = E^\infty(f)$$

and

$$\lim_{n\to\infty} \frac{1}{n} E_s\left\{\left[\sum_{j=1}^{n}(f(E_n) - E^\infty(f))\right]^2\right\} = \sigma^2$$

exist for any real valued bounded measurable function $f$ on $E$ and do not depend on $s$. If $\sigma^2 > 0$, then $\sum_{j=1}^{n} f(E_j)$ is asymptotically normally distributed with mean $nE^\infty(f)$ and variance $n\sigma^2$.



To apply these results to Suppes' model we take the state space $S$ to be the Borel probability measures on the rim $R$ of the disc, the distance $d(\mu, \mu')$ between two states to be the total variation of their difference, the event space $E$ to be $R$, and $\Gamma$ to be the Borel subsets of $R$. The operator for the event $z$ is

$$f_z(\mu) = (1-\theta)\mu + \theta\tau(z, \cdot)$$

where $\theta > 0$. Then $\mu(f_e) = (1-\theta) < 1$ for all events $e$, so (iii) holds. The measurability condition (iv) on $f.(\mu)$ is satisfied if $\tau(\cdot, A)$ is measurable for each Borel set $A$. Assuming $\pi(\cdot, G)$ measurable for any $G$, we take

$$\phi_G(\mu) = \int \mu(dy)\pi(y, G).$$

Then $m(\phi_G) \leqq 1$ for all $G$ so (vi) holds. Finally, (vii) is satisfied with $j = 0$ if $\pi(y, \cdot)$ has a nontrivial response independent component: $\pi(y, G) \geqq a\xi(G)$ for some probability measure $\xi$ and $a > 0$, and all $y$ and $G$. For in that case

$$\phi_G(\mu) \geqq \int \mu(dy)a\xi(G) = a\xi(G)$$

for all $\mu$ and $G$.

$CL$ includes functions of the form

$$\psi(\mu) = \int \chi(y)\,\mu(dy)$$

where $\chi$ is bounded and measurable on $R$. Hence it includes finite products of such functions. Thus, when Theorem 2 applies, we can conclude that

$$\lim_{n\to\infty} E_{\mu_1}\left[\prod_{j=1}^{k}\int \chi_j(y)\,\mu_n(dy)\right]$$

exists and does not depend on $\mu_1$ for bounded measurable functions $\chi_1, \cdots, \chi_k$. The central limit theorem mentioned above is not very useful in this case, since the subject's response on a trial is not a function of the event that occurred on that trial.

Theorem 2 (with $j = 1$) and Iosifescu's central limit theorem are also applicable to my linear model for operant conditioning (Norman [17]).

## Bibliography

1. R. C. Atkinson and W. K. Estes, "Stimulus sampling theory" in *Handbook of mathematical psychology*. vol. II edited by R. D. Luce et al, Wiley, New York, 1963, pp. 121-268.

2. R. R. Bush, "Sequential properties of linear models" in *Studies in mathematical learning theory* edited by R. R. Bush and W. K. Estes, Stanford Univ. Press, Stanford, Calif., 1959, pp. 215-227.

3. ———, "Identification learning" in *Handbook of mathematical psychology*. vol. III edited by R. D. Luce et al, Wiley, New York, 1965, pp. 161-203.

4. R. R. Bush and F. Mosteller, "A comparison of eight models" in *Studies in mathematical learning theory* edited by R. R. Bush and W. K. Estes, Stanford Univ. Press, Stanford, Calif., 1949, pp. 293-307.

5. K. L. Chung, *Markov chains*. 2nd ed., Springer-Verlag, New York, 1967.

6. J. L. Doob, *Stochastic processes*, Wiley, New York, 1953.

7. W. K. Estes, "Probability learning" in *Categories of human learning* edited by A. W. Melton, Academic Press, New York, 1964, pp. 89-128.

8. C. T. Ionescu Tulcea, *On a class of operators occurring in the theory of chains of infinite order*, Canad. J. Math. **11** (1959), 112-121.

9. C. T. Ionescu Tulcea and G. Marinescu, *Theorie ergodique pour des classes d'operations noncompletement continues*, Ann. of Math. **52** (1950), 140-147.

10. M. Iosifescu, *Random systems with complete connections with an arbitrary set of states*, Rev. Math. Pures Appl. **8** (1963), 611-645.

11. B. Jamison, *Asymptotic behavior of successive iterates of continuous functions under a Markov operator*, J. Math. Anal. Appl. **9** (1964), 203-214.

12. ———, *Ergodic decompositions induced by certain Markov operators*, Trans. Amer. Math. Soc. **117** (1965), 451-468.

13. L. Kanal, *A functional equation analysis of two learning models*, Psychometrika **27** (1962), 89-104.

14. ———, *The asymptotic distribution for the two-absorbing-barrier beta model*, Psychometrika **27** (1962), 105-109.

15. J. Lamperti and P. Suppes, *Some asymptotic properties of Luce's beta learning model*, Psychometrika **25** (1960), 233-241.

16. R. D. Luce, "Some one-parameter families of commutative learning operators" in *Studies in mathematical psychology* edited by R. C. Atkinson, Stanford Univ. Press, Stanford, Calif., 1964, pp. 380-398.

17. M. F. Norman, *An approach to free-responding on schedules that prescribe reinforcement probability as a function of interresponse time*, J. Math. Psychology **3** (1966), 235-268.

18. ———, *Some convergence theorems for stochastic learning models with distance diminishing operators*, J. Math. Psychology **5** (1968), 61-101.

19. ———, *On the linear model with two absorbing barriers*, J. Math. Psychology **5** (1968), 225-241.

20. M. F. Norman and N. V. Graham, *A central limit theorem for families of stochastic processes indexed by a small average step size parameter, and some applications to learning models*, Psychometrika **33** (1968), 441-449.

21. E. Parzen, *On asymptotically efficient consistent estimates of the spectral density function of a stationary time series*, J. Roy. Statist. Soc. B. **20** (1958), 303-322.

22. M. Rosenblatt, *Equicontinuous Markov operators*, Theor. Probability Appl. **9** (1964), 205-222.



23. ———, *Almost periodic transition operators acting on the continuous functions on a compact space*, J. Math. Mech. **13** (1964), 837-847.

24. S. Sternberg, "Stochastic learning theory" in *Handbook of mathematical psychology*. vol. II edited by R. D. Luce et al, Wiley, New York, 1963, pp. 1-120.

25. P. Suppes, "A linear model for a continuum of responses" in *Studies in mathematical learning theory* edited by R. R. Bush and W. K. Estes, Stanford Univ. Press, Stanford, Calif., 1959, pp. 400-414.

26. M. Tatsuoka and F. Mosteller, "A commuting-operator model" in *Studies in mathematical learning theory* edited by R. R. Bush and W. K. Estes, Stanford Univ. Press, Stanford, Calif., 1959, pp. 228-247.

27. K. Yosida and S. Kakutani, *Operator-theoretical treatment of Markoff's process and mean ergodic theorem*, Ann. of Math. **42** (1941), 188-228.

## Addendum (January, 1970)

Proofs of the theorems in II.C and II.D are given in [**30**]. The theory of slow learning, some aspects of which are considered in III.A, is further developed in [**29**], [**31**, §6], and [**32**]. Some new methods and new results for *additive models*, such as the beta model (IV.A), are presented in [**31**]. §IV.B is superseded by [**33**].

A comprehensive presentation of the theory of random systems with complete connections is given in [**28**].

## Supplementary Bibliography

28. M. Iosifescu and R. Theodorescu, *Random processes and learning*, Springer-Verlag, New York, 1969.

29. M. F. Norman, *Slow learning*, British J. Math. Statist. Psychology **21** (1968), 141-159.

30. ———, *Compact Markov processes*, Technical Report No. 2, NSF grant GP-7335, August, 1968.

31. ———, *Limit theorems for additive learning models*, J. Math. Psychology **7** (1970), (to appear).

32. ———, *Slow learning with small drift in two-absorbing-barrier models*, J. Math. Psychology **7** (1970), (to appear).

33. ———, *A uniform ergodic theorem for certain Markov operators on Lipschitz functions on bounded metric spaces*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete. **15** (1970), 51-56.

University of Pennsylvania
Philadelphia, Pennsylvania