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On the Linear Model with Two Absorbing Barriers

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A family of linear models for learning in two-choice situations is considered. These models have in common the assumption that nonreward has no effect on response probability. The function $\gamma(p)$ that relates asymptotic probability of one of the responses to its initial probability is studied intensively. It is shown to be closely related to the total number $\chi(p)$ of response alternations. The asymptotic probability of making the less favorable response is shown to be small when the learning rates associated with reward are small. Finally, some of the basic analytic function theoretic properties of $\gamma(p)$ are presented.

1. INTRODUCTION

Throughout this section we consider a two-choice (A_1 or A_2) animal learning experiment where, on any trial, response A_i is followed by reward (event E_i) with probability $\pi_i > 0$. Occurrence of A_i or E_i on trial n is denoted $A_{i,n}$, or $E_{i,n}$, and a subject's probability of $A_{1,n}$ is denoted p_n. When correction or retracing is permitted after a nonreinforced response (an "error"), asymptotic A_1 response probability is found to depend on a number of factors, among them the species of the subject and the type of discrimination required (Bitterman, 1965). In such experiments reward has always been set up for exactly one of the two responses before each trial so that reinforcement is noncontingent, i.e., $\pi_1 = 1 - \pi_2 = \pi$. Noncorrection experiments using both contingent and noncontingent reinforcement, on the other hand, have consistently yielded very high asymptotic proportions of choices of the response with the highest probability of being rewarded (Behrend and Bitterman, 1961; Bitterman, Wodinsky, and Candland, 1958; Brody, 1965; Meyer, 1960; Parducci and Polt, 1958; Stanley, 1950; and Weinstock, North, Brody, and LoGuidice, 1965).

The possibility has often been considered that the effect of nonreinforcement in these experiments might be nil or practically nil. This assumption, embedded within linear learning models, has interesting consequences. When coupled with the auxiliary assumption that a reinforced correction response has the same effect on response probability as a reinforced original response, it leads to the transition rules

$$p_{n+1} = \begin{cases} (1-\theta)\, p_n + \theta, & \text{if} \quad A_{1,n}E_{1,n} \quad \text{or} \quad A_{2,n}\tilde{E}_{2,n} \\ (1-\theta)\, p_n, & \text{if} \quad A_{2,n}E_{2,n} \quad \text{or} \quad A_{1,n}\tilde{E}_{1,n}, \end{cases} \tag{1}$$

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for p_n. The probabilities that the first and second rows are applicable are easily seen to be π and $1 - \pi$ respectively. This model is quite useful psychologically and quite well understood mathematically. It predicts probability matching:

$$\lim_{n\to\infty} P_p(A_{1,n}) = \pi$$

for any value p of p_1.

The corresponding linear model for the noncorrection experiment has transition equations

$$p_{n+1} = \begin{cases} (1-\theta_1)\,p_n + \theta_1\,, & \text{if} \quad A_{1,n}E_{1,n} \\ (1-\theta_2)\,p_n\,, & \text{if} \quad A_{2,n}E_{2,n} \\ p_n\,, & \text{if} \quad A_{1,n}\bar{E}_{1,n} \quad \text{or} \quad A_{2,n}\bar{E}_{2,n}\,, \end{cases} \tag{2}$$

$1 \geqslant \theta_1, \theta_2 > 0$. For the sake of the present discussion we suppose that $\theta_1 = \theta_2 = \theta$, but the more general model, which might arise, for instance, if the magnitudes or delays of the rewards E_1 and E_2 were unequal, is also treated below. The identity operator or absorbing barrier linear model (2) has been considered by many investigators (Brody, 1965; Bush and Mosteller, 1955, pp. 291–294; Weinstock *et al.*, 1965) in this setting with considerable success. (See the last two paragraphs of this paper for an observation that suggests that the model may have been more successful than the authors of the last paper realized.) There are no special difficulties in deriving predictions from this model either by mathematical approximation techniques (Bush and Mosteller, 1955, pp. 286–291; Mosteller and Tatsuoka, 1960) or by Monte Carlo methods (Brody, 1965; Weinstock *et al.*, 1965). Nevertheless there are some large gaps in our knowledge about this model. The purpose of this paper is to fill some of these.

The quantities

$$\gamma(p) = \lim_{n\to\infty} P_p(A_{1,n}), \tag{3}$$

and

$$\chi(p) = E_p\,[\text{total number of response alternations}] \tag{4}$$

are of basic interest. There are practically no cases (i.e., special parameter values) for which simple formulas for γ and χ are known. The research reported below began as an attempt to provide the first proof that, in the equal theta case, the asymptotic probability of choosing the unfavorable side is small, at least when θ is small; that is, $\gamma(p) \to 0$ as $\theta \to 0$ if $\pi_2 > \pi_1$, $\theta_1 = \theta_2 = \theta$, and $0 \leqslant p < 1$. It developed into a fairly general investigation of the functions γ and χ. The lemmas of Sec. 2 provide the foundation for the subsequent development. The main result of Sec. 3 is a relation between χ and γ that essentially reduces the study of the former to that of the latter. The two special cases $\pi_1 = \pi_2 = 1$ and $\theta_1 = \theta_2$ of this result are easily derivable from formulas relating the asymptotic probability of A_1 to the total number of runs of A_1's that Bush (1959, Secs. 5 and 6) obtained by another method. In Sec. 4 bounds are obtained for γ that are sufficiently precise to permit deduction that the asymptotic probability of choosing the unfavorable side is small when the learning rates are small. The theorem of Sec. 5 is concerned with the analytic character of γ. Most of the results of that section are extensions or refinements of those of Karlin (1953, Sec. 1 and 5), who concentrated on the case $\pi_1 = \pi_2 = 1$.

2. FUNDAMENTALS

Throughout the remainder of the paper we will be concerned with the identity operator model (2) under the conditions $1 \geqslant \theta_1, \theta_2, \pi_1, \pi_2 > 0$.

For any (real valued) function ψ on [0,1] we define $|\psi|$ by

$$|\psi| = \sup_{0 \leqslant p \leqslant 1} |\psi(p)|.$$

We let D be the class of all differentiable functions with bounded derivative on [0,1]. Such functions are necessarily continuous. If $\psi \in D$, its norm $\|\psi\|$ is defined by

$$\|\psi\| = |\psi| + |\psi'|.$$

Thus the relation $\lim_{n\to\infty} \|\psi_n - \psi\| = 0$ for $\psi_n, \psi \in D$ is tantamount to the uniform convergence of ψ_n to ψ and of ψ_n' to ψ'. The operator

$$U\psi(p) = E[\psi(p_{n+1}) | p_n = p],$$

i.e.,

$$U\psi(p) = \psi((1-\theta_1)p + \theta_1)\pi_1 p + \psi((1-\theta_2)p)\pi_2(1-p) + \psi(p)(1 - \pi_1 p - \pi_2(1-p)) \tag{5}$$

maps D into D and is linear and positive (that is, it preserves nonnegative functions). It can be shown by a simple inductive argument that

$$U^n\psi(p) = E_p[\psi(p_{n+1})], \tag{6}$$

$0 \leqslant p \leqslant 1$, where U^n is the nth iterate of the operator U.

Lemmas 1 and 2 below follow easily from the results of Norman (1968, see especially Subsection *a* of Sec. 3).

LEMMA 1. *The only absorbing states (barriers) of the Markov process* $\{p_n\}_{n=1}^{\infty}$ *are* 0 *and* 1. *The process converges with probability* 1 *to a random absorbing state* p_∞. *The sequence* $\{P_p(A_{1,n})\}_{n=1}^{\infty}$ *converges as* $n \to \infty$ *to*

$$\gamma(p) = P_p(p_\infty = 1). \tag{7}$$

The function γ *belongs to* D *and is the only continuous solution of the functional equation*

$$U\gamma = \gamma \tag{8}$$

that has the boundary values $\gamma(0) = 0$ *and* $\gamma(1) = 1$. *The function*

$$\gamma_{a,b}(p) = a(1 - \gamma(p)) + b\gamma(p)$$

is the only continuous function such that $U\gamma_{a,b} = \gamma_{a,b}$, $\gamma_{a,b}(0) = a$, *and* $\gamma_{a,b}(1) = b$. *There are* $\alpha < 1$ *and* $C < \infty$ *such that*

$$\| E_.[\psi(p_n)] - E_.[\psi(p_\infty)] \| \leqslant C\alpha^n \| \psi \| \tag{9}$$

for all $\psi \in D$ *and* $n \geqslant 1$.

LEMMA 2. *The total number* Y *of alternations between responses is finite with probability* 1. *In fact, for any* $0 \leqslant p \leqslant 1$, $E_p[Y] < \infty$. *The function* $\chi(p) = E_p[Y]$ *belongs to* D *and is the unique continuous solution of the functional equation*

$$\chi(p) = (2 - \theta_1\pi_1 - \theta_2\pi_2)p(1 - p) + U\chi(p) \tag{10}$$

for which $\chi(0) = \chi(1) = 0$.

Let X_n be the indicator random variable for the event $A_{1,n}$. Since

$$Y = \Sigma_{n=1}^{\infty} | X_{n+1} - X_n | < \infty,$$

it follows that $\{X_n\}$ converges with probability 1. Let X_∞ be the limiting random variable. Clearly $X_\infty \in \{0,1\}$ with probability 1, and it is plausible that $X_\infty = p_\infty$ with probability 1. To prove this we note that

$$\begin{aligned} P_p(X_\infty \neq p_\infty) &= P_p(X_\infty = 1, p_\infty = 0) + P_p(X_\infty = 0, p_\infty = 1) \\ &= E_p[X_\infty(1 - p_\infty)] + E_p[(1 - X_\infty)\, p_\infty] \\ &= \lim_{n\to\infty} E_p[X_n(1 - p_n)] + \lim_{n\to\infty} E_p[(1 - X_n)\, p_n] \\ &= \lim_{n\to\infty} E_p[E[X_n(1 - p_n) | p_n]] + \lim_{n\to\infty} E_p[E[(1 - X_n)\, p_n \mid p_n]] \\ &= \lim_{n\to\infty} E_p[(1 - p_n)\, E[X_n \mid p_n]] + \lim_{n\to\infty} E_p[p_n(1 - E[X_n \mid p_n])] \\ &= 2 \lim_{n\to\infty} E_p[(1 - p_n)\, p_n] = 2E_p[(1 - p_\infty)\, p_\infty] = 0. \end{aligned}$$

Therefore $\gamma(p)$ has the following behavioral interpretation:

$$\gamma(p) = P_p(X_\infty = 1).$$

Since X_n is an indicator, $X_\infty = 1$ means that $X_n = 1$ for all but a finite number of n, i.e., $A_{1,n}$ for all but a finite number of n. A standard notation for the latter event is lim inf $A_{1,n}$. Therefore $\gamma(p) = P_p(\liminf A_{1,n})$.

If $\psi \in D$ with $\psi(0) = 0$ and $\psi(1) = 1$ (e.g., $\psi(p) = p$) then

$$E_p[\psi(p_\infty)] = \psi(0)(1 - \gamma(p)) + \psi(1)\gamma(p) = \gamma(p).$$

Thus (9) and (6) imply that $\| U^n\psi - \gamma \|$ converges geometrically to 0 as $n \to \infty$. This gives us an iterative method of approximating γ that should be useful in numerical computations.

The functions γ and χ and the operator U depend on four parameters. When it is necessary to call attention to the dependence on some of these parameters we use notations such as $\gamma(p; \pi_1, \pi_2)$, $\gamma(p; \theta_1, \theta_2, \pi_1, \pi_2)$, and $U_{\theta_1,\theta_2,\pi_1,\pi_2}$. The study of the dependence of γ upon its parameters is considerably simplified by the following lemma which states the obvious fact that the probability of absorption on response A_1 when the parameters are θ_1, θ_2, π_1, and π_2 and p is the initial probability of A_1 is the same as the probability of absorption on A_2 when the parameters are θ_2, θ_1, π_2, and π_1 and p is the initial probability of A_2 so that $1 - p$ is the initial probability of A_1.

LEMMA 3.

$$\gamma(p; \theta_1, \theta_2, \pi_1, \pi_2) = 1 - \gamma(1 - p; \theta_2, \theta_1, \pi_2, \pi_1).$$

Another proof of this equality is obtained by verifying that the function

$$\varphi(p) = 1 - \gamma(1 - p; \theta_2, \theta_1, \pi_2, \pi_1)$$

belongs to D, and satisfies the functional equation

$$U_{\theta_1,\theta_2,\pi_1,\pi_2}\phi = \phi,$$

and the boundary conditions $\varphi(0) = 0$, $\varphi(1) = 1$. We use a similar method to prove the following lemma, which is only slightly less obvious than the preceding one.

LEMMA 4. *For any* $0 < x \leqslant 1/\max(\pi_1, \pi_2)$,

$$\gamma(p; x\pi_1, x\pi_2) = \gamma(p; \pi_1, \pi_2).$$

In other words, γ depends on π_1 and π_2 only through their ratio.

Proof. The functional equation (8) for $\gamma(p; \pi_1, \pi_2)$ is equivalent to

$$\begin{aligned}(\pi_1 p + \pi_2(1 - p))\gamma(p; \pi_1, \pi_2) \\ = \pi_1 p\gamma((1 - \theta_1)p + \theta_1; \pi_1, \pi_2) + \pi_2(1 - p)\gamma((1 - \theta_2)p; \pi_1, \pi_2).\end{aligned}$$

Multiplying both sides by x we see that $\gamma(p; \pi_1, \pi_2)$ satisfies an equation equivalent to the functional equation for $\gamma(p; x\pi_1, x\pi_2)$. Q.E.D. In particular, the absorption probability $\gamma(p; \nu, \nu)$ for the case $\pi_1 = \pi_2 = \nu$ does not depend on ν.

We now describe a method that permits us to obtain bounds on the solutions γ and χ of the functional equations (8) and (10) by solving corresponding functional inequalities.

DEFINITION. *A function ψ on* [0,1] *is superregular* (*regular, subregular*) *if and only if* $\psi(p) \geqslant (=, \leqslant)\ U\psi(p)$ *for all* $0 \leqslant p \leqslant 1$. These concepts are standard in the potential theory of Markov processes (see, for instance, Kemeny, Snell, and Knapp, 1966). The next lemma shows the usefulness of these notions and justifies the terminology.

LEMMA 5. *Let $\psi \in D$ be superregular* (*subregular*) *with* $\psi(0) = a$ *and* $\psi(1) = b$. *Then* $\psi(p) \geqslant (\leqslant)\ \gamma_{a,b}(p)$, *where* $\gamma_{a,b}(p)$ *is the continuous regular function with* $\gamma_{a,b}(0) = a$ *and* $\gamma_{a,b}(1) = b$.

Proof. Since $\psi(p) \geqslant (\leqslant)\ U\psi(p)$ and U^n is a positive linear operator, $U^n\psi(p) \geqslant (\leqslant)\ U^{n+1}\psi(p)$ for all $n \geqslant 0$. But

$$\lim_{n\to\infty} U^n\psi(p) = E_p[\psi(p_\infty)] = \gamma_{a,b}(p)$$

by Lemma 1. Thus $\psi(p) \geqslant (\leqslant)\ \gamma_{a,b}(p)$ for all $0 \leqslant p \leqslant 1$. Q.E.D.

Lemma 5 implies the slightly more general Lemma 6.

LEMMA 6. *Suppose that ψ, φ, and $g \in D$, all three functions vanish at* 0 *and at* 1, *and*

$$\psi(p) \geqslant (\leqslant)\ g(p) + U\psi(p),$$

while

$$\varphi(p) = g(p) + U\varphi(p).$$

Then $\psi(p) \geqslant (\leqslant)\ \varphi(p)$ *for all* $0 \leqslant p \leqslant 1$.

Proof. Let $\Delta(p) = \psi(p) - \varphi(p)$. Then $\Delta \in D$, Δ is superregular (subregular), and $\Delta(0) = \Delta(1) = 0$. Hence by Lemma 5,

$$\Delta(p) \geqslant (\leqslant)\ \gamma_{0,0}(p) = 0,$$

for all $0 \leqslant p \leqslant 1$. Q.E.D.

We remark that Lemmas 5 and 6 generalize immediately to the general absorbing distance diminishing models treated by Norman (1968).

3. ALTERNATIONS

Let I be the identity function on the real line: $I(p) \equiv p$, and let $B(p) \equiv p(1-p)$. A simple computation yields the following important result.

LEMMA 7. $UI = I + (\theta_1\pi_1 - \theta_2\pi_2)B$.

Taking $\theta_1\pi_1 = \theta_2\pi_2$ and using the characterization of γ given in Lemma 1, we obtain

THEOREM 1. *If* $\theta_1\pi_1 = \theta_2\pi_2$, *then* $\gamma = I$.

When $\theta_1\pi_1 \neq \theta_2\pi_2$ Lemma 7 leads to a relation between γ and χ.

THEOREM 2. *If* $\theta_1\pi_1 \neq \theta_2\pi_2$, *then*

$$\chi = \frac{2 - \theta_1\pi_1 - \theta_2\pi_2}{\theta_2\pi_2 - \theta_1\pi_1}(I - \gamma). \tag{11}$$

Proof. Let the function on the right be denoted F. Since I and γ are continuous, F is too. And since I and γ agree at 0 and 1, $F(0) = F(1) = 0$. Finally

$$\begin{aligned} UF &= \frac{2 - \theta_1\pi_1 - \theta_2\pi_2}{\theta_2\pi_2 - \theta_1\pi_1}(UI - U\gamma) \\ &= \frac{2 - \theta_1\pi_1 - \theta_2\pi_2}{\theta_2\pi_2 - \theta_1\pi_1}(I + (\theta_1\pi_1 - \theta_2\pi_2)B - \gamma), \end{aligned}$$

by (8) and Lemma 7. So

$$UF = F - (2 - \theta_1\pi_1 - \theta_2\pi_2)B,$$

which is equivalent to (10). Thus Lemma 2 implies $\chi = F$. Q.E.D.

By evaluating both sides of (11) at the random point p_1 and taking expectations we obtain a relation

$$E[Y] = \frac{2 - \theta_1\pi_1 - \theta_2\pi_2}{\theta_2\pi_2 - \theta_1\pi_1}(P(A_{1,1}) - P(p_\infty = 1))$$

between the three quantities $E[Y]$, $P(A_{1,1})$, and $P(p_\infty = 1)$ that does not depend on the distribution of p_1. This relation can be tested empirically once θ_1 and θ_2 have been estimated. Alternatively, when $\theta_1 = \theta_2 = \theta$, it can be used to estimate θ. Since the absorption probabilities are not changed when π_1 and π_2 are both multiplied by the same constant, the quantity $P(A_{1,1}) - P(p_\infty = 1)$ can be cancelled to obtain

$$\frac{E^{(x)}[Y]}{E^{(x')}[Y]} = \frac{x'}{x}\frac{(2 - x(\theta_1\pi_1 + \theta_2\pi_2))}{(2 - x'(\theta_1\pi_1 + \theta_2\pi_2))},$$

where the reinforcement probabilities corresponding to $E^{(z)}[Y]$ are $z\pi_1$ and $z\pi_2$. In particular, if x and x' are small, or θ_1 and θ_2 are small, or some combination of the two,

$$\frac{E^{(x)}[Y]}{E^{(x')}[Y]} \doteq \frac{x'}{x}.$$

In Sec. 4 we use (11) and our results on the asymptotic behavior of γ as the thetas become small to obtain an asymptotic expression for χ.

The next lemma will help us derive bounds for χ when $\theta_1\pi_1 = \theta_2\pi_2$.

LEMMA 8. *If* $\theta_1\pi_1 = \theta_2\pi_2$ *then*

$$(1 - \theta_1\theta_2 \max(\pi_1, \pi_2))B(p) \leqslant UB(p) \leqslant (1 - \theta_1\theta_2 \min(\pi_1, \pi_2))B(p).$$

Proof. A straightforward computation shows that, for any θ_1, θ_2, π_1, and π_2,

$$UB(p) = B(p)\{1 - [\theta_1\pi_1 - \theta_2(1 - \theta_2)\pi_2]p - [\theta_2\pi_2 - \theta_1(1 - \theta_1)\pi_1](1 - p)\}. \quad (12)$$

When $\theta_1\pi_1 = \theta_2\pi_2$ this reduces to

$$UB(p) = B(p)\{1 - \theta_2^2\pi_2 p - \theta_1^2\pi_1(1 - p)\}.$$

Since $\theta_2^2\pi_2 = \theta_2\theta_1\pi_1$ and $\theta_1^2\pi_1 = \theta_1\theta_2\pi_2$ the lemma follows. Q.E.D.

THEOREM 3. *If* $\theta_1\pi_1 = \theta_2\pi_2$, *then*

$$\frac{2 - \theta_1\pi_1 - \theta_2\pi_2}{\theta_1\theta_2 \max(\pi_1, \pi_2)} B(p) \leqslant \chi(p) \leqslant \frac{2 - \theta_1\pi_1 - \theta_2\pi_2}{\theta_1\theta_2 \min(\pi_1, \pi_2)} B(p).$$

Proof. It follows from Lemma 8 that the function $F(p)$ on the right (left) satisfies

$$F(p) \geqslant (\leqslant)\ (2 - \theta_1\pi_1 - \theta_2\pi_2)B(p) + UF(p).$$

Since these are the functional inequalities corresponding to (10), an application of Lemma 6 completes the proof. Q.E.D.

These bounds are tight when $\pi_1 - \pi_2$ is small. In the important special case $\pi_1 = \pi_2$ (and $\theta_1 = \theta_2$) they yield χ exactly.

COROLLARY. When $\theta_1 = \theta_2 = \theta$ *and* $\pi_1 = \pi_2 = \nu$,

$$\chi = \frac{2(1 - \theta\nu)}{\theta^2\nu} B.$$

4. SMALL LEARNING RATES

In this section we will show that if π_1 and π_2 are fixed and (θ_1, θ_2) approaches (0,0) along a line in the θ_1, θ_2 plane for which $\theta_2\pi_2 > \theta_1\pi_1$, then $\gamma(p; \theta_1, \theta_2) \to 0$ for all $0 \leqslant p < 1$. Moreover the convergence is extremely rapid. The inequality $\theta_2\pi_2 > \theta_1\pi_1$ may be thought of as indicating that A_2 is the most favorable response. When $\theta_1 = \theta_2$ this inequality reduces to $\pi_2 > \pi_1$ as, intuitively, it should. Analogous results for $\theta_1\pi_1 > \theta_2\pi_2$ may be obtained by applying the results established below to $\gamma(1 - p; \theta_2, \theta_1, \pi_2, \pi_1)$ and using Lemma 3.

We assume throughout this section that (θ_1, θ_2) is confined to a line through the origin in the θ_1, θ_2 plane on which $\theta_2\pi_2 > \theta_1\pi_1$. This line is characterized by the ratio $\zeta = \theta_1/\theta_2$, and θ_1 and θ_2 are of the form

$$\theta_1 = \zeta\theta, \theta_2 = \theta, \tag{13}$$

$0 < \theta \leqslant \min(1, 1/\zeta)$. Clearly $\zeta > 0$ and, if

$$\omega = \pi_1/\pi_2,$$

then

$$1 > \omega\zeta. \tag{14}$$

For any $x, \theta > 0$, let $\psi_{x,\theta}$ be the function on [0,1] defined by

$$\psi_{x,\theta}(p) = e^{xp/\theta}.$$

Most of our effort in this section goes into the proof of the following lemma.

Lemma 9. *There are positive constants $y = y(\omega, \zeta)$ and $z = z(\omega, \zeta)$ such that $\psi_{y,\theta}$ is subregular and $\psi_{z,\theta}$ is superregular for all $0 < \theta \leqslant \min(1, 1/\zeta)$.*

Proof. For any $x > 0$ and $0 \leqslant p \leqslant 1$,

$$\begin{aligned}(U_{\theta_1,\theta_2}\psi_{x,\theta})(p) &= e^{xp/\theta}[e^{x\theta_1(1-p)/\theta}\pi_1 p + e^{-x\theta_2 p/\theta}\pi_2(1-p) + (1 - \pi_1 p - \pi_2(1-p))] \\ &= \psi_{x,\theta}(p)[1 + (e^{x\zeta(1-p)} - 1)\,\pi_1 p - (1 - e^{-xp})\,\pi_2(1-p)]\end{aligned}$$

by (13). Thus $\psi_{x,\theta}$ is subregular (superregular) for all $0 < \theta \leqslant \min(1, 1/\zeta)$ if and only if

$$(e^{x\zeta(1-p)} - 1)\,\pi_1 p \geqslant (\leqslant)(1 - e^{-xp})\,\pi_2(1-p) \tag{15}$$

for all $0 \leqslant p \leqslant 1$. (The reader should note that these inequalities do not involve θ.) However the difference between the two sides of (15) is continuous throughout [0,1], therefore these inequalities hold throughout [0,1] if and only if they hold throughout (0,1), i.e.,

$$\frac{(e^{x\zeta(1-p)}-1)}{(1-e^{-xp})}\frac{p}{1-p} \geqslant (\leqslant) \frac{1}{\omega},$$

for all $0 < p < 1$. In terms of the function V on $(-\infty,\infty)$ defined by

$$V(u) = \begin{cases} (e^u - 1)/u, & \text{if} \quad u \neq 0 \\ 1, & \text{if} \quad u = 0, \end{cases} \tag{16}$$

$\psi_{x,\theta}$ is subregular (superregular) for all $0 < \theta \leqslant \min(1, 1/\zeta)$ if and only if

$$f_\zeta(p, x) = \frac{V(x\zeta(1-p))}{V(-xp)} \geqslant (\leqslant) \frac{1}{\zeta\omega} \tag{17}$$

for all $0 < p < 1$.

Now

$$V(u) = \int_0^1 e^{uw}\, dw,$$

for all real u. Since the integrand is positive, $V(u) > 0$ for all real u. By taking the derivatives under the integral sign we see that

$$V^{(k)}(u) = \int_0^1 w^k e^{uw}\, dw,$$

for $k = 1$ or 2. Thus $V'(u) > 0$ for all real u, so V is strictly increasing. It follows that

$$H(u) = \ln V(u), \tag{18}$$

is also strictly increasing. Furthermore

$$\begin{aligned} H''(u) &= \frac{V(u)\, V''(u) - (V'(u))^2}{V^2(u)} \\ &= \int_0^1 (w - V'(u)/V(u))^2\, e^{uw}\, dw / V(u), \end{aligned}$$

so $H''(u) > 0$ for all real u, and H is convex.

Writing

$$\Delta_\zeta(p, x) = H(\zeta x(1-p)) - H(-xp), \tag{19}$$

(17) and (18) give

$$f_\zeta(p,x) = e^{\Delta_\zeta(p,x)}. \tag{20}$$

We must now distinguish two cases.

Case 1: $\zeta \geqslant 1$. For $0 \leqslant p \leqslant 1$,

$$(\partial/\partial p)\Delta_\zeta(p, x) = -x[\zeta H'(\zeta x(1-p)) - H'(-xp)]. \tag{21}$$

Since $H'(\zeta x(1-p)) \geqslant 0$, and $\zeta \geqslant 1$,

$$\begin{aligned}\zeta H'(\zeta x(1-p)) - H'(-xp) &\geqslant H'(\zeta x(1-p)) - H'(-xp)\\ &\geqslant 0\end{aligned}$$

since the convexity of H implies that H' is nondecreasing. Equation 21 then yields

$$(\partial/\partial p)\Delta_\zeta(p, x) \leqslant 0,$$

so that

$$\Delta_\zeta(1, x) \leqslant \Delta_\zeta(p, x) \leqslant \Delta_\zeta(0, x),$$

for all $0 < p < 1$. But $\Delta_\zeta(1, x) = -H(-x)$ and $\Delta_\zeta(0, x) = H(\zeta x)$, so

$$-H(-x) \leqslant \Delta_\zeta(p, x) \leqslant H(\zeta x),$$

or

$$\frac{1}{V(-x)} \leqslant f_\zeta(p, x) \leqslant V(\zeta x), \tag{22}$$

for all $0 < p < 1$ and $x > 0$.

Case 2: $\zeta \leqslant 1$. In this case

$$\Delta_1(p, \zeta x) \leqslant \Delta_\zeta(p, x) \leqslant \Delta_1(p, x),$$

since H is nondecreasing. We saw in Case 1, though, that $\Delta_1(\cdot, x)$ is nonincreasing. Hence

$$\Delta_1(1, \zeta x) \leqslant \Delta_\zeta(p, x) \leqslant \Delta_1(0, x),$$

i.e.,

$$-H(-\zeta x) \leqslant \Delta_\zeta(p, x) \leqslant H(x).$$

Therefore

$$\frac{1}{V(-\zeta x)} \leqslant f_\zeta(p, x) \leqslant V(x), \tag{23}$$

for all $0 < p < 1$ and $x > 0$.

Returning for the moment to the general case, note that $\lim_{u\to-\infty} V(u) = 0$, $V(0) = 1$, and $\lim_{u\to\infty} V(u) = \infty$, and recall that V is continuous and strictly increasing. Since $\zeta\omega < 1$, the equation

$$V(x') = 1/\zeta\omega, \tag{24}$$

has a unique root $x' = x'(\omega, \zeta)$ in $(0,\infty)$, while the equation

$$V(x'') = \zeta\omega, \tag{25}$$

has a unique root $x'' = x''(\omega, \zeta)$ in $(-\infty, 0)$. Now consider again the cases discussed above. In Case 1 ($\zeta \geqslant 1$) let

$$y = -x'' \text{ and } z = x'/\zeta. \tag{26}$$

Then from (22),

$$\frac{1}{V(x'')} = \frac{1}{V(-y)} \leqslant f_\zeta(p, y),$$

while

$$f_\zeta(p, z) \leqslant V(\zeta z) = V(x'),$$

for all $0 < p < 1$. In Case 2 ($\zeta \leqslant 1$) let

$$y = -x''/\zeta \text{ and } z = x'. \tag{27}$$

Then from (23)

$$\frac{1}{V(x'')} \leqslant f_\zeta(p, y) \qquad \text{and} \qquad f_\zeta(p, z) \leqslant V(x'),$$

for all $0 < p < 1$. Therefore, in either case,

$$\frac{1}{\zeta\omega} \leqslant f_\zeta(p, y) \qquad \text{and} \qquad f_\zeta(p, z) \leqslant \frac{1}{\zeta\omega}$$

for all $0 < p < 1$. Referring back to the sentence containing Eq. 17 we see that $\psi_{y,\theta}$ is subregular and $\psi_{z,\theta}$ is superregular for all $0 < \theta \leqslant \min(1, 1/\zeta)$. Q.E.D.

Though we will not need them below, we note that the proof gives simple formulas for y and z. The function V is defined by (16). The points x' and x'' are defined in terms of V by (24) and (25), and y and z are defined in terms of x' and x'' by (26) when $\zeta \geqslant 1$ and by (27) when $\zeta \leqslant 1$.

It is easy to see that the classes of superregular and subregular functions are closed under addition and multiplication by nonnegative constants. Further, the constant functions are regular, hence both superregular and subregular. For any $x > 0$, $\psi_{x,\theta}(1) > \psi_{x,\theta}(0) = 1$, therefore

$$\phi_{x,\theta}(p) = \frac{\psi_{x,\theta}(p) - 1}{\psi_{x,\theta}(1) - 1}, \tag{28}$$

is superregular or subregular if $\psi_{x,\theta}$ is. Also $\varphi_{x,\theta} \in D$, with $\varphi_{x,\theta}(0) = 0$ and $\varphi_{x,\theta}(1) = 1$. Thus, combining Lemma 9 and Lemma 5 we obtain the following theorem.

THEOREM 4. *There are positive constants $y = y(\omega, \zeta)$ and $z = z(\omega, \zeta)$ such that*

$$\varphi_{y,\theta}(p) \leqslant \gamma(p; \theta_1, \theta_2) \leqslant \varphi_{z,\theta}(p) \tag{29}$$

for all $0 < \theta \leqslant \min(1, 1/\zeta)$ and $0 \leqslant p \leqslant 1$.

COROLLARY. *For any $0 < \theta \leqslant \min(1, 1/\zeta)$ and $0 \leqslant p \leqslant 1$,*

$$\gamma(p) \leqslant 1/e^{z(1-p)/\theta}, \tag{30}$$

so that

$$\lim_{\theta \to 0} \gamma(p) = 0, \tag{31}$$

if $0 \leqslant p < 1$.

Proof. A simple calculation shows that

$$\phi_{z,\theta}(p) = \frac{1}{e^{z(1-p)/\theta}} \frac{1 - e^{-zp/\theta}}{1 - e^{-z/\theta}},$$

and the second factor on the right clearly does not exceed unity. Q.E.D.

Equation 30 suggests that when the learning rates are small the probability of being absorbed on the unfavorable side is very small.

Combining (31) with Theorem 2 we immediately obtain

THEOREM 5. *For $0 < p < 1$,*

$$\chi(p) \sim \frac{2p}{\theta_2 \pi_2 - \theta_1 \pi_1},$$

as $\theta \to 0$.

Thus when $\theta_2 \pi_2 > \theta_1 \pi_1$ (or, more generally, when $\theta_2 \pi_2 \neq \theta_1 \pi_1$) and θ is small, the mean number of alternations is of the order of magnitude of $1/\theta$, and tends to be inversely proportional to the difference $| \theta_2 \pi_2 - \theta_1 \pi_1 |$ in favorability between the two responses. When $\theta_2 \pi_2 = \theta_1 \pi_1$ and θ is small the mean number of alternations is of the order of magnitude of $1/\theta^2$ by Theorem 3. Thus if the learning rates are small we expect many more alternations when the two responses are approximately equally favorable than when one is much more favorable than the other.

5. γ AS AN ANALYTIC FUNCTION

In this section we assume that the reader is familiar with the elements of the theory of analytic functions as presented, for example, by Knopp (1945), whose terminology we follow.

In order to motivate our results, consider the case $\theta_2 = 1$, $\theta_1 < 1$. Then, since $\gamma(0) = 0$, (8) reduces to

$$\gamma(p) = \gamma(g_1(p))\pi_1 p/[\pi_1 p + \pi_2(1-p)], \tag{32}$$

where

$$g_1(p) = (1-\theta_1)p + \theta_1.$$

(We will also use the notation

$$g_2(p) = (1-\theta_2)p$$

below.) Iterating (32), and recalling that γ is continuous with $\gamma(1) = 1$, we obtain

$$\gamma(p) = A(p)/J(p), \tag{33}$$

where

$$A(p) = \prod_{n=0}^{\infty} g_{1,n}(p),$$

$$J(p) = \prod_{n=0}^{\infty} [g_{1,n}(p) + \frac{\pi_2}{\pi_1}(1 - g_{1,n}(p))],$$

$g_{1,0}(p) = p$, and $g_{1,n}$ is the nth iterate of g_1, $n \geqslant 1$. Since $g_{1,n}(p) = 1 - (1-\theta_1)^n(1-p)$, $n \geqslant 0$, the infinite products $A(p)$ and $J(p)$ converge for all complex numbers p. So the formula (33) serves to continue γ analytically into the complex plane C. If $\pi_2 = \pi_1$ then $J(p) \equiv 1$ and γ is entire. If $\pi_2 \neq \pi_1$ then γ has a pole of order 1 at each point that is a zero of one of the factors of $J(p)$. Since $\pi_1 p + \pi_2(1-p) = 0$ if and only if $p = c$ where

$$c = \pi_2/(\pi_2 - \pi_1),$$

these zeros are just the points $g_{1,n}^{-1}(c)$ that $g_{1,n}$ maps into c for some $n \geqslant 0$. Note that $c > 1$ or $c < 0$ depending on whether $\pi_2 > \pi_1$ or $\pi_2 < \pi_1$, and that the sequence c, $g_1(c), g_{1,2}(c),\ldots$ of poles is confined to $[c,\infty)$ or $(-\infty, c]$, respectively, in these two cases.

These examples and that given by Theorem 1 suggest all of the possibilities for the qualitative analytic character of γ that arise in the general case treated in Theorem 6, which summarizes our results. The theorem shows that γ is always meromorphic and occasionally entire. Whenever there are poles, c is the one closest to [0,1].

If c is a pole and its distance from [0,1] is less than 1 then there will be points x in [0,1] such that the Taylor series

$$\gamma(p) = \sum_{n=0}^{\infty} \frac{\gamma^{(n)}(x)}{n!}(p-x)^n, \tag{34}$$

about x does not converge for all $0 \leqslant p \leqslant 1$. In such cases an attempt to compute the sequence $\{\gamma^{(n)}(x)\}$ by the conventional method of substituting (34) into (8) and equating coefficients of $(p - x)^n$ on the two sides of the resulting equation seems doomed to failure. In fact, little progress has been made to date in computing these coefficients even in cases where this obstacle is not present. Theorem 6 gives some information about them. For instance, when c is a pole the standard formula for the radius of convergence yields

$$\limsup_{n\to\infty} \sqrt[n]{|\gamma^{(n)}(x)|/n!} = \frac{1}{|x - c|}$$

for $0 \leqslant x \leqslant 1$. Moreover the theorem specifies the sign of $\gamma^{(n)}(x)$ for all $n \geqslant 0$.

To state Theorem 6, some new notation will be needed. The function Δ is defined by

$$\Delta(y) = \pi_2(1 - (1 - \theta_2)^y) - \pi_1(1 - (1 - \theta_1)^y), \tag{35}$$

for $y \geqslant 1$. E is the set of all points that map into c under repeated application of g_1 and g_2, i.e.,

$$E = \{g_{i_n}^{-1}(g_{i_{n-1}}^{-1}(\cdots(g_{i_1}^{-1}(c)))) : n \geqslant 1 \quad \text{and} \quad i_j = 1 \quad \text{or} \quad 2 \quad \text{for all} \quad 1 \leqslant j \leqslant n\} \cup \{c\}. \tag{36}$$

For any real y, $[y]$ is the smallest integer greater than y. Finally, we note from (5) that U may be regarded as an operator on complex valued functions of a complex variable. We shall sometimes so regard it below.

THEOREM 6. *Suppose* $\theta_2\pi_2 > \theta_1\pi_1$ *and* $1 > \theta_1, \theta_2$.

a. *If* $\pi_2 > \pi_1$ *then* γ *can be continued analytically throughout* $C - E$. *The point* c *is a pole of order* 1 *and every other point of* E *is either a pole of order* 1 *or a regular point of* γ. *For all* $n \geqslant 1$ *and* $0 \leqslant p \leqslant 1$, $\gamma^{(n)}(p) > 0$.

b. *If* $\pi_2 = \pi_1$ *then* γ *can be continued analytically over the entire complex plane. For all* $n \geqslant 1$ *and* $0 \leqslant p \leqslant 1$,

$$0 < \gamma^{(n)}(p) \leqslant n!\, 2^n \frac{\eta^{n(n-1)/2}}{\prod_{j=1}^{n} (1 - \eta^j)}, \tag{37}$$

where $\eta = 1 - \min(\theta_1, \theta_2)$.

c. *If* $\pi_2 < \pi_1$ *then there is a unique* $x > 1$ *for which* $\Delta(x) = 0$.

i. *If* x *is an integer, then* γ *is a polynomial of degree* x, *and* $\gamma^{(n)}(p) > 0$ *for all* $1 \leqslant n \leqslant x$ *and* $0 \leqslant p \leqslant 1$.

ii. *If* x *is not an integer then* γ *can be continued analytically throughout* $C - E$. *The point* c *is a pole of order* 1, *and every point of* E *is either a pole of order* 1 *or a regular* *point of* γ. *For any* $0 \leqslant p \leqslant 1$, $\gamma^{(n)}(p) > 0$ *if* $1 \leqslant n \leqslant [x]$ *or if* $n - [x]$ *is a positive even integer, and* $\gamma^{(n)}(p) < 0$ *if* $n - [x]$ *is a positive odd integer.*

The functional equation (8) *holds for all of the extensions described above.*

The proof of Theorem 6 is fairly routine, but rather intricate. Limitations of space prevent inclusion of more than the following brief sketch.

Sketch of proof. An inequality is obtained relating the quantities $|(U^m\psi)^{(n)}|$ for a smooth function ψ. When $\psi(0) = 0$ and $\psi(1) = 1$ this inequality yields a bound for $|\gamma^{(n)}|$ on taking the limit as $m \to \infty$. When $\pi_1 = \pi_2$ this bound is the expression on the right side of (37), and it follows that γ is entire. When $\pi_1 \neq \pi_2$ this bound shows that γ can be extended to a function analytic in an oval including [0,1] and having c on its boundary. The functional equation (8) can be rewritten $W\gamma = \gamma$ where

$$W\psi(z) = \psi(g_1(z))w(z) + \psi(g_2(z))(1 - w(z)),$$

and $w(z) = \pi_1 z/(\pi_1 z + \pi_2(1 - z))$. The functions $W^n\gamma$ provide a sequence of analytic continuations of γ into regions whose union is $C - E$. From the equation $W\gamma(z) = \gamma(z)$ it can be seen that c is at worst a pole of order 1 of γ, and from this it follows that each point of E is either a pole of order 1 or a regular point. If c is regular, all are regular. To prove the assertion about the signs of the derivatives of γ, we consider, for each hypothesis about the parameters of the model, the class of nonnegative functions having derivatives with the pattern of signs indicated by the theorem, with, however, strict inequalities replaced by weak inequalities. It is shown that U preserves this class. Since I belongs to this class and $(U^mI)^{(n)}(p) \to \gamma^{(n)}(p)$ as $m \to \infty$, γ belongs to this class also. A supplementary argument using the analyticity of γ yields strict inequalities. Finally, the above results on the signs of the derivatives of γ and the entirety of γ if c is a regular point are shown to preclude regularity of c in cases (a) and (cii). Q.E.D.

Surveying the cases treated in Theorem 6 we see that in all of them $\gamma''(p) > 0$ for $0 \leqslant p \leqslant 1$. This implies that if p_1 has a distribution F with mean μ and positive variance, and if corresponding probabilities and expectations are indicated by subscript F's, then

$$\begin{aligned} P_F(\liminf A_{1,n}) &= E_F[\gamma(p_1)] \\ &\geqslant \gamma(\mu) + \min_{0 \leqslant p \leqslant 1} \gamma''(p)\, E_F[(p_1 - \mu)^2]/2 \\ &> \gamma(\mu) = P_\mu(\liminf A_{1,n}). \end{aligned}$$

Thus, if the distribution of p_1 over a group of animals has mean 1/2 and positive variance, the proportion absorbed on the unfavorable side will tend to exceed the corresponding proportion for zero variance.

Since all of the stat rats for the .75 group of Experiment 1 of Weinstock, *et al.* (1965)

had $p_1 = 1/2$, and since there is no reason to believe that this condition was met by all of the real rats, the above result may help to explain why a few more real rats than stat rats were absorbed on the unfavorable side.

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