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What Can One Learn from a Strength–Duration Experiment?

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A physical, physiological, or behavioral system, G, has a fundamentally continuous response to temporally varying stimuli, but this system is hidden behind a threshold. One can readily observe only whether or not, for a given input, the peak output exceeds this threshold. Given an input of a certain form, its intensity can be adjusted to the "critical intensity," the smallest value that achieves threshold. If the system, G, is linear and time-invariant, it is characterized by its response to an impulse. This paper addresses the problem of inferring the impulse response from critical intensity data.

1. INTRODUCTION

A wide variety of physical, physiological, and behavioral processes can be modeled by time-invariant linear systems (Schwarz & Friedland, 1965; Grodins, 1963; McFarland, 1971). Such a system, G, maps an input, x, into an output, $y = Gx$. Both x and y are functions of time, and the action of G is determined by another function, g, of time, according to the equation

$$y(t) = Gx(t) = \int_{-\infty}^{t} g(t-u)\, x(u)\, du. \tag{1.1}$$

The inputs considered in this paper all start at time 0 ($x(u) = 0$ for $u < 0$), in which case (1.1) reduces to

$$y(t) = Gx(t) = \int_{0}^{t} g(t-u)\, x(u)\, du,$$

for $t \geqslant 0$.

If x approximates a unit impulse, δ, at time 0 (equivalently, $x(u)\, du$ approximates the probability distribution, $\delta(u)\, du$, concentrated at 0), and if g is suitably regular, then $y(t)$ approximates

$$\int_{-\infty}^{t} g(t-u)\, \delta(u)\, du = g(t-0) \cdot 1 = g(t),$$

i.e., $y = Gx$ approximates g. For this reason, we write $g = G\delta$ and term g the *impulse response* of G. Equation (1.1) says that the output, $y(t)$, at time t is the sum of or "superposition" of responses, $g(t-u)$, to impulses of magnitude $x(u)\, du$ occurring at a con-

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tinuum of past times, u. Since $y(t)$ depends only on prior values of x, G is said to be *nonanticipating*.

It is clear from (1.1) that G is *linear*, i.e., $G(a_1x_1 + a_2x_2) = a_1Gx_1 + a_2Gx_2$ for inputs x_1 and x_2 and constants a_1 and a_2. If, moreover, we take $g(t) = 0$ for $t < 0$, then we can rewrite (1.1) in the form

$$y(t) = \int_{-\infty}^{\infty} g(t - u)\, x(u)\, du, \tag{1.2}$$

or, changing the variable of integration,

$$y(t) = \int_{-\infty}^{\infty} g(u)\, x(t - u)\, du. \tag{1.3}$$

It follows from (1.3) that the only effect of advancing or delaying the input by time τ is to advance or delay the output by same amount. This property of G is termed *time-invariance*.

Since a time-invariant linear system is completely determined by its impulse response, experimental determination of this function is a central problem in linear modeling. It can be found by subjecting the system to an impulse input, if this is practical, or by a technique based on the response of the system to sinusoidal inputs of various frequencies. Both of these methods presuppose that the entire output is observable. Thus neither suffices if the output is hidden behind a threshold, as in the models with which this paper is concerned.

We have in mind situations, like detection experiments, where the subject's response to an input is one of two alternatives, for example, "yes" or "no." A natural model to consider is one in which some characteristic, $c(y)$, of $y = Gx$, is compared to a threshold, K. Possibilities for $c(y)$ include

$$c_{\mathrm{p}}(y) = \sup_{t>0} y(t),$$

$$c_{\mathrm{d}}(y) = \sup_{t>0} |y(t)|,$$

and

$$c_{\mathrm{i}}(y) = \int_0^{T} y(t)^2\, dt.$$

(Readers unfamiliar with "sup" may safely read it as "max") The subject says "yes" if and only if $c(y) \geqslant K$. For $c_{\mathrm{p}}(y)$, $c_{\mathrm{d}}(y)$, or $c_{\mathrm{i}}(y)$, this response rule is a peak detector, a deviation detector, or an integrated squared deviation detector. Models of this kind, incorporating $c_{\mathrm{d}}(y)$ and $c_{\mathrm{i}}(y)$, have appeared in the literature on visual detection (Sperling, 1964; Rashbass, 1970), and we shall shortly give an example of a model for the generation of nerve impulses the incorporates $c_{\mathrm{p}}(y)$. Moreover, it is apparent that models of this type underlie much unformalized theoretical thinking (e.g., Gallistel, 1974, 1976).

This paper is concerned with the problem of finding the impulse response, g, *when the subject's response is determined by the peak detector*, $c_P(y)$.[1] An important example of such a scheme is Hill's (1936) model for the generation of nerve impulses, which was the culmination of a very extensive experimental and theoretical development to which many people contributed. A nerve is stimulated with current, x. This produces a (hypothetical) excitation, $V = V_0 + G_V x$. The departure, $V - V_0$, of V from its resting value, V_0, is the input for an accommodating threshold process, $U = U_0 + G_U(V - V_0)$. Both G_V and G_U are time-invariant linear systems with exponentially decreasing impulse response functions,

$$g_V(t) = b e^{-t/k} \tag{1.4}$$

and

$$g_U(t) = \lambda^{-1} e^{-t/\lambda},$$

where $\lambda > k$ (accommodation is slower than excitation). The nerve fires if and when the excitation "catches" the threshold, in the sense that $V(t) = U(t)$. Clearly this is equivalent to

$$V_0 + G_V x(t) = U_0 + G_U G_V x(t)$$

($G_U G_V$ represents a composite or "cascade" of G_V and G_U), or

$$Gx(t) = K,$$

where $K = U_0 - V_0$, and $G = G_V - G_U G_V$ is a time-invariant linear system, whose impulse response can be shown to be

$$g(t) = b \frac{(\lambda/k)\, e^{-t/k} - e^{-t/\lambda}}{(\lambda/k) - 1}. \tag{1.5}$$

Thus Hill's "two process" model falls within our framework. But, whereas Hill's specification of g_V and g_U leaves only three parameters of g to be determined experimentally, we shall provide an approach to determining the entire function, g, subject only to qualitative restrictions on its form.

[1] Our "unrestricted" peak detector must have access to $y(t)$ for *all* positive t in order to respond "no." A more plausible peak detector would respond "no" if and only if $y(t) < K$ for all $t \leqslant \tau$, where τ is a finite constant. It seems clear that restricted and unrestricted detectors would behave similarly in most experiments of the type we envisage. Thus our choice of an unrestricted detector, based on its mathematical simplicity, entails little sacrifice in emperical accuracy. On the other hand, an integrated squared deviation detector might well behave rather differently and yield a more appropriate model in many cases. In such cases, our techniques for determining g would be inapplicable.

2. Preliminaries

Assumptions Concerning g

We begin by listing our assumptions concerning the impulse response, g. An illustrative function consistent with these assumptions is given in Fig. 1. The assumptions about g revolve around three points, p, z, and z', satisfying $0 \leqslant p < z \leqslant z' \leqslant \infty$. We assume that $g(0) \geqslant 0$, and, if $0 < p$, then g is strictly increasing on $[0, p]$. This is termed the *upramp* of g. Absence of the upramp is equivalent to $p = 0$. On the *downramp*, $[p, z]$, g is assumed to be strictly descreasing, with $g(z) = 0$. The upramp and downramp together constitute the *positive phase* of g. If $z < z'$, g is strictly negative on (z, z'), the *negative phase*, and $g(z') = 0$. If $z' < \infty$, then g vanishes on the *null phase*, $[z', \infty)$.

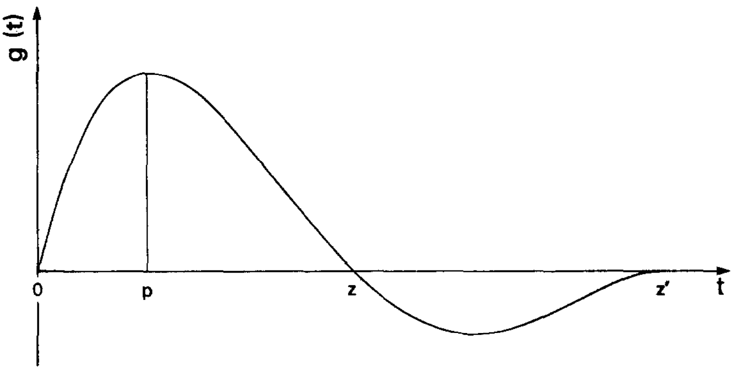

FIG. 1. An impulse response. The peak occurs at p, and (z, z') is the negative phase. For this function, $0 < p < z < z' < \infty$. Under our assumptions, all inequalities except the second may be replaced by equalities.

If $z = \infty$, then

$$\int_0^\infty g(u)\, du < \infty. \tag{2.1}$$

If $z < \infty$ and $z' = \infty$, then

$$\int_0^\infty |\, g(u) \,|\, e^{au}\, du < \infty$$

for some $a > 0$. These assumptions prevent $g(t)$ from being inconveniently large for large t. They imply that, in any case,

$$\int_0^\infty |\, g(u)|\, du < \infty.$$

Finally, we assume that g is continuous throughout $[0, \infty)$.

The impulse response functions (1.4) and (1.5), for example, satisfy all of these conditions. In both cases, $p = 0$. For (1.4), $z = \infty$, while, for (1.5), $z < z' = \infty$.

However, our assumptions exclude the important class of "oscillatory" impulse response functions, with multiple positive and negative phases.

Sketch of the Argument

The classical strength–duration experiment uses rectangular inputs, inputs whose intensity is constant throughout their duration. The experiment determines the *critical intensity*, $I_r(D)$, the intensity the input must have in order that the peak of the output reach threshold. It is easily shown that the output at any time, t, for a rectangular input of duration D, is proportional to the integral of $g(t)$ over the interval from $t - D$ to t (see Section 7). Since this proportionality holds for all t, it holds for the *critical time*, $t_r(D)$, the time at which the output reaches its peak. In other words, the peak of the response to an input of unit intensity is:

$$\int_{t_r(D)-D}^{t_r(D)} g(t)\, dt.$$

The critical intensity indicates how much this integral must be scaled up or down so as to just equal the criterion (threshold) of the peak detector, because the critical intensity is inversely proportional to this integral.

As one varies D from zero to infinity, one expands the interval over which g is integrated. As $D \to 0$, the interval becomes infinitesimal; as $D \to \infty$, the interval of integration encompasses all of g. At each increment in D, the critical intensity declines by an amount proportional to the increment, if any, in the integral. It seems at first blush that differentiating the reciprocal of the critical intensity function might yield the positive portion of g. This approach fails because one does not know the critical time corresponding to each critical intensity. Because the stength–duration experiment with rectangular inputs does not determine the critical time, no portion of the impulse response is determined by such data (*indeterminancy theorem*, Section 8).

However, our *truncation theorem* establishes a way of determining the critical times in a slightly different class of critical-intensity experiments, experiments in which the input intensity either increases or decreases continuously after the onset of the input. It seems intuitively obvious that with an input whose intensity decays slowly and continuously toward zero, one must be able to cut off some portion of the "tail" of the input without affecting the peak of the output. The truncation theorem establishes that, for either decaying or increasing inputs, truncating the input fails to affect the peak of the output *if and only if* the truncation occurs after the output has reached its peak, that is, after the critical time. Truncating the input prior to the critical time decreases the peak reached by the output, necessitating an increase in critical intensity (Section 4). This result gives a handle on the critical time.

The problem then is to choose decaying or increasing inputs felicitously, i.e., so that the resulting critical intensity functions are simply related to g and reveal information about every portion of g. Section 5 proves that the critical intensity function for an exponentially decaying input, in conjunction with a truncation experiment, determines the downramp. Section 6 proves that the critical intensity function for an increasing input that approaches its upper asymptote exponentially determines much if not all of the negative phase—again, in conjunction with a truncation experiment. Once the downramp is known, the critical intensity function for a rectangular input, i.e., the traditional strength–duration function, determines the upramp (Section 7).

The mathematical unfolding of this argument proceeds as follows. In the balance of Section 2, we deal with certain, largely notational, preliminaries, namely: (a), the development of rigorous definitions of and notations for critical intensity, critical time, and truncation; (b), the introduction of four types of input functions; (c), the asymptotic values of the critical intensity and critical time functions for these inputs; and, (d), the inability of our method to determine pure delays. Thus, Section 2 develops all of our notation and can serve as a dictionary for later reference. In Section 3 we give rigorous statements of the principal results and a summary of the methods for determining g. The remaining sections provide proofs, except for Section 9, which offers concluding remarks on the practical difficulties of applying our results.

Definitions

Critical intensity. In a typical experiment of the kind we are considering, an input, x, is chosen, and its amplitude is adjusted via a multiplicative constant, I, yielding an input, Ix. The basic measurement for x is the *critical intensity*, $I(x)$. This is the smallest I for which the output, $G(Ix) = I \times Gx$, attains threshold. Assume, without loss of generality, that the threshold is $K = 1$. Then the critical intensity, $I(x)$, satisfies

$$\sup_{t>0} I(x)\, Gx(t) = 1$$

or

$$\sup_{t>0} Gx(t) = 1/I(x). \tag{2.2}$$

We restrict our attention to bounded nonnegative inputs ($0 \leqslant x(t)$, $\sup_{t>0} x(t) < \infty$) that are positive in some interval, so that the supremum in this equation is positive and finite, as is $I(x)$.

Critical time. The output of the system described by (1.1) is continous, even if x has discontinuities. Consider first the case in which the output, Gx, attains its maximum at some finite time. By continuity, there is a first time that the maximum is attained, and we denote it $t(x)$. Thus $0 < t(x) < \infty$, and $t(x)$ *is the smallest solution of*

$$Gx(t(x)) = \sup_{t>0} Gx(t)$$

or, equivalently,

$$Gx(t(x)) = 1/I(x). \tag{2.3}$$

It is also possible that $Gx(t)$ does not attain its maximum for any finite t. In that case, the maximum can only be approached asymptotically as $t \to \infty$. More precisely, there is a sequence, t_n, such that

$$\lim_{n\to\infty} Gx(t_n) = \sup_{t>0} Gx(t),$$

and any sequence, t_n, having this property converges to infinity,

$$\lim_{n\to\infty} t_n = \infty.$$

In this case, we write $t(x) = \infty$. Hence $t(x) = \infty$ *if and only if* (2.3) *has no finite solution.* Whether or not $t(x)$ is finite, we shall refer to it as the *critical time for x*.

Truncation. An input, x, can be truncated at time $T > 0$. The truncated input, x^T, is given by

$$\begin{aligned} x^T(t) &= x(t), \qquad \text{if} \quad t \leqslant T, \\ &= 0, \qquad \text{if} \quad t > T. \end{aligned} \tag{2.4}$$

Input Functions

Our techniques employ four basic types of inputs: the unit step input,

$$x_s(t) = 1 \qquad \text{for all} \quad t \geqslant 0;$$

the rectangular input of duration $D > 0$,

$$\begin{aligned} x_r(t) &= 1, \qquad \text{if} \quad 0 \leqslant t \leqslant D, \\ &= 0, \qquad \text{if} \quad t > D; \end{aligned}$$

the exponentially decaying input with time constant D,

$$x_e(t) = e^{-t/D}, \qquad \text{for all} \quad t \geqslant 0;$$

and the exponentially increasing input with time constant D,

$$x_i = x_s - x_e .$$

The justification for using the same parameter, D, in x_r and x_e is the equality

$$\int_0^\infty x_r(t)\,dt = D = \int_0^\infty x_e(t)\,dt,$$

which shows that D can be regarded as a "duration" parameter for x_e as well as x_r. This equality also shows that the integrals of $D^{-1}x_r$ and $D^{-1}x_e$ are unity. These functions behave like unit impulses when D is small,

$$\lim_{D\to 0} D^{-1}x_r(t) = \lim_{D\to 0} D^{-1}x_e(t) = \delta(t). \tag{2.5a}$$

In addition, we note that

$$\lim_{D\to\infty} x_r(t) = \lim_{D\to\infty} x_e(t) = x_s(t), \tag{2.5b}$$

and, for $t > 0$,

$$\lim_{D\to 0} x_i(t) = x_s(t). \tag{2.6}$$

We employ the special notation, $I_r(D)$ and $t_r(D)$, for the critical intensity, $I(x_r)$, and the critical time, $t(x_r)$, corresponding to the input, x_r. Similar notation is used in connection with x_e and x_i. Thus

$$I_r(D) = I(x_r),\ t_r(D) = t(x_r), \tag{2.7}$$

$$I_e(D) = I(x_e),\ t_e(D) = t(x_e), \tag{2.8}$$

$$I_i(D) = I(x_i),\ t_i(D) = t(x_i). \tag{2.9}$$

Asymptotes at $D = 0$ and $D = \infty$

The limits in (2.5a), (2.5b), and (2.6) suggest simple expressions for asymptotic values of the critical intensity and critical time functions in (2.7), (2.8), and (2.9). If $f(D)$ is a function (like I_r or t_r) defined originally for $0 < D < \infty$, let $f(0)$ and $f(\infty)$ be defined by

$$f(0) = \lim_{D\to 0} f(D)$$

and

$$f(\infty) = \lim_{D\to\infty} f(D),$$

provided that these limits are well defined (though, perhaps, infinite). Then (2.5b) and (2.6) suggest that

$$I(x_s) = I_r(\infty) = I_e(\infty) = I_i(0) \tag{2.10}$$

and

$$t(x_s) = t_r(\infty) = t_e(\infty) = t_i(0). \tag{2.11}$$

As we shall see in Sections 5–7, all of these equalities are valid without qualification, except for the last one in (2.11), which holds unless $z = z' < \infty$ (in which case $t(x_s) = z < \infty = t_i(\infty)$). Moreover,

$$Gx_s(t) = \int_0^t g(u)\, du, \qquad \text{for} \quad t \geqslant 0,$$

by (1.3), so $t(x_s) = z$ and

$$\frac{1}{I(x_s)} = \int_0^z g(u)\, du.$$

Combining these results with (2.10) and (2.11), we obtain

$$\frac{1}{I_r(\infty)} = \frac{1}{I_e(\infty)} = \frac{1}{I_i(0)} = \int_0^z g(u)\, du \tag{2.12}$$

and

$$t_r(\infty) = t_e(\infty) = t_l(0) = z, \tag{2.13}$$

with the understanding that $t_l(0)$ must be dropped from (2.13) if $z = z' < \infty$.

In view of (2.5a), we expect the peak value of $G(D^{-1}x_r)$ and the time when this peak is attained to converge to the corresponding characteristics, $g(p)$ and p, of $G\delta = g$. The peak value of $G(D^{-1}x_r) = D^{-1}Gx_r$ is $D^{-1}I_r(D)^{-1}$, and it is attained at $t_r(D)$. The same argument applies to x_e. Hence

$$\lim_{D\to 0} \frac{1}{DI_r(D)} = \lim_{D\to 0} \frac{1}{DI_e(D)} = g(p) \tag{2.14}$$

and

$$t_r(0) = t_e(0) = p. \tag{2.15}$$

Detailed proofs of most of these results will be given later.

Insentivity to Pure Delay

All of our techniques for determining g are based on measurements of critical intensity. We make no use of direct measurements of reaction time. Thus it is not surprising that our techniques are insensitive to "transmission times" or other pure delays. These would be reflected in temporal displacements of the impulse response, g, which, under our assumptions, starts at time zero ($g(t) > 0$ for $0 < t < z$). It is evident from (1.2) that if such a function were displaced to the right by τ, then the ontput, y, would be similarly displaced, the supremum of $y(t)$ over $t > 0$ would be unaffected, and thus, by (2.2), $I(x)$ would be unaffected. Consequently, the "zero-delay" impulse response determined by our procedures may differ from the "true" impulse response by a temporal displacement.

3. Overview

In this section we survey our main findings. Proofs appear in subsequent sections.

Fundamental Formulas

Most of the impulse response can be derived from the following formulas:

$$g(t_e(D)) = \frac{1}{DI_e(D)}, \qquad \text{for all } D; \tag{3.1}$$

$$g(t_l(D)) = \frac{d}{dt_l(D)}\left(\frac{1}{I_l(D)}\right), \qquad 0 < D < \alpha; \tag{3.2}$$

$$g(t_r(D)) = \frac{d}{dD}\left(\frac{1}{I_r(D)}\right), \qquad \text{for all } D; \tag{3.3}$$

$$g(t_r(D) - D) = g(t_r(D)), \qquad 0 < D < \delta; \tag{3.4}$$

where α and δ will be defined momentarily.[2] The critical times, $t_e(D)$ and $t_r(D)$, in (3.1), (3.3), and (3.4) fall on the downramp of g; $t_r(D) - D$ in (3.4) is on the upramp; and $t_i(D)$ in (3.2) falls in the negative phase, if there is one.

Truncation

Formulas (3.1)–(3.4) can be used to determine portions of g, provided that the critical times can be found. The following result permits $t_e(D)$ and $t_i(D)$ to be obtained by truncating the corresponding inputs, x_e and x_i.

TRUNCATION THEOREM. *Suppose that $x(t) > 0$ for all $t > 0$. Then truncation of x decreases peak output (hence increases critical intensity) if and only if the truncation point, T, is below the critical time, $t(x)$. In other words,*

$$I(x^T) = I(x), \qquad \text{if} \quad T \geqslant t(x), \tag{3.5}$$

$$I(x^T) > I(x), \qquad \text{if} \quad T < t(x). \tag{3.6}$$

(If $t(x) = \infty$, (3.5) is vacuous. For x^T, see (2.4).)

Thus, for example, $t_e(D)$ is the value of T at which $I((x_e)^T)$ attains its asymptote. Note that this determination of $t_e(D)$ requires only critical intensity data. The theorem also applies to $t_i(D)$, but not to $t_r(D)$, since x_r does not satisfy the strict positivity assumption ($x_r(t) = 0$ for $t > D$).

Determining g: A Brief Summary

We have now introduced all of the tools required for determining g. The main points of our technique may be described as follows.

(a) Negative phase. This phase is determined by experiments with increasing inputs. Such experiments yield critical intensities, $I_i(D)$, and, via truncation, critical times, $t_i(D)$. These quantities, in conjunction with (3.2), determine either the entire negative phase, or, at least, its leftmost (small t) portion.

(b) Downramp. This phase is determined by experiments with exponentially decreasing inputs, which yield values of $I_e(D)$ and $t_e(D)$. These quantities, together with (3.1), determine the downramp of g.

(c) Upramp. First the downramp is found as in (b), and the classical strength–duration curve, $I_r(D)$, is obtained from experiments with rectangular inputs. The strength-duration curve determines $g(t_r(D))$ via (3.3). Since $t_r(D)$ falls on the downramp, which has already been determined, $t_r(D)$ (and thus $t_r(D) - D$) can be inferred from $g(t_r(D))$. But $t_r(D) - D$ is on the upramp so the upramp can now be calculated from (3.4). (This complex approach to determining the upramp is necessitated by the fact that $t_r(D)$ cannot be found by truncation.)

[2] This "δ" has nothing to do with the unit impulse function. No confusion should arise from our use of the same letter for both.

We now consider in more detail the roles of exponential, increasing, and rectangular inputs in the determination of g.

Exponentially Decreasing Inputs

The critical time function, $t_e(D)$, maps $(0, \infty)$ continuously into (p, z), and, as we have already noted in (2.13) and (2.15), $t_e(0) = p$ and $t_e(\infty) = z$. Thus $t_e(D)$ takes on all possible values between p and z, and the entire downramp of g is determined by (3.1). Thus we may henceforth regard the downramp of g (and, in particular, its endpoints, p and z) as "known."

Increasing Inputs

As we have noted, the increasing input, x_i, is used to determine the negative phase of g. Let A^+ and A^- be the areas of the positive and negative phases of g,

$$A^+ = \int_0^z g(u)\, du,$$

$$A^- = \int_z^{z} |\, g(u)|\, du,$$

and note that

$$A^+ - A^- = \int_0^\infty g(u)\, du.$$

It will emerge from the ensuing discussion that we can determine the entire negative phase if $A^+ \geqslant A^-$. If, on the other hand, $A^+ < A^-$, we can determine the portion up to the point, $Z(=t_i(\infty))$ that cuts off area A^+ of the negative phase,

$$\int_z^Z |\, g(u)|\, du = A^+.$$

There are only three possibilities for the gross behavior of I_i: (1) I_i is constant, (2) there is an $\alpha \in (0, \infty)$ such that $I_i(D)$ is strictly increasing for $D \leqslant \alpha$ and constant for $D \geqslant \alpha$, and (3) $I_i(D)$ is strictly increasing for all D (in which case we take $\alpha = \infty$). Case 1 occurs if and only if there is no negative phase ($z = z'$). In Cases 2 and 3, $t_i(0) = z$, $t_i(D)$ is continuous and strictly increasing for $D < \alpha$, and (3.2) applies for $D < \alpha$. Moreover, $t_i(D)$ converges to $t_i(\alpha)$ as D converges to α from below. Since $t_i(D)$ can be found by truncation, (3.2) determines the portion of g between z and $t_i(\alpha)$. In Case 2, $t_i(\alpha) = z'$, so this portion is the entire negative phase. Moreover, knowledge of $z' = t_i(\alpha)$ is tantamount to knowledge of the null phase. Finally, the maximum value, $I_i(\alpha)$, of I_i is finite and satisfies

$$\frac{1}{I_i(\alpha)} = \int_0^\infty g(u)\, du. \tag{3.7}$$

Since $0 < I_i(\alpha) < \infty$, this shows that $A^+ > A^-$ in Case 2.

In Case 3 ($\alpha = \infty$), $I_i(\infty) = \infty$, $z < t_i(\infty) \leqslant z'$, and

$$\int_0^{t_i(\infty)} g(u)\, du = 0, \tag{3.8}$$

which implies that $\int_0^\infty g(u)\, du \leqslant 0$, so $A^+ \leqslant A^-$. Letting D approach ∞ in (3.2), we find that

$$g(t_i(\infty)) = \lim_{D\to\infty} \frac{d}{dt_i(D)} \left(\frac{1}{I_i(D)}\right). \tag{3.9}$$

Since $z < t_i(\infty) \leqslant z'$, this quantity is at most zero. If $g(t_i(\infty)) = 0$ (e.g., if $t_i(\infty) = \infty$) then $t_i(\infty) = z'$, so that (3.2) determines the entire negative phase of g. Also, $\int_0^\infty g(u) du = 0$ by (3.8), so (3.7) applies with $I_i(\infty) = \infty$, and $A^+ = A^-$. This situation seems unlikely, a priori, but Hill's impulse response function, (1.5), is of this type. If $g(t_i(\infty)) < 0$, then $t_i(\infty) < z'$. Thus, by (3.8), $A^+ < A^-$, and (3.2) determines the negative phase of g only up to the point, $t_i(\infty)$, at which the negative phase achieves the same area as the entire positive phase.

Rectangular Inputs

We note first that the peak value, $g(p)$; the boundary, z, between the positive and negative phases; and the area, $\int_0^z g(u)\, du$, of the positive phase, can all be deduced directly from the classical strength–duration function, I_r. In fact,

$$g(p) = \lim_{D\to 0} \frac{1}{DI_r(D)}; \tag{3.10}$$

I_r is continuous and hits its asymptote at z, in the sense that $I_r(D)$ is strictly decreasing for $D < z$ and (if $z < \infty$) constant for $D \geqslant z$; and, finally,

$$\int_0^z g(u)\, du = 1/I_r(z). \tag{3.11}$$

Equation (3.10) is contained in (2.14), and, since $I_r(z) = I_r(\infty)$, (3.11) follows from (2.12). In those cases to which (3.7) applies, (3.7) and (3.11) can be used to calculate the area of the negative phase,

$$\int_z^{z'} |g(u)|\, du = \frac{1}{I_r(z)} - \frac{1}{I_i(\alpha)}. \tag{3.12}$$

Similarly, once the downramp of g has been found, the area of the upramp can be calculated from

$$\int_0^p g(u)\, du = \frac{1}{I_r(z)} - \int_p^z g(u)\, du. \tag{3.13}$$

(In view of (2.12), $I_r(z) = I_r(\infty)$ can be replaced by $I_e(\infty)$ or $I_i(0)$ in (3.11), (3.12), or (3.13).)

We have noted that rectangular inputs can also be used to determine the upramp of g. Recall that (3.3) cannot be used to find the downramp of g, since $t_r(D)$, which falls on the downramp, cannot be found by truncation. But, once the downramp of g is known (from experiments with exponentially decreasing inputs), (3.3) can be used to determine $t_r(D)$, since g is strictly monotonic over the downramp. This function, and (3.4) can then be used to find the upramp of g, as we shall see momentarily. Either (1) $t_r(D) = D$ for all $D < z$, or (2) there is a $\delta \in (0, z]$ such that $t_r(D) > D$ for $D < \delta$ and (if $\delta < z$) $t_r(D) = D$ for $\delta \leqslant D < z$. The first case arises if and only if there is no upramp ($p = 0$), hence we restrict our attention to the second case. For $D < \delta$, we have

$$0 < t_r(D) - D < p < t_r(D) < \delta.$$

Since $g(t_r(D))$ is determined by (3.3), $g(t_r(D) - D)$, which is on the upramp of g, is determined by (3.4). Moreover the entire upramp of g can be found in this manner, for $t_r(D) - D$ decreases continuously from

$$p = \lim_{D \to 0} t_r(D) - D$$

(see (2.15)) to

$$0 = \lim_{D \to \delta} t_r(D) - D \tag{3.14}$$

as D goes from 0 to δ. (Letting $D \to \delta$ in (3.4) and using (3.14), we see that $g(0) = g(\delta)$. This equation and $p < \delta \leqslant z$ characterize δ.) In Section 5 we will sketch another method of determining the upramp, using only t_e and I_e.

Indeterminacy

The reader may be surprised that the classical strength–duration curve, I_r, plays such a limited role in our theory. One might expect this function, by itself, to determine much of g. The following theorem shows that this is not the case.

INDETERMINACY THEOREM. *Let g be any impulse response function (satisfying our assumptions), let z be the corresponding boundary between its positive and negative phases, and let p^* be any point in $[0, z)$. (Thus p^* may be on either the upramp or downramp of g.) Then there is an impulse response function, g^*, that achieves its maximum at p^*, has no negative phase, and predicts the same strength-duration curve, I_r, as g.*

Summary

The results presented in this section provide an answer to the question posed in the title, at least as it pertains to determining the linear component of the threshold model. The situation can be summarized as follows. The classical strength–duration experiment provides little information about the impulse response function, g. However, when this information is supplemented by critical intensity information from a number of similar experiments, a nearly complete picture of g emerges.

The next five sections provide proofs of most of the results presented in this section. The reader may skip directly to the concluding section (Section 9) without loss of continuity.

4. Proof of the Truncation Theorem

Since $x^T(u) = x(u)$ for $u \leqslant T$, and since G is nonanticipating,

$$Gx^T(t) = Gx(t), \qquad \text{for} \quad t \leqslant T. \tag{4.1}$$

Our proof of the Truncation Theorem is based on (4.1) and on the following inequalities:

$$Gx^T(t) < Gx(t), \qquad \text{for} \quad T < t < T + z, \tag{4.2}$$

and

$$Gx^T(t) \leqslant 0, \qquad \text{for} \quad T + z \leqslant t. \tag{4.3}$$

Let us assume (4.2) and (4.3) for the moment. We will prove them at the end of the section.

It follows immediately from (4.1), (4.2), and (4.3) that

$$\sup Gx^T \leqslant \sup Gx, \qquad \text{for all } T, \tag{4.4}$$

where "sup y" is an abbreviation for $\sup_{t>0} y(t)$. If $T \geqslant t(x)$, then we take $t = t(x)$ in (4.1) to obtain

$$Gx^T(t(x)) = \sup Gx,$$

hence

$$\sup Gx^T \geqslant \sup Gx.$$

In combination with (4.4), this yields

$$\sup Gx^T = \sup Gx, \qquad \text{if} \quad T \geqslant t(x),$$

which is equivalent to (3.5).

Suppose now that $T < t(x)$. There are, a priori, three possibilities for $t(x^T)$: (1) $t(x^T) \leqslant T$, (2) $T < t(x^T) < T + z$, and (3) $t(x^T) \geqslant T + z$. The third is ruled out by (4.3). If the second obtains, then, taking $t = t(x^T)$ in (4.2), we find that

$$\sup Gx^T < \sup Gx. \tag{4.5}$$

Finally, in the first case, we may take $t = t(x^T)$ in (4.1) to obtain

$$\sup Gx^T = Gx(t(x^T)). \tag{4.6}$$

But $t(x^T) \leqslant T$, and we are assuming that $T < t(x)$, hence $t(x^T) < t(x)$. Since $t(x)$ is the first point at which Gx hits its peak value,

$$Gx(t(x^T)) < \sup Gx,$$

which, in combination with (4.6), yields (4.5). Therefore (4.5) holds whenever $T < t(x)$, and this is equivalent to (3.6).

It remains only to establish (4.2) and (4.3). Note first that, by (1.3),

$$Gx(t) = \int_0^t g(u)\, x(t-u)\, du, \qquad \text{for} \quad t \geqslant 0 \tag{4.7}$$

(assuming, as usual, that $g(t) = x(t) = 0$ for $t < 0$), and

$$Gx^T(t) = \int_{t-T}^t g(u)\, x(t-u)\, du, \qquad \text{for} \quad t \geqslant T, \tag{4.8}$$

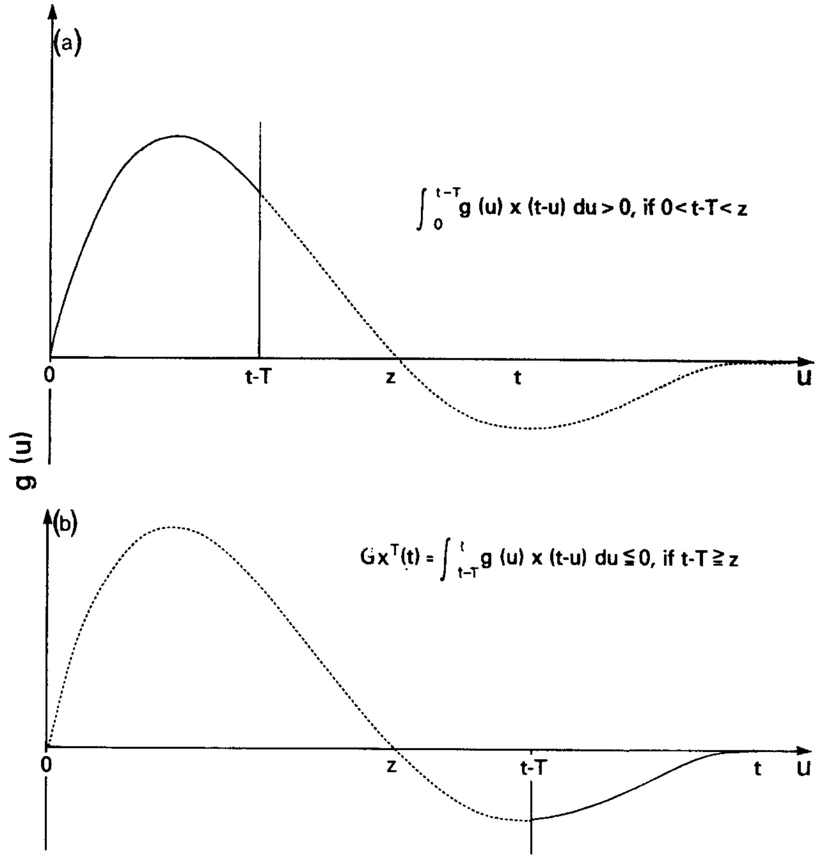

FIG. 2. Graphs of $g(u)$. Values that contribute to the integrals in (4.9) and (4.8) are indicated by solid curves in Panels (a) and (b), respectively.

since $0 \leqslant t - u \leqslant T$ if and only if $t - T \leqslant u \leqslant t$. For $t \geqslant T$, we can divide the range of integration in (4.7) at $u = t - T$ to obtain

$$Gx(t) = \int_0^{t-T} g(u)\, x(t-u)\, du + Gx^T(t). \tag{4.9}$$

If $T < t < T + z$, then $0 < t - T < z$, as in Panel (a) of Fig. 2. The values of g that contribute to the integral in (4.9) are indicated by a solid curve. These values are positive, as is $g(u)\, x(t-u)$, so the integral is positive, as (4.2) asserts. If $t \geqslant T + z$, then $t - T \geqslant z$ as in Panel (b) of Fig. 2. The values of g that contribute to the integral in (4.8) are indicated by a solid curve. These values are nonpositive, so the integral in (4.8) is nonpositive, as claimed in (4.3).

5. Exponentially Decreasing Inputs

Taking $x = x_e$ in (4.7), we obtain

$$Gx_e(t) = \int_0^t g(u)\, x_e(t-u)\, du, \tag{5.1}$$

or

$$Gx_e(t) = e^{-t/D} \int_0^t g(u)\, e^{u/D}\, du. \tag{5.2}$$

It follows that

$$(Gx_e)'(t) = g(t) - D^{-1} Gx_e(t), \tag{5.3}$$

for all $t > 0$. But $\int_0^\infty |g(u)|\, du < \infty$, $|x_e(t-u)| \leqslant 1$, and $x_e(t-u) \to 0$ as $t \to \infty$ for each $u > 0$; hence, (5.1) implies $Gx_e(\infty) = 0$. Consequently, $t_e(D) < \infty$. Therefore $(Gx_e)'$ vanishes at $t_e(D)$, and, in view of (2.3), (5.3) yields

$$g(t_e(D)) = 1/(DI_e(D)), \tag{5.4}$$

which agrees with (3.1).

Since the right-hand side of (5.4) is clearly positive, this equation implies that $t_e(D) < z$. Moreover, for $0 < t \leqslant p$, it can be seen from (5.1) that

$$\begin{aligned} Gx_e(t) &< g(t) \int_0^t x_e(t-u)\, du \\ &< Dg(t); \end{aligned}$$

hence, by (5.3), $(Gx_e)'(t) > 0$. Consequently, $t_e(D) > p$. In summary,

$$p < t_e(D) < z. \tag{5.5}$$

We shall now show that I_e is continuous,

$$\lim_{D\to 0} 1/(DI_e(D)) = g(p) \tag{5.6}$$

(as asserted in (2.14)), and

$$\lim_{D\to\infty} 1/(DI_e(D)) = 0. \tag{5.7}$$

These results, in combination with (5.4), (5.5), and the continuity and monotonicity of g on $[p, z]$, show that t_e is continuous, $t_e(0) = p$, and $t_e(\infty) = z$. Thus $t_e(D)$ attains all values between p and z.

Let $x_e(t) = e^{-t/D}$ and $x^*(t) = e^{-t/D^*}$. As $D \to D^*$, $x_e(t) \to x^*(t)$, uniformly over t. It follows that $Gx_e(t) \to Gx^*(t)$ uniformly over t, sup $Gx_e \to$ sup Gx^*, and $I_e(D) \to I_e(D^*)$. Thus I_e is continuous.

To prove (5.6), we note first that

$$D^{-1}Gx_e(t) < g(p)$$

for all t, hence

$$1/(DI_e(D)) < g(p) \tag{5.8}$$

for all D. But, in view of (2.5a), $D^{-1}Gx_e(t) \to G\delta(t) = g(t)$ as $D \to 0$, if $t > 0$. Consequently,

$$\liminf_{D\to 0} 1/(DI_e(D)) \geqslant g(t)$$

for all $t > 0$, so that

$$\liminf_{D\to 0} 1/(DI_e(D)) \geqslant g(p).$$

This inequality and (5.8) yield (5.6).

To prove (5.7), we need only observe that

$$\begin{aligned} 1/I_e(D) &= Gx_e(t_e(D)) \\ &\leqslant \int_0^z g(u)\,du, \end{aligned} \tag{5.9}$$

since $t_e(D) < z$ and $x_e(t-u) \leqslant 1$, and the integral on the right is finite. (It is only a short step from (5.9) to the equality,

$$\frac{1}{I_e(\infty)} = \int_0^z g(u)\,du,$$

given in (2.12).)

Evaluating (5.2) at $t = t_e(D)$, using (2.3), and splitting the integral into two parts, we obtain

$$\int_0^p g(u)\,e^{u/D}\,du = e^{t_e(D)/D}/I_e(D) - \int_p^{t_e(D)} g(u)\,e^{u/D}\,du.$$

Once the downramp of g has been determined, the right-hand side involves only known quantities, hence the Laplace-transform-like function of D^{-1} on the left is also known. This function determines the upramp of g, which can, in principle, be calculated via minor modifications of Laplace inversion techniques. Thus I_e and t_e determine the entire positive phase of g.

All of our assertions concerning exponentially decreasing inputs have now been proved. In closing this section, we mention the following open problem. Determine whether or not t_e must be an increasing function under our assumptions on g.

6. Increasing Inputs

We begin this section with the observation that it is necessary to go outside the class of nonincreasing inputs to get information about the negative phase of g. To see this, assume $z < \infty$ and consider

$$Gx(t) = \int_0^z g(u)\, x(t-u)\, du + \int_z^t g(u)\, x(t-u)\, du$$

for $t > z$. If x is nonincreasing, the first term is at most $Gx(z)$, while the second is at most zero, so $Gx(t) \leqslant Gx(z)$. This implies $t(x) \leqslant z$, hence

$$\frac{1}{I(x)} = \max_{t \leqslant z} Gx(t).$$

Thus, if x is nonincreasing, $I(x)$ depends only on the positive phase of g. Moreover x^T is also nonincreasing, so the same conclusion applies to $I(x^T)$. Therefore none of our critical intensity measurements depend on the negative phase, if x is nonincreasing.

Since $x_i = x_s - x_e$, and $(Gx_s)' = g$, it follows from (5.3) that

$$(Gx_i)'(t) = D^{-1}(Gx_e)(t). \tag{6.1}$$

Hence

$$Gx_e(t_i(D)) = 0, \tag{6.2}$$

if $t_i(D) < \infty$. If $t_i(D) = \infty$, (6.2) states that $Gx_e(\infty) = 0$, which is also valid. Consequently,

$$Gx_i(t_i(D)) = Gx_s(t_i(D)),$$

or

$$\int_0^{t_i(D)} g(u)\, du = \frac{1}{I_i(D)}. \tag{6.3}$$

Equation (3.2) is obtained by differentiating (6.3) with respect to $t_i(D)$ over an appropriate range of values of D.

Observe that (5.2) can be written in the form

$$Gx_e(t) = e^{-bt} Y(t, b), \tag{6.4}$$

where $b = D^{-1}$ and

$$Y(t, b) = \int_0^t g(u)\, e^{bu}\, du.$$

Hence $Gx_e(t)$ and $(Gx_i)'(t)$ are positive, negative, or zero, depending on whether $Y(t, b)$ is positive, negative, or zero. Moreover $Y(t, b)$ is strictly increasing for $0 \leqslant t \leqslant z$, strictly decreasing for $z \leqslant t \leqslant z'$, and constant for $t \geqslant z'$. Since $Y(0, b) = 0$, either $Y(t, b) > 0$ for all $t > 0$, or $Y(t, b) = 0$ for some $t > 0$. In the former case, it is clear from (6.4) and (6.1) that $t_i(D) = \infty$. In the latter case, let $\tau(b)$ be the smallest positive zero of $Y(t, b)$. Then $Y(t, b) > 0$ for $t < \tau(b)$, and $Y(t, b) \leqslant 0$ for $t \geqslant \tau(b)$. It then follows from (6.4) and (6.1) that $t_i(D) = \tau(b)$. In other words, $t_i(D) < \infty$ if and only if $Y(t, b) = 0$ for some $t > 0$, and, in that case,

$$\begin{aligned} Y(t, b) &> 0, \quad &&\text{for} \quad t < t_i(D) \\ &= 0, &&\text{for} \quad t = t_i(D) \\ &\leqslant 0, &&\text{for} \quad t > t_i(D). \end{aligned} \tag{6.5}$$

The same statement holds if $Y(t, b)$ is replaced by $Gx_e(t)$ or $(Gx_i)'(t)$. It follows immediately from (6.5) that

$$z < t_i(D) \leqslant z', \qquad \text{if} \quad t_i(D) < \infty. \tag{6.6}$$

In order to establish the monotonicity of t_i, we need the following simple inequality for $Y(t, b)$ when $z < \infty$:

$$e^{-zb_0} Y(t, b_0) < e^{-zb_1} Y(t, b_1), \tag{6.7}$$

for all t, if $b_0 > b_1$ (or $D_0 < D_1$, where $b_i = D_i^{-1}$). To prove this, we note that

$$\begin{aligned} \frac{\partial}{\partial b} e^{-zb} Y(t, b) &= -ze^{-zb} Y(t, b) + e^{-zb} \frac{\partial}{\partial b} Y(t, b) \\ &= -e^{-zb} \int_0^t g(u)(z - u)\, e^{bu}\, du. \end{aligned}$$

The integral is positive, since $g(u)(z - u) > 0$ for $0 < u < z$ and $z < u < z'$, and (6.7) follows.

We now show that, if $D_0 < D_1$, then either (a) $t_i(D_0) = t_i(D_1) = \infty$, or (b) $t_i(D_0) < t_i(D_1)$. For if $t_i(D_0) < \infty$, then (6.5) and (6.7) imply $Y(t_i(D_0), b_1) > 0$, so, by (6.5), $t_i(D_0) < t_i(D_1)$. Similarly, if $t_i(D_1) < \infty$, then (6.7) yields $Y(t_i(D_1), b_0) < 0$; hence, by (6.5), $t_i(D_1) > t_i(D_0)$.

In view of the monotonicity of t_i, there are three possible cases: (1) $t_i(D) = \infty$ for all D, (2) there is an $\alpha \in (0, \infty)$ such that $t_i(D) < \infty$ for $D < \alpha$ and $t_i(D) = \infty$ for $D > \alpha$, or (3) $t_i(D) < \infty$ for all D, in which case we take $\alpha = \infty$. In Case 1, (6.3) shows that I_i is constant. In Case 2, $I_i(D)$ is constant for $D > \alpha$, and, by (6.3) and (6.6), strictly increasing for $D < \alpha$. Finally, in Case 3, I_i is strictly increasing throughout $(0, \infty)$.

We omit the proof that $t_i(D)$ is continuous over $D < \alpha$ in Cases 2 and 3. It follows that (6.3) can be differentiated with respect to $t_i(D)$ for $D < \alpha$ to obtain (3.2).

Next we show that

$$\frac{1}{I_i(0)} = \int_0^z g(u)\, du, \tag{6.8}$$

as indicated in (2.12). If $z = z'$, then $Y(t, b) > 0$ for all $t > 0$ and $b > 0$, so $t_i(D) = \infty$ for all $D > 0$, and (6.8) follows trivially from (6.3). Suppose, then, that $z < z'$, so that $z < \infty$. Clearly $Gx_e(z) \to 0$ as $D \to 0$, hence $Gx_i(z) \to Gx_s(z)$. Therefore

$$\liminf_{D\to 0} \sup Gx_i \geqslant \int_0^z g(u)\, du.$$

But

$$\sup Gx_i \leqslant \int_0^z g(u)\, du$$

for all D, so (6.8) holds.

We have just noted that, if $z = z'$, then $t_i(D) = \infty$ for all D. Conversely, if $t_i(D) = \infty$ for all D, (6.3) and (6.8) imply that

$$\int_0^\infty g(u)\, du = \int_0^z g(u)\, du,$$

from which it follows that $z = z'$. Thus $t_i(D) = \infty$ for all D (Case 1) if and only if $z = z'$.

Consequently, if $z < z'$, then $t_i(0) < \infty$; indeed, in view of (6.6), $z \leqslant t_i(0) < z'$. By (6.3) and (6.8),

$$\int_0^{t_i(0)} g(u)\, du = \int_0^z g(u)\, du;$$

hence

$$t_i(0) = z, \qquad \text{if} \quad z < z' \quad (\text{or } z = z' = \infty).$$

Let $t_i(\alpha^-)$ be the limit of $t_i(D)$ as D approaches α from the left. We next show that

$$t_i(\alpha^-) = t_i(\alpha) = z', \qquad \text{if} \quad 0 < \alpha < \infty \quad (\text{Case 2}). \tag{6.9}$$

(Equation (3.7) follows from $t_i(\alpha) = z'$ and (6.3).) Clearly $t_i(\alpha^-) \leqslant t_i(\alpha)$, and, by (6.6), $t_i(\alpha^-) \leqslant z'$; thus (6.9) certainly holds if $t_i(\alpha^-) = \infty$. Suppose, then, that $t_i(\alpha^-) < \infty$. If $D < \alpha$, then

$$Y(t_i(D), D^{-1}) = 0. \tag{6.10}$$

Letting $D \to \alpha$, we obtain

$$Y(t_i(\alpha^-), \alpha^{-1}) = 0,$$

hence $t_i(\alpha^-) \geqslant t_i(\alpha)$, so that $t_i(\alpha^-) = t_i(\alpha)$. But $Y(t, D^{-1}) > 0$ for all t if $D > \alpha$, so $Y(t, \alpha^{-1}) \geqslant 0$ for all t. Therefore, $Y(t, \alpha^{-1}) = 0$ for $t \geqslant t_i(\alpha)$. Differentiating this equality, we find that $g(t) = 0$ for $t \geqslant t_i(\alpha)$. This implies $t_i(\alpha) \geqslant z'$, which completes the proof of (6.9).

If, in Case 3 ($\alpha = \infty$), we let $D \to \infty$ in (6.10) we obtain (3.8), which, in conjunction with (6.3), yields $I_i(\infty) = \infty$. It follows from (6.6) that $z < t_i(\infty) \leqslant z'$ in Case 3, and our other assertions concerning increasing inputs are straightforward consequences of these inequalities, (3.8), and (3.9).

7. Rectangular Inputs

As a consequence of (4.7),

$$Gx_r(t) = \int_0^t g(u)\, du, \qquad \text{for} \quad t \leqslant D, \tag{7.1}$$

and

$$Gx_r(t) = \int_{t-D}^t g(u)\, du, \qquad \text{for} \quad t \geqslant D. \tag{7.2}$$

Thus

$$(Gx_r)'(t) = g(t) - g(t - D), \qquad \text{for} \quad t > D. \tag{7.3}$$

We shall now give a thorough analysis of t_r. Suppose, first, that $z < \infty$ and $D \geqslant z$. By (7.1), $Gx_r(t) < Gx_r(z)$ if $t < z$ and $Gx_r(t) \leqslant Gx_r(z)$ if $z \leqslant t \leqslant D$, while (7.2) implies that $Gx_r(t) \leqslant Gx_r(z)$ for $t \geqslant D$. Therefore

$$t_r(D) = z \qquad \text{if} \quad D \geqslant z. \tag{7.4}$$

If $z = \infty$, (7.4) says that $t_r(\infty) = \infty$, which follows from (7.5) below.

Suppose, until further notice, that $D < z$. We first show that

$$D \leqslant t_r(D) < z, \qquad \text{if} \quad D < z. \tag{7.5}$$

For $Gx_r(t) < Gx_r(D)$, if $t < D$, by (7.1), so $t_r(D) \geqslant D$. If $t \geqslant z + D$, then $Gx_r(t) \leqslant 0$ by (7.2). If $z \leqslant t < z + D$, we have $0 < t - D < z$, so $g(t) \leqslant 0$ and $g(t - D) > 0$. Thus $(Gx_r)'(t) < 0$, by (7.3). Therefore, if $D < z < \infty$, then $t_r(D) < z$. Since $Gx_r(\infty) = 0$, $t_r(D) < z$ is obvious if $z = \infty$. This completes the proof of (7.5).

There is a unique point, $\delta \in [p, z]$, such that $g(\delta) = g(0)$. See Fig. 3. Clearly $\delta = p$ if and only if $p = 0$, and $\delta = z$ if and only if $g(0) = 0$ ($\delta = z = \infty$ is a possibility). If $\delta \leqslant D < t < z$, it can be seen from Fig. 3 that $g(t) < g(t - D)$. Hence, by (7.3) and (7.5),

$$t_r(D) = D \qquad \text{for} \quad \delta \leqslant D < z. \tag{7.6}$$

If $D < \delta$ (hence $\delta > p > 0$), Fig. 3 shows that there is a unique point, $\tau \in (D, z)$, such

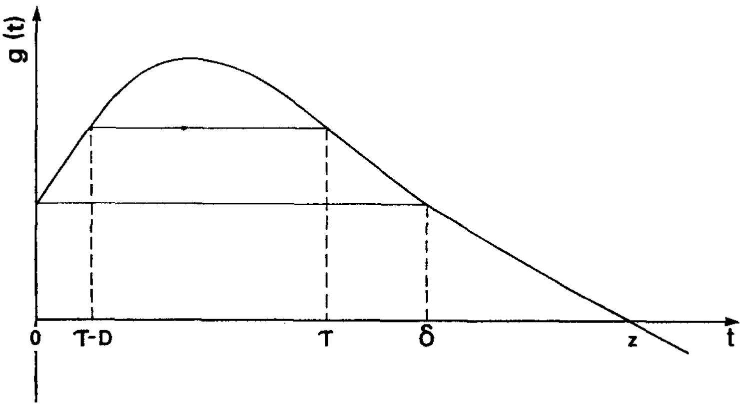

FIG. 3. The positive phase of g. By definition, $g(0) = g(\delta)$, and, for $D < \delta$, $g(\tau - D) = g(\tau)$. It is shown in the text that $\tau = t_r(D)$. If $D \geqslant \delta$, and $z > t > D$, then $g(t - D) > g(t)$.

that $g(\tau) = g(\tau - D)$. Moreover, $g(t) > g(t - D)$ for $D < t < \tau$ and $g(t) < g(t - D)$ for $\tau < t < z$. By (7.3) and (7.5), $\tau = t_r(D)$. Thus (3.4) holds and characterizes $t_r(D)$, given $D < t_r(D) < z$. A number of other facts are clear from Fig. 3: $t_r(D) - D < p < t_r(D)$, $t_r(0) = p$, $t_r(D)$ is strictly increasing and continuous, $t_r(D) - D$ is strictly decreasing, $t_r(\delta^-) = \delta$, and $t_r(D) - D \to 0$ as $D \to \delta$.

Combining these observations with (7.6) and (7.4), we find that t_r is continuous throughout $(0, \infty)$, strictly increasing for $D < z$ and constant for $D \geqslant z$.

We shall now use these properties of t_r to derive the basic equation,

$$\int_0^D g(t_r(u))\, du = \frac{1}{I_r(D)}, \tag{7.7}$$

which, together with continuity of g and t_r, implies (3.3). By (7.2) and (7.5),

$$\frac{1}{I_r(D)} = \int_{t_r(D)-D}^{t_r(D)} g(t)\, dt \tag{7.8}$$

for $D \leqslant z$. Splitting the range of integration at $u = p$, and making the changes of variables $t = t_r(u) - u$ and $t = t_r(u)$ on the lower and upper portions, we obtain

$$\begin{aligned} \frac{1}{I_r(D)} &= -\int_{0^+}^{D} g(t_r(u) - u)\, d(t_r(u) - u) + \int_{0^+}^{D} g(t_r(u))\, dt_r(u) \\ &= \int_0^D g(t_r(u))\, du + \int_{0^+}^{D} [g(t_r(u)) - g(t_r(u) - u)]\, d(t_r(u) - u). \end{aligned}$$

The second integral vanishes, since the integrand is zero for $u < \delta$, by (3.4), while $t_r(u) - u = 0$ for $\delta \leqslant u \leqslant D$, by (7.6). Thus (7.7) holds for $D \leqslant z$. If $D > z$ then (7.4) implies

$$\int_0^D g(t_r(u))\, du = \int_0^z g(t_r(u))\, du$$

and $I_r(D) = I_r(z)$. Thus (7.7) for $D > z$ follows from (7.7) for $D = z$. Consequently, (7.7) holds for all D.

It remains only to note that (3.10) follows from (7.7) and $t_r(0) = p$, that (3.3) and (7.5) imply that $I_r(D)$ is strictly decreasing for $D < z$, and that (3.11) is a consequence of (7.8).

8. Proof of the Indeterminacy Theorem

We are given a strength–duration curve, I_r, arising from an impulse response function, g. Let

$$g^*(t) = \frac{d}{dt}\left(\frac{1}{I_r(t)}\right), \tag{8.1}$$

that is,

$$g^*(t) = g(t_r(t)), \tag{8.2}$$

for all $t > 0$. We shall show that this function satisfies all of our assumptions, has no upramp ($p^* = 0$) or negative phase, and predicts the same strength-duration curve as g ($I_r^* = I_r$). We shall omit the construction of an impulse response function, g^*, with $I_r^* = I_r$ and $0 < p^* < z$. We note, however, that it is possible to construct this function in such a way that $g^*(0) = 0$.

As a consequence of (3.3) and properties of t_r established in Sec. 7, $g^*(t)$ is continuous, strictly decreasing for $t < z$, and equal to zero for $t \geqslant z$. Thus $p^* = 0$, $z^* = z$, and g^* has no negative phase. Since $p < t_r(t) < z$ and $t \leqslant t_r(t)$, it follows from (8.2) that $g^*(t) \leqslant g(t)$ for $t < z$. Thus, if $z^* = z = \infty$, (2.1) implies the same inequality for g^*. Finally, since $p^* = 0$, we have $\delta^* = 0$, hence, by (7.6), $t_r^*(t) = t$ for $0 < t < z$. Therefore (3.3) yields

$$g^*(t) = \frac{d}{dt}\left(\frac{1}{I_r^*(t)}\right)$$

for $t < z$. Comparing this with (8.1) we find that

$$\frac{d}{dt}\left(\frac{1}{I_r^*(t)}\right) = \frac{d}{dt}\left(\frac{1}{I_r(t)}\right) \tag{8.3}$$

for $t < z$, hence for all t, since both sides vanish for $t \geqslant z$. But $I_r^*(0) = \infty = I_r(0)$, so it follows from (8.3) that $I_r^* = I_r$, as was to be shown.

9. Concluding Comments

In Section 3 we saw how, under idealized conditions, one can calculate most values of the impulse response function from critical intensity data. We have described our methods in terms of completely and precisely known functions, like I_e and t_e, but, in practice, only estimates of a moderate number of function values are available.

The uncertainties thus engendered become more and more worrisome as we move further and further from our critical intensity data-base while progressing through a complex computational structure. For example, the value, $t_e(D)$, is identified as the point at which a certain curve based on critical intensity data appears to reach asymptote. This value, in conjuntion with a value of $I_e(D)$, yields an estimate of a point on the downramp of g. Such estimates are combined, in turn, with estimates of the derivative of $1/I_r(D)$ to obtain estimates of points on the upramp.

Thus it is clear that adaptation of our procedures to a laboratory setting will be a challenging problem, to say the least. We hope that presentation of the ideal or "noiseless" form of these methods will stimulate other to share this challenge with us.

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