The Strategic Equivalence of Games with Unawareness

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Abstract

An equivalence relation is more transparent when complemented by an explicit specification of the set of transformations under which equivalence is preserved. In the case of extensive games equivalent with respect to their strategic normal form, the set of transformations was provided by Thompson (1952). We extend Thompson's result to the case of games with unawareness (Feinberg (2009))

1 Introduction

When are two games equivalent? Given a set of formal objects, an equivalence relation partitions this set into subsets. Any two members of a subset are then equivalent with respect to the defined partition. For example, one can partition the set of extensive games into subsets which share a corresponding reduced normal form strategic game. Any two extensive games which fall into the same equivalence class are equivalent with respect to *strategic structure*, but not in general equivalent with respect to *temporal structure*.

Thompson (1952) defines a set of four transformations on extensive games which preserve the underlying strategic structure. He proves that any two extensive games which share strategic structure can be transformed into one another by some sequence of his transformations. This paper extends Thompson's transformations and his result to games with unawareness (Feinberg (2009)). In games with unawareness, players may be ignorant of aspects of game structure (including moves available to them, other players, etc.).

The *epistemic structure* of a game with unawareness is modeled by a set of standard games, each of which is indexed by a *view*. Each game in this set represents the perspective of some player on the awareness of others. Effectively, Feinberg has made the hierarchy of "higher-order expectations" discussed in Lewis (1969) explicit by assigning a game to each. In the limiting case, where all players are aware of the complete game structure (and aware that others are aware, ad infinitum) we have *common knowledge*, and games with unawareness simply reduce to standard strategic and extensive games. Awareness is just one way of cashing out the informational relationship between agents and the world. As such, it exhibits compelling parallels with belief and knowledge; however, we should be careful not to reduce awareness to either of these concepts. Unlike knowledge, awareness is not factive; unlike belief, awareness directly constrains the agent's perception of the game.

What is added to an equivalence notion by explicitly specifying the transformations under which it remains invariant? One answer is *transparency*. Consider, for example, geometry, in which we have many different notions of equivalence (e.g. topological equivalence, affine equivalence, similarity, congruence)—what is the relationship between these equivalence notions? Felix Klein's Erlangen program sought to answer this question by organizing geometries into a hierarchy in accordance with the transformations under which their objects remain invariant (Klein (1893)). Topological structure remains invariant under arbitrary stretching of the plane, rotations, and translations. Affine structure is invariant under uniform stretching in a single direction, rotations, translations, and dilations. Similarity is invariant under translations, rotations, and uniform dilation. Finally, congruence is invariant only under translation and rotation. Stated in terms of permissible transformations, the relationship between these various equivalence relations (and the corresponding geometries) becomes more transparent; we can clearly see, for example, that congruence is a restriction of similarity which takes size to be meaningful.

We also find the close relationship between equivalence and transformations in logic, which illustrates a second benefit to making transformational rules explicit: *precision*. Proof systems provide a set of permissible transformations over syntactic objects. Two syntactic objects which share truth value in all circumstances are logically equivalent. In order to ensure precision of our logical system, we demand proofs of its *soundness* and *completeness*. Soundness demonstrates that the transformations are indeed permissible, i.e. if one sentence can be transformed into another *and vice versa*, they are indeed logically equivalent. Completeness demonstrates that the transformations are adequate to take any sentence into any other which is logically equivalent. Only once we have proved both soundness and completeness do we have a satisfactory account of a logical system. Thompson's proof demonstrates the soundness and completeness of his transformations for strategic equivalence. He proves that they preserve underlying strategic form (soundness), and that for any games which do share strategic form, they can be reached by applications of his transformations (completeness). Just as the local manipulations of deductive rules give us insight into the nature of truth (insight not necessarily provided by direct model checking), transformations on extensive games can give us insight into the nature of strategic structure.

In the context of game theory, much ink has been spilt over the correct analysis of game equivalence. Our view is that there is room in game theory for many notions of equivalence; ideally, they will eventually be organized into a hierarchy in terms of the increasingly strict transformations under which game structure remains invariant. From this perspective Thompson's transformations provide a fruitful starting point for the development of more refined equivalence notions. Contributions to this project can be found scattered throughout the literature. Kohlberg and Mertens (1986) supplement Thompson's transformations with two which respectively introduce superfluous chance moves

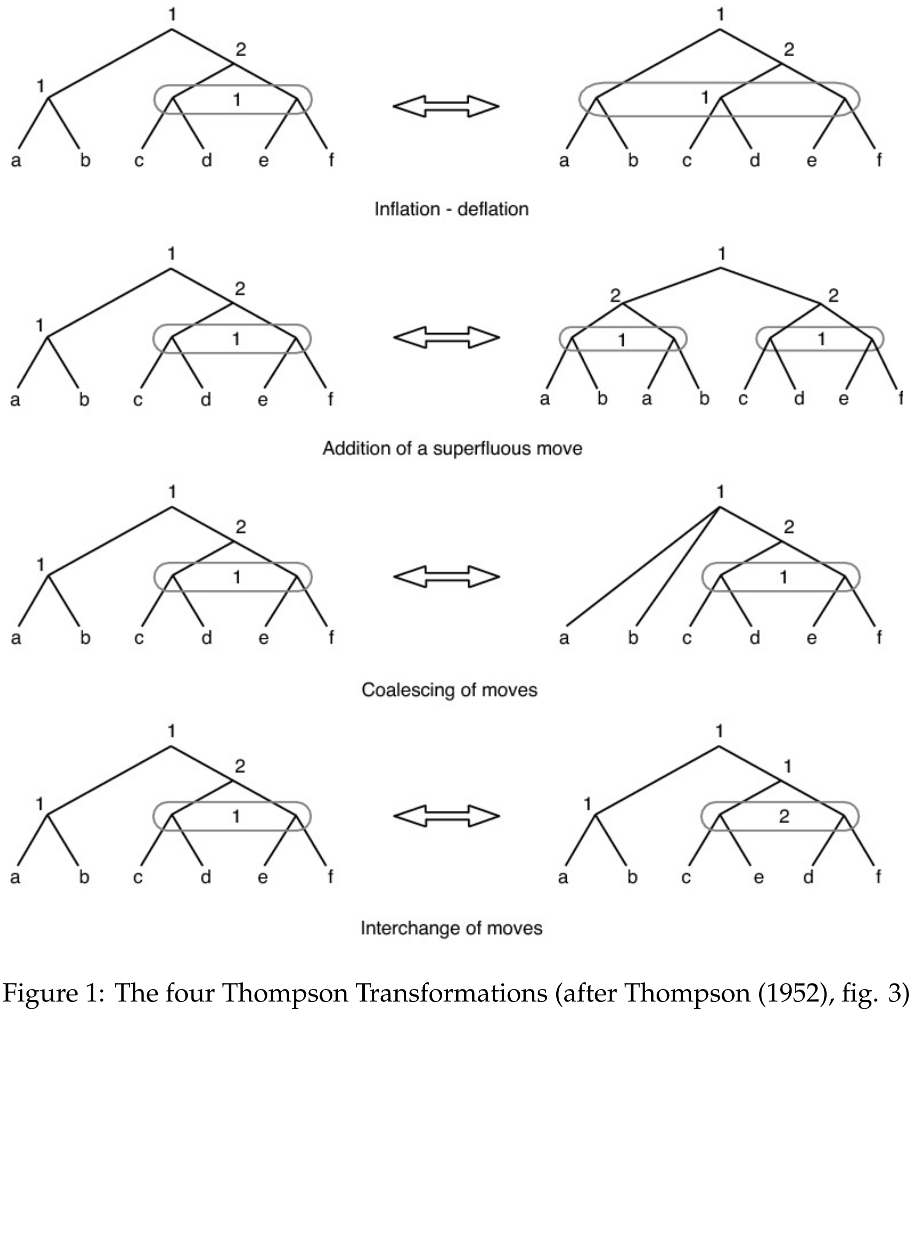

Figure 1: The four Thompson Transformations (after Thompson (1952), fig. 3)

and remove chance moves. This extends Thompson's result to games with chance players (see discussion and refinement in de Bruin (1999)). (Since this problem has been solved, we omit discussion of chance players in our presentation below.) Elmes and Reny (1994) address the worry that the transformation *inflation-deflation* (see Figure 1) does not preserve perfect information. They demonstrate that a modification of *addition of a superfluous move* allows any two games *with perfect information* which share a corresponding strategic form to be transformed into each other without appealing to *inflation-deflation*. Finally, Bonanno (1992) investigates the notion of game equivalence which arises from applications of *interchange of moves* only; such games are equivalent with respect to their *set-theoretic form*. Bonanno's motivation is the preservation of temporal structure without introducing an (arbitrary?) ordering over moves which, from an informational standpoint, are simultaneous. Our hope with this paper is to contribute to this general endeavor by proving soundness and completeness for transformations on a richer structure than standard extensive games.

Section 2 outlines Thompson's basic result and extends it with a new transformation, *coalescing of players*. The insight behind this addition is that players who share payoffs are strategically equivalent. Such redundant players do not usually appear in standard game applications, but they arise in a natural way when considering games with unawareness. Section 3 introduces strategic and extensive games with unawareness, closely following the treatment of Feinberg (2009). In Section 3.3 we extend the concept of strategic equivalence to extensive games with unawareness. Two extensive games with unawareness are strategically equivalent if they share a reduced strategic form with unawareness (modulo coalescing of players). The insight here is that the awareness of a player may change over the course of a temporally extended game. Such players must be treated as distinct from a strategic standpoint because they literally perceive themselves to be playing different games. In situations where the awareness of a player does not change, however, we prefer to treat them as a single strategic agent, and this is ensured by coalescing of players. Finally, Sections 4 and 5 give the main result of the paper. We first introduce transformations over extensive games with unawareness which correspond to those provided by Thompson in the standard case. We prove that two extensive games with unawareness are equivalent with respect to their reduced strategic form if and only if they can be transformed into each other by some sequence of these transformations.

2 The Equivalence of Extensive Games

In this section, we summarize Thompson's work on game equivalence and develop some basic technical apparatus. Our presentation of Thompson's result will be minimal and readers are referred to Thompson (1952) for more details (see also Osborne and Rubinstein (1994), Section 11.2).

We start with basic definitions for strategic games and extensive games.

Definition 2.1 (Strategic Games)**.** A *strategic game* (*SG*) is a tuple $g = \langle I, \{A_i\}_{i\in I}, \{u_i\}_{i\in I}\rangle$ such that:

- I is a set of players.

- A_i is the set of actions available to each player.
- u_i is a utility function that associates payoffs with action profiles $(a_i)_{i\in I}$ in $\Pi_{i\in I} A_i$.

For strategic games g, g', we say g is a *restriction* of g', if the sets of players and actions in g are subsets of those in g' and the utility function in g is a restriction of that in g' with respect to the set of actions in g.

Also two strategic games g, g' are *isomorphic* if there is an isomorphic map between g and g' in the standard sense. In our notation, isomorphism is characterized by the following definition.

Definition 2.2 (Isomorphism on SG)**.** Let $g = \langle I, \{A_i\}_{i\in I}, \{u_i\}_{i\in I}\rangle$ and $g' = \langle I', \{A'_{i'}\}_{i'\in I'}, \{u'_{i'}\}_{i'\in I'}\rangle$. g is *isomorphic* to g', written as $g \cong g'$, if there are functions ρ, α, υ such that

1. $\rho : I \to I'$ is bijective.
2. $\alpha : \bigcup_{i\in I} A_i \to \bigcup_{i'\in I'} A'_{i'}$ is bijective and, for all i and $a \in A_i$, $\alpha(a) \in A'_{\rho(i)}$.
3. $\upsilon : \mathbb{R} \to \mathbb{R}$ is such that $\upsilon(u_i((a_i)_{i\in I})) = u'_{\rho(i)}((\alpha(a_i))_{i\in I})$.

If a given set of functions, ρ, α, υ, satisfy the above conditions, we say they *isomorphically map* g to g'.

Definition 2.3 (Extensive Games)**.** An *extensive game* (EG) is a tuple $G = \langle (W, \prec), I, \{A_i\}_{i\in I}, \{F_i\}_{i\in I}, \{u_i\}_{i\in I}\rangle$ where:

1. $(W, \prec)$ is a finite tree with a disjoint union of vertices $W = \bigcup_{i\in I} V_i \cup Z$ where V_i denotes the set of player i's decision points and Z is the set of terminal vertices and the order $w' \prec w$ denotes w' occurs before w on the tree. We denote the set of immediate successors of $w \in W$ by $Succ(w)$. Also we write $w \preceq w'$, if $w \prec w'$ or $w = w'$.
2. I is a set of players.
3. A_i maps (w, w'), where $w' \in Succ(w)$, to the action that i can play at w which leads to w'. It is required that $u \neq v$ implies $A_i(w, u) \neq A_i(w, v)$. We define $A_i(w, \cdot) = \{A_i(w, v) | v \in Succ(w)\}$. We define $\mathbf{A}_i = \{A_i(w, w') \mid w' \in Succ(w) \text{ and } w \in V_i\}$.
4. F_i partitions the set V_i and induces the function f_i that maps $w \in V_i$ to the information set $f_i(w) \in F_i$ that contains w. If $w \in f(w')$, we say w is indistinguishable from w': otherwise, w is distinguishable from w' for i.
5. It is required that $w' \in f_i(w) \in F_i$ implies $A_i(w', \cdot) = A_i(w, \cdot)$.
6. $u_i : Z \to \mathbb{R}$ is the payoffs for the player i defined on the terminal vertices.

For extensive games G, G', we say G is a *restriction* of G' if information sets, sets of nodes, players, and actions in G are subsets of those in G' and the tree relation $\prec$ and the utility function in G are restrictions of those in G' with respect to the sets of nodes and actions in G.

We also say that two extensive games, G, G', are *isomorphic* if there is an isomorphic map between G and G' in the standard sense. In our notation:

Definition 2.4 (Isomorphism on EG)**.** Let $G = \langle (W, \prec), I, \{A_i\}_{i\in I}, \{F_i\}_{i\in I}, \{u_i\}_{i\in I}\rangle$ with $W = \bigcup_{i\in I} V_i \cup Z$ and $G' = \langle (W', \prec'), I', \{A'_i\}_{i\in I'}, \{F'_i\}_{i\in I'}, \{u'_i\}_{i\in I'}\rangle$ with $W' = \bigcup_{i'\in I'} V'_{i'} \cup Z'$ be extensive games. G is isomorphic to G', written as $G \cong G'$, if there are $\rho, \alpha, \upsilon, \phi, \iota$ such that

1. $\alpha : I \to I'$ is bijective.
2. $\phi : \bigcup_{i\in I} V_i \cup Z \to \bigcup_{i'\in I'} V'_{i' \in I'} \cup Z'$ is bijective and
   (a) for all $i \in I$ and v$\in V_i$, $\phi(\mathrm{v}) \in V'_{\rho(i)}$
   (b) for all $v \in Z$, $\phi(v) \in Z'$
   (c) for all v, w, if $v \prec w$, then $\phi(v) \prec' \phi(w)$
3. $\alpha : \bigcup_{i\in I} \mathbf{A}_i \to \bigcup_{i'\in I'} \mathbf{A}_{i'}$ is bijective and, for all $a = A_i(w, v)$, $\alpha(a) = A'_{\rho(i)}(\phi(w), \phi(v))$.
4. $\iota : \bigcup_{i\in I} F_i \to \bigcup_{i'\in I'} F'_{i'}$ is bijective and, for all $f \in F_i$, $\iota(f) = \{\phi(\mathrm{v}) \mid \mathrm{v} \in f\} \in F'_{\rho(i)}$.
5. $\upsilon_i : \mathbb{R} \to \mathbb{R}$ is such that, for all $z \in Z$, $\upsilon(u_i(z)) = u'_{\rho(i)}(\phi(z))$.

If a given set of functions, $\rho, \alpha, \upsilon, \phi, \iota$, satisfy the above conditions, we say that they *isomorphically map* G to G'.

A *strategy* s_i *of a player* $i \in I$ in an extensive game $\langle (W, \prec), I, \{A_i\}_{i\in I}, \{F_i\}_{i\in I}, \{u_i\}_{i\in I}\rangle$ is a function that assigns to every $f \in F_i$ an action in $A_i(w, \cdot)$. We denote the set of i's strategies in an extensive game by S_i. A strategy profile $\mathbf{s}$ of players in I is a sequence $(\mathbf{s}_i)_{i\in I}$ where $\mathbf{s}_i \in S_i$. We denote the set of strategy profiles $\Pi_{i\in I} S_i$ by S. Given $\mathbf{s} \in S$, $\mathbf{s}_i$ is the strategy for i in the strategy profile $\mathbf{s}$. The *outcome* $O(\mathbf{s})$ *of* $\mathbf{s} \in S$ is the terminal node that results when each player i plays the corresponding game by following s_i, i.e. $O(\mathbf{s}) = w_1 \ldots w_K$ such that, for each k $(1 \le k \le K)$, $w_k \in V_i$ for some $i \in I$ and $A_i(w_k, w_{k+1}) = s_i(f(w_k))$.

Definition 2.5 (Strategic Form)**.** The strategic form of an extensive game $G = \langle (W, \prec), I, \{A_i\}_{i\in I}, \{F_i\}_{i\in I}, \{u_i\}_{i\in I}\rangle$ is a strategic game $sf(G) = \langle I, S, \{U_i\}_{i\in I}\rangle$, where $U_i : S \to \mathbb{R}$ is such that $U_i(\mathbf{s}) = u_i(O(\mathbf{s}))$ for $\mathbf{s} \in S$.

Let $G = \langle (W, \prec), I, \{A_i\}_{i\in I}, \{F_i\}_{i\in I}, \{u_i\}_{i\in I}\rangle$. For $i \in I$ and $s, t \in S_i$, $sR_i t$ (s is equivalent to t for i) if for all $\mathbf{s}, \mathbf{t} \in S$, if $\mathbf{s}_i = s$, $\mathbf{t}_i = t$, and $\mathbf{s}_j = \mathbf{t}_j$ for all $j \in I - \{i\}$, then $u_i(O(\mathbf{s})) = u_i(O(\mathbf{t}))$. Denote the equivalence class under R_i that contains $s \in S_i$ by $\bar{s}$ and the set of equivalence classes of S_i under R_i by $\bar{S}_i$. Let $\bar{S} = \Pi_{i\in I} \bar{S}_i$. Given $\mathbf{s} \in S$, write $\bar{\mathbf{s}}$ for the sequence $(\bar{\mathbf{s}}_i)_{i\in I}$, where each $\bar{\mathbf{s}}_i$ is in $\bar{S}_i$.

Definition 2.6 (Reduced Strategic Form)**.** The *reduced strategic form red*(G) of an extensive game $G = \langle (W, \prec), I, \{A_i\}_{i\in I}, \{F_i\}_{i\in I}, \{u_i\}_{i\in I}\rangle$ is the strategic game $\langle I, \bar{S}, \{\bar{U}_i\}_{i\in I}\rangle$ where $\bar{U}_i(\bar{\mathbf{s}}) = u_i(O(\mathbf{s}))$ for every $\bar{\mathbf{s}} \in \bar{S}$ with $\mathbf{s} \in S$.

Thus the reduced strategic form of an extensive game does not have redundancy in the sense that no two strategies of a player result in the same payoff against all strategies of the other players. The reduced strategic form can be obtained from the strategic form by simply removing such redundant strategies.

Definition 2.7 (Strategic Equivalence)**.** An extensive game G_1 is *strategically equivalent* to an extensive game G_2, written as $G_1 \approx G_2$, if *red*$(G_1) \cong$ *red*(G_2).

Thompson (1952) considers two extensive games to be equivalent if they are strategically equivalent. He introduces a set of four local manipulations which are sufficient to transform an extensive game into any strategically equivalent game. We will not give formal definitions of the Thompson transformations here, but examples of each are depicted graphically in Figure 1. See Thompson (1952) for more details. (See also our discussion of the extended versions of Thompson's transformations in Definitions 4.4–4.6 and 4.8.)

Definition 2.8 (Transformability)**.** An extensive game G_1 is transformable into G_2, written as $G_1 \sim G_2$, if there is a sequence of extensive games, $G_1^*, \ldots, G_n^*$, such that $G_1^* = G_1$, $G_n^* \cong G_2$, and G_i^* $(1 \leq i \leq n-1)$ is a result of applying one of Thompson's transformation rules to G_{i+1}^*.

It is easy to see the four transformation rules preserve strategic equivalence. Therefore,

Lemma 1. Suppose G_1 is transformed into G_2 by one of Thompson's four transformations. Then $G_1 \approx G_2$. Moreover, if $G_1 \sim G_2$, then $G_1 \approx G_2$.

For the other direction, we need the following lemmas.

Lemma 2. For every extensive game G, there is an extensive game G' such that $G \sim G'$ and $sf(G') \cong red(G')$.

Let us say G is in *canonical form*, if $sf(G) \cong red(G')$.

Lemma 3. Let G_1, G_2 be extensive games in canonical form. $red(G_1) \cong red(G_2)$ iff there is an extensive game G^* such that $G_1 \cong G^*$ with $G_2 \sim G^*$.

Theorem 1 (Thompson (1952))**.** For two extensive games, G_1, G_2, $G_1 \approx G_2$ iff $G_1 \sim G_2$.

Proof. For the left-to-right direction, suppose $G_1 \approx G_2$. By Lemma 2, G_1 and G_2 can be transformed into their canonical forms, G_1', G_2'. If $red(G_1) \cong red(G_2)$, then $red(G_1') \cong red(G_2')$ (Lemma 1). The desired claim immediately follows from Lemma 3. □

This result can be extended slightly based on the following consideration. Strategic equivalence is a plausible analysis of game equivalence if we take strategies to be differentiated by their payoffs. Strategies which produce the exact same payoffs are treated as equivalent and all but one of them as redundant. Analogously, we may also take players to be differentiated by their payoffs. Since all strategic considerations apply equally to such players, we may remove "redundant" players without changing the essential strategic structure of the game.

For illustration, consider the games A and B in Figure 2. The only difference between these two games is that they are played by different sets of players: A is played by 1 and 2; B is played by 1, 2, and 3. Despite the different numbers of players, there is a sense in which these two games are the same. Note that in B, players 2 and 3 share the same utility function. Since their payoffs are the same, any strategic analysis of player 3's performance at the rightmost node in B must suggest the same action as that applied to player 2's performance at the rightmost node in A. This general point applies at every decision point

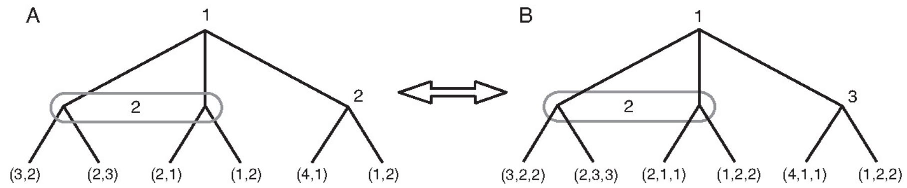

Figure 2: Coalescing of Players

in an extensive game; so long as players with identical payoffs are held to the same standard of rationality, they will seek to ensure the same outcomes obtain. These considerations license a stronger notion than strategic equivalence and the introduction of a new transformation.

Let $G = \langle (W, \prec), I, \{A_i\}_{i\in I}, \{F_i\}_{i\in I}, \{u_i\}_{i\in I}\rangle$ be an extensive game. We define an equivalence relation R_I on I so that, for every $i, j \in I$, $iR_I j$ iff $u_i = u_j$. We write $\bar{I}(i)$ for the equivalence class under R_I containing i and $\bar{I}$ for the set of equivalence classes. The following defines a new transformation that *coalesces* players with the same utility function.

Definition 2.9 (Coalescing of Players)**.** Let G_1 and G_2 be two extensive games with $G_1 = \langle (W, \prec), I, \{A_i\}_{i\in I}, \{F_i\}_{i\in I}, \{u_i\}_{i\in I}\rangle$. We define the equivalence relation $\sim_0$ so that $G_1 \sim_0 G_2$ if $G_2 \cong G^* = \langle (\bigcup_{\mathbf{j}\in I^*} V^*_{\mathbf{j}} \cup Z^*, \prec^*), I^*, \{A^*_{\mathbf{i}}\}_{\mathbf{i}\in I^*}, \{F^*_{\mathbf{i}}\}_{\mathbf{i}\in I^*}, (u^*_{\mathbf{i}})_{\mathbf{i}\in I^*}\rangle$, where

1. $I^* = \bar{I}$
2. $V^*_{\mathbf{i}} = \bigcup_{j\in\mathbf{i}} V_j$
3. $Z^* = Z$
4. $\prec^* = \prec$
5. $F^*_{\mathbf{i}} = \bigcup_{j\in\mathbf{i}} F_j$
6. $A^*_{\mathbf{i}} = \bigcup_{j\in\mathbf{i}} A_j$

Essentially, this transformation treats different players with the same utility function as the same player and, conversely, the same player at distinguishable points of a game as distinct players with identical utility functions.

We now need a new notion of reduced strategic form, corresponding to this extended sense of equivalence.

Definition 2.10 (p-Reduced Strategic Form)**.** Let $G = \langle (W, \prec), I, \{A_i\}_{i\in I}, \{F_i\}_{i\in I}, \{u_i\}_{i\in I}\rangle$ be an extensive game. Let the reduced strategic form of G be $red(G) = \langle I, \bar{S}, \{\bar{U}_i\}_{i\in I}\rangle$. For every $\bar{\mathbf{s}} = (\bar{\mathbf{s}}_i)_{i\in I} \in \bar{S}$, define $\bar{\mathbf{s}}^p = (\bigcup_{j\in\mathbf{i}} s_j)_{\mathbf{i}\in\bar{I}}$. Define $\bar{S}^p = \{\bar{\mathbf{s}}^p \mid \bar{\mathbf{s}} \in S\}$. The p-reduced strategic form $red^p(G)$ of G is a strategic game $\langle \bar{I}, \bar{S}^p, \{\bar{U}^p_{\mathbf{i}}\}_{\mathbf{i}\in\bar{I}}\rangle$ where $\bar{U}^p_{\mathbf{i}}(\bar{\mathbf{s}}^p) = \bar{U}_i(\mathbf{s})$.

Next we extend the notions of transformability and equivalence based on the above definitions. Let G_1 and G_2 be two extensive games.

Definition 2.11 (p-Transformability)**.** G_1 is p-transformable into G_2, written as $G_1 \sim^p G_2$ if there is a sequence of extensive games, $G^*_1, \ldots, G^*_n$, such that $G^*_1 = G_1$, $G^*_n \cong G_2$, and G^*_i $(1 \leq i \leq n-1)$ is a result of applying one of Thompson's transformation rules or coalescing of players to G^*_{i+1}.

Definition 2.12 (p-Equivalence)**.** G_1 is *p-equivalent* to G_2, written as $G_1 \approx^p G_2$ if $red^p(G_1) \cong red^p(G_2)$.

Theorem 2. For all extensive games, G_1, G_2, $G_1 \approx^p G_2$ iff $G_1 \sim^p G_2$.

Proof. For the right-to-left direction, suppose $G_1 \sim^p G_2$. Then there is a sequence $G_1^*, \ldots, G_k^*$ such that $G_1 = G_1^*$, $G_2 = G_k^*$ and, for all i $(1 \le i \le k)$ G_i^* is transformable into G_{i+1}^* by one application of some transformation rule X. If X is one of Thompson's four rules, then $red(G_i) \cong red(G_{i+1})$ (Lemma 1). If X is coalescing of players, it is clear from Definition 2.9 and 2.10 that $red^p(G_i) \cong red(G_{i+1})$ and thus to $red^p(G_{i+1})$.

For the left-to-right direction, let I be the set of players in G_1. By Lemma 2, G_1 is transformable by Thompson's four rules into some game G' such that $sf(G') = red(G')$. Take an extensive game G'' such that $G' \sim^p G''$ where the set of players in G'' is $\bar{I}$. It is clear that $sf(G'') = red^p(G'')$. The rest of the proof follows that for Thompson's theorem. □

3 Games with Unawareness

Our aim is to find corresponding equivalence notions for games with unawareness. We start with basic definitions. Our presentation is minimal; for a full justification and explication of the definitions, readers are referred to Feinberg (2009).

Let X be a non-empty set. X^* is the set of finite sequences of elements in X. We denote the empty sequence in X by λ. When a sequence v is an initial segment of a sequence u, we write $v \preceq u$. If $v \preceq u$ but $u \npreceq v$, we write $v \prec u$. Also $v\hat{\ }u$ is the concatenation of the sequences, v and u, in that order. When there is no danger of confusion, we will use set theoretical notation for sequences. For instance, for a sequence v and $\mathrm{v} \in X$, $\mathrm{v} \in v$ just means that v appears in the sequence v.

3.1 Strategic Games with Unawareness (SGU)

The key idea of strategic games with unawareness is to assign strategic games to sequences of players, which are called *views*. More precisely, given a strategic game $g = \langle I, \{A_i\}_{i \in I}, \{u_i\}_{i \in I} \rangle$, a *view* is a sequence of agents $v \in I^*$. A strategic game γ with unawareness is a collection of restrictions of g that are assigned in some coherent way (specified in the definition below) to views in a given set $\mathcal{V} \subseteq I^*$. We say a game assigned to a view is the game that view *is aware of* or the view *perceives*. Each game then constitutes the *perspective* of the view to which it is assigned. For example, the game assigned to the view 12 is the game that player 1 perceives player 2 to be aware of; the game assigned to 121 is that which player 1 perceives player 2 to perceive that player 1 is aware of.

Definition 3.1 (Strategic Games with Unawareness)**.** Let $g = \langle I, \{A_i\}_{i \in I}, \{u_i\}_{i \in I} \rangle$. Given a set of views $\mathcal{V} \subseteq I^*$ that includes λ, a collection of strategic games, $\gamma = \{g_v\}_{v \in \mathcal{V}}$, is a *strategic game with unawareness* (SGU), if $g_\lambda = g$ (we call this game *the master game* of γ) and the following conditions are satisfied, where $g_v = \langle I_v, \{(A_i)_v\}_{i \in I_v}, \{(u_i)_v\}_{i \in I_v} \rangle$:

C1 For every $v \in \mathcal{V}$, $v\hat{\ }\mathrm{v} \in \mathcal{V}$ iff $\mathrm{v} \in I_v$.

C2 For every $v\hat{}\tilde{v} \in \mathcal{V}$,

1. $v \in \mathcal{V}$
2. $\emptyset \neq I_{v\hat{}\tilde{v}} \subseteq I_v$
3. $\emptyset \neq (A_i)_{v\hat{}\tilde{v}} \subseteq (A_i)_v$ for all $i \in I_{v\hat{}\tilde{v}}$

C3 If $v\hat{}\mathrm{v}\hat{}\tilde{v} \in \mathcal{V}$, then $g_{v\hat{}\mathrm{v}\hat{}\tilde{v}} = g_{v\hat{}\mathrm{v}\hat{}\mathrm{v}\hat{}\tilde{v}}$ and $v\hat{}\mathrm{v}\hat{}\mathrm{v}\hat{}\tilde{v} \in \mathcal{V}$.

C4 For every action profile $(a)_{v\hat{}\tilde{v}} = \{(a_j)\}_{j \in I_{v\hat{}\tilde{v}}}$, there exists a completion to an action profile $(a)_v = \{a_j, a_k\}_{j \in I_{v\hat{}\tilde{v}}, k \in I_v \setminus I_{v\hat{}\tilde{v}}}$ such that

$$(u_i)_{v\hat{}\tilde{v}}((a)_{v\hat{}\tilde{v}}) = (u_i)_v((a)_v).$$

Intuitively, **C1** says that players must assign perspectives on the game to all the players they can see. **C2** ensures that perspectives only include aspects of the game that can be seen from that view; if I don't know that X is an option, then I can't perceive you as seeing X as an option. **C3** stipulates common knowledge of reflexivity; players are always aware of what they are aware of (and other players know this and assign perspectives accordingly). Finally **C4** is a consistency condition; it ensures that if a player cannot see the entire game, the outcomes that player can see do all obtain in the master game.

Definition 3.2 (Isomorphism on SGU)**.** Let $\gamma = \{g_v\}_{v \in \mathcal{V}}$ and $\gamma' = \{g_{v'}\}_{v' \in \mathcal{V}'}$ be SGU's with g, g' their master games. We say γ is *isomorphic* to γ', written as $\gamma \cong \gamma'$, if there are functions, ρ, α, υ between g and g', as specified in Definition 2.2, such that $g \cong g'$, and:

1. $\mathcal{V}' = \{\rho(\mathrm{v}_1) \ldots \rho(\mathrm{v}_k) \,|\mathrm{v}_1 \ldots \mathrm{v}_k \in \mathcal{V}\}$
2. for each $g_v \in \gamma$, the restrictions of ρ, α, υ with respect to g_v isomorphically map g_v to $g_{\rho(v)}$, where $\rho(v) = \rho(i_1) \ldots \rho(i_n)$ with $v = i_1 \ldots i_n$.

When a given set of functions, ρ, α, υ, satisfy the above conditions, we say that they *isomorphically map* γ to γ'.

3.2 Extensive Games with Unawareness (EGU)

Strategic games with unawareness assigned strategic games to sequences of players. Since the awareness of a player may change throughout the course of a game, this strategy will not work in the context of extensive games with unawareness. Instead, we assign extensive games to sequences of decision points. This allows different states of awareness of a single player to be distinguished by the decision point at which they occur. Let $G = \langle (W, \prec), I, \{A_i\}_{i \in I}, \{F_i\}_{i \in I}, \{u_i\}_{i \in I} \rangle$ be an extensive game. Then $W = \bigcup_{i \in I} V_i \cup Z$, where each V_i is the set of player i's decision points. The *set of viewpoints* is the set of all decision points $V = \bigcup_{i \in I} V_i$. The set of *views* is the set of all finite sequences of viewpoints is $\bar{V} = \bigcup_{n=0}^{\infty} V^{(n)}$. We denote views by italicized letters, such as v, u, etc. and viewpoints by unitalicized letters, such as v, u, etc.

An extensive game with unawareness is a collection of restrictions of G that are assigned in a coherent way to views in $\mathcal{V} \subseteq \bar{V}$. We say a game assigned to a view is the game that view *is aware of* or the view *perceives*, as in the case of SGU's. The game assigned to $\mathrm{v}\hat{}\mathrm{u}$, where $\mathrm{v} \in V_1$ and $\mathrm{u} \in V_2$, is the game that player 1 at v perceives player 2 at u to be aware of; the game assigned to v^u, where v, u ∈ V_1 is the game that player 1 at v perceives himself to be aware of at u. When it is clear, we will say a viewpoint v perceives a game, etc., ignoring the distinction between viewpoints and players.

Definition 3.3 (Extensive Games with Unawareness)**.** Let $G = \langle (W, <), I, \{A_i\}_{i\in I}, \{F_i\}_{i\in I}, \{u_i\}_{i\in I}\rangle$. Given a set of views $\mathcal{V} \subseteq \bar{V}$ that includes λ, a collection of extensive games, $\Gamma = \{G_v\}_{v\in\mathcal{V}}$ is an *extensive game with unawareness* (EGU) if $G_\lambda = G$ (we call this game *the master game*) and the following conditions are satisfied, where $G_v = \langle (W_v, <_v), I_v, \{(A_i)_v\}_{i\in I_v}, \{(F_i)_v\}_{i\in I_v}, \{(u_i)_v\}_{i\in I_v}\rangle$:

CE1 For every $v \in \mathcal{V}$, v ∈ V_i, v^v ∈ $\mathcal{V}$ iff $i \in I_v$, v ∈ $(V_i)_v$

CE2 For every $v\hat{}\bar{v} \in \mathcal{V}$,

1. $v \in \mathcal{V}$
2. $\emptyset \neq W_{v\hat{}\bar{v}} \subseteq W_v$
3. $\emptyset \neq I_{v\hat{}\bar{v}} \subseteq I_v$.

CE3 For every $v\hat{}\bar{v} \in \mathcal{V}$, $i \in I_{v\hat{}\bar{v}}$, and $w \in (V_i)_{v\hat{}\bar{v}}$,

1. $(V_i)_{v\hat{}\bar{v}} = (V_i)_v \cap (W_{v\hat{}\bar{v}})\backslash Z_{v\hat{}\bar{v}}$
2. $(F_i)_{v\hat{}\bar{v}} = \{f \cap (W_{v\hat{}\bar{v}}\backslash Z_{v\hat{}\bar{v}}) | f \in (F_i)_v\}$
3. $(A_i)_{v\hat{}\bar{v}}(w, w') = (A_i)_v(w, w'')$ for the unique successor w'' of w in W_v such that $w'' \leq w'$, where w' is the successor of w in $W_{v\hat{}\bar{v}}$.

CE4 If $v\hat{}\mathrm{v}\hat{}\bar{v} \in \mathcal{V}$ with v ∈ V_i, then:

1. $f_i(\mathrm{v}) \cap (V_i)_{v\hat{}\mathrm{v}} \neq \emptyset$, and for any $\tilde{\mathrm{v}} \in f_i(\mathrm{v})$, $G_{\mathrm{v}} = G_{\tilde{\mathrm{v}}}$
2. for every sequence $\tilde{v}$ all of whose elements are from $f_i(\mathrm{v}) \cap (V_i)_{v\hat{}\mathrm{v}}$, $G_{v\hat{}\mathrm{v}\hat{}\bar{v}} = G_{v\hat{}\tilde{v}\hat{}\bar{v}}$, and
3. $v\hat{}\tilde{v}\hat{}\bar{v} \in \mathcal{V}$.

CE5 Let $v\hat{}\bar{v} \in \mathcal{V}$. For every terminal node $w \in Z_{v\hat{}\bar{v}}$, there exists a node $w' \in Z_v$ such that $w < w'$ and $(u_i)_{v\hat{}\bar{v}}(w) = (u_i)_v(w')$.

We denote G_λ by $G = \langle (W, <), I, \{A_i\}_{i\in I}, \{F_i\}_{i\in I}, \{u_i\}_{i\in I}\rangle$.

Conditions **CE1**, **CE2**, and **CE3** do essentially the same work for EGU as **C1** and **C2** did for SGU, ensuring that nodes assign views to all nodes they can see and do so in a consistent manner. **CE5** corresponds to **C4**, ensuring that payoffs in restricted games are always consistent with outcomes which might obtain in the master game. A remark is in order concerning condition **CE4**. Feinberg (2009) has a weaker condition, which states that $v\hat{}\mathrm{v}\hat{}\bar{v} \in \mathcal{V}$ with v ∈ V_i implies $f_i(\mathrm{v}) \cap (V_i)_{v\hat{}\mathrm{v}} \neq \emptyset$, and if $\tilde{\mathrm{v}} \in f_i(\mathrm{v}) \cap (V_i)_{v\hat{}\mathrm{v}}$ then $v\hat{}\mathrm{v}\hat{}\tilde{\mathrm{v}}\hat{}\ldots\hat{}\tilde{\mathrm{v}}\hat{}\bar{v} \in \mathcal{V}$ and

$$G_{v\hat{}\mathrm{v}\hat{}\bar{v}} = G_{v\hat{}\mathrm{v}\hat{}\tilde{\mathrm{v}}\hat{}\bar{v}} = \cdots = G_{v\hat{}\mathrm{v}\hat{}\tilde{\mathrm{v}}\hat{}\ldots\hat{}\tilde{\mathrm{v}}\hat{}\bar{v}}$$

Discussing the condition, Feinberg writes "This definition follows the interpretation of an information set as representing indistinguishable information and indistinguishable awareness" (22). However, the condition is still weak for the stated purpose, since it allows for the possibility that two nodes indistinguishable in the master game assign different perspectives to the game *so long as neither can see each other*. If this possibility obtained, however, it would violate the intuitive restriction Feinberg states that indistinguishable nodes should exhibit the exact same awareness of game structure. Our stronger condition conforms more closely to **C3**, stipulating not only that nodes within an information set must have the same perspective (reflexivity), but also that all players are aware of this property and assign perspectives accordingly (common knowledge of reflexivity). This stronger condition is needed for the proof of Theorem 4.

Definition 3.4 (Isomorphism on EGU)**.** Let $\Gamma = \{G_v\}_{v \in \mathcal{V}}$ and $\Gamma' = \{G_{v'}\}_{v' \in \mathcal{V}'}$ be EGU's with G, G' their master games. We say Γ is *isomorphic* to Γ', written as $\Gamma \cong \Gamma'$, if there are functions, $\rho, \alpha, \upsilon, \phi, \iota$ between G and G', as specified in Definition 2.4, such that $G \cong G'$, and:

1. $\mathcal{V}' = \{\phi(\mathrm{v}_1)\ldots\phi(\mathrm{v}_k)\ |\mathrm{v}_1\ldots\mathrm{v}_k \in \mathcal{V}\}$
2. for each $G_v \in \Gamma$, the restrictions of $\rho, \alpha, \upsilon, \phi, \iota$ with respect to G_v isomorphically map G_v to $G_{\phi(v)}$, where $\phi(v) = \phi(\mathrm{v}_1)\ldots\phi(\mathrm{v}_n)$ with $v = \mathrm{v}_1\ldots\mathrm{v}_n$.

When a given set of functions, $\rho, \alpha, \upsilon, \phi, \iota$, satisfy the above conditions, we say that they *isomorphically map* Γ to Γ'.

3.3 Strategic Forms and Equivalence on EGU

When are two EGU's strategically equivalent? We must begin by defining the SGU which corresponds to a given EGU. At a first pass, we might reduce all extensive games in the EGU to their corresponding strategic forms. If one adopts this approach, the strategic form of an EGU Γ will be the collection of strategic forms of extensive games in Γ that are assigned to views in G. However, views in an EGU are sequences of *viewpoints*, whereas views in an SGU are sequences of *players*. A natural move here is to induce sequences of players from sequences of viewpoints in terms of the associated players. Unfortunately, this strategy still does not work, since different viewpoints in a view can be associated with a single player. For illustration, consider the possibility that v and u are distinct (distinguishable) viewpoints for player 1, but $G_{\mathrm{v}\hat{}\mathrm{u}}$ and $G_{\mathrm{u}\hat{}\mathrm{v}}$ do not share the same strategic form. In an SGU, we cannot associate two distinct games with the corresponding sequence of players, $1\hat{}1$.

Therefore, for the strategic form of an EGU Γ, we consider distinguishable viewpoints of player i (in the master game G) as being played by distinct players with the same utility function as i. This decision mirrors the conceptual point that, if a player's state of awareness changes during the game, then the strategic analysis of that player must change as well. Furthermore, if player 1 perceives player 2's state of awareness as changing (even if it does not actually change), player 1 must play as if the two states of player 2's awareness are strategically distinct. (For a related discussion, see the analysis of solution concepts for games with unawareness in Feinberg (2009).)

Given an EGU $\Gamma = \{G_v\}_{v \in \mathcal{V}}$, define the equivalence relation R_{V_i} so that, for all v, u $\in V_i$, $\mathrm{u}R_{V_i}\mathrm{v}$ iff $G_{v\hat{}\mathrm{v}\hat{}\tilde{v}} = G_{v\hat{}\mathrm{u}\hat{}\tilde{v}}$ for all $v\hat{}\mathrm{v}\hat{}\tilde{v}, v\hat{}\mathrm{u}\hat{}\tilde{v} \in \mathcal{V}$. The relation R_{V_i} partitions the set V_i of player i's viewpoints. We denote by $\bar{V}_i(\mathrm{v})$ the equivalence class under R_{V_i} that contains $\mathrm{v} \in V_i$. Note that, by **CE4** in Definition 3.3, every information set $f \in F_i$ is such that $f \subseteq \bar{V}_i(\mathrm{v})$ for some v. The following operation relabels players in an EGU with the equivalence classes $\bar{V}_i(\mathrm{v})$.

Definition 3.5 (p-Normal Form)**.** Let $\Gamma = \{G_v\}_{v\in\mathcal{V}}$ be an EGU and for each $v \in \mathcal{V}$, $G_v = \langle(\bigcup_{i\in I_v}(V_i)_v \cup Z, \prec_v), I_v, \{(A_i)_v\}_{i\in I_v}, \{(F_i)_v\}_{i\in I_v}, \{(u_i)_v\}_{i\in I_v}\rangle$. The *p-normal form* Γ^p of Γ is a collection $\{G^p_v\}_{v\in\mathcal{V}}$, where $G^p_v = \langle(\bigcup_{\mathbf{i}\in I^p_v}(V^p_{\mathbf{i}})_v \cup Z^p_v, \prec^p_v), I^p_v, \{(A^p_{\mathbf{i}})_v\}_{\mathbf{i}\in I^p_v}, \{(F^p_{\mathbf{i}})_v\}_{\mathbf{i}\in I^p_v}, \{(u^p_{\mathbf{i}})_v\}_{\mathbf{i}\in I^p_v}\rangle$ is defined by:

1. $I^p_v = \{\bar{V}_i(\mathrm{v}_i) \mid \mathrm{v}_i \in (V_i)_v\}$, and for each $\mathbf{i} \in I^p_v$
2. $(V^p_{\mathbf{i}})_v = \mathbf{i} \cap (V_i)_v$, $Z^p_v = Z_v$ and $\prec^p_v = \prec_v$
3. $(A^p_{\mathbf{i}})_v(w, w') = (A_i)_v(w, w')$ with $w \in (V^p_{\mathbf{i}})_v$ and $\mathbf{i} \subseteq V_i$
4. $(F^p_{\mathbf{i}})_v = \{f \cap \mathbf{i} \mid f \in (F_i)_v \text{ and } f \cap \mathbf{i} \neq \emptyset\}$ with $\mathbf{i} \subseteq V_i$
5. $(u^p_{\mathbf{i}})_v = (u_i)_v$ where $\mathbf{i} \subseteq V_i$

To find the strategic form of an EGU Γ, first transform it into its p-normal form, then take the strategic form of each game in Γ^p.

Definition 3.6 (Strategic Form for EGU)**.** Let $\Gamma = \{G_v\}_{v\in\mathcal{V}}$ be an EGU with G its master game. The *strategic form of* Γ is a collection of strategic games $sf(\Gamma) = \{g_{v'}\}_{v'\in\mathcal{V}'}$ defined as follows:

1. $\mathbf{i}_1 \ldots \mathbf{i}_n \in \mathcal{V}'$ iff $\mathrm{v}_1 \ldots \mathrm{v}_n \in \mathcal{V}$ and $\mathbf{i}_k$ $(1 \leq k \leq n)$ is the player at the viewpoint v_k in Γ^p.
2. Each $g_v = sf(G^p_v)$.

As discussed above, the definition of Γ^p ensures that the game assigned to each view v in $sf(\Gamma)$ is always unique. Therefore, the strategic form of an EGU is well-defined. Indeed let $v, u \in \mathcal{V}$ be distinct views and set $v = \mathrm{v}_1 \ldots \mathrm{v}_n$ and $u = \mathrm{u}_1 \ldots \mathrm{u}_n$ with $\mathrm{v}_k \in V^p_{\mathbf{i}_k}$ and $\mathrm{u}_k \in V^p_{\mathbf{j}_k}$ $(1 \leq k \leq n)$. If $\mathbf{i}_1 \ldots \mathbf{i}_n = \mathbf{j}_1 \ldots \mathbf{j}_n$, then v_k and u_k are either identical or in the same equivalence class under R_{V_i}. By the definition of p-normal form, $G^p_v = G^p_u$.

To show that $sf(\Gamma)$ is indeed a strategic game with unawareness, we need one proposition. Let G, G' be extensive games such that G' is a restriction of G. The following structural result is straightforward.

Proposition 1. $sf(G')$ is a restriction of $sf(G)$.

Given this proposition, it is easy to check that the conditions **CE1-5** on an EGU Γ in Definition 3.3 guarantee that $sf(\Gamma)$ satisfies the conditions **C1-4** in Definition 3.1. If $G_{v\hat{\ }\mathrm{v}}$ is a restriction of G_v, as required by **CE1-5**, the strategic form $red(G_{v\hat{\ }\mathrm{v}})$ (defined in Definition 2.5) is a restriction of $red(G_v)$, as required by **C1-4**. Therefore, we have:

Proposition 2. $sf(\Gamma)$ is a strategic game with unawareness.

We can similarly define the reduced strategic form for EGU.

Definition 3.7 (Reduced Strategic Form for EGU)**.** Let $\Gamma = \{G_v\}_{v\in\mathcal{V}}$ be an EGU with G its master game. The *reduced strategic form of* Γ is a collection of strategic games $red(\Gamma) = \{g_{v'}\}_{v'\in\mathcal{V}'}$ defined as follows:

1. $\mathbf{i}_1 \ldots \mathbf{i}_n \in \mathcal{V}'$ iff $\mathrm{v}_1 \ldots \mathrm{v}_n \in \mathcal{V}$ and $\mathbf{i}_k$ $(1 \leq k \leq n)$ is the player at the viewpoint v_k in Γ^p.
2. Each g_v is the restriction g^* of $red(G^p)$ such that $g^* \cong red(G^p_v)$.

(The reason that we can't simply take g_v to be $red(G^p_v)$ is that the operation *red* takes the equivalence classes of strategies in G^p_v, which may be different from those in G^p since G^p may contain more strategies than its restriction, G^p_v.)

A similar argument to the one above applies to $red(\Gamma)$. Thus it is straightforward to show:

Proposition 3. $red(\Gamma)$ is a strategic game with unawareness.

Next let us consider the notion that corresponds to p-reduced strategic form. As we saw in Section 2, the intuition behind the p-reduced strategic form is that it removes redundant players (i.e. players who share the same payoffs) from the reduced strategic form. For a game with unawareness, players may not be redundant, even if they share the same payoffs, since they may have different perspectives on the game. For example, there may be players $i, j \in I$ such that $u_i = u_j$ yet $A_i \neq A_j$. We can interpret such players in two ways. They may represent different epistemic states of the same player (say, his perspective on the game at different times), or they may represent players on the same "team" who share interests but have different states of knowledge about the game. In either case, the difference between such players is strategically significant, so we wish to keep both in the p-reduced strategic form. Therefore, in order to be considered redundant, players must share both payoffs *and* perspectives on the game.

We now make this idea precise for the definition of p-reduced strategic form on EGU. Let Γ be a p-normal EGU $\{G_v\}_{v \in \mathcal{V}}$. Let I be the set of players in the master game G. Define an equivalence relation R_I on I such that, for $i, j \in I$, iR_Ij iff

1. $u_i = u_j$
2. for all $v\hat{\ }\mathrm{v}_i\hat{\ }\tilde{v}, v\hat{\ }\mathrm{v}_j\hat{\ }\tilde{v} \in \mathcal{V}$ with $\mathrm{v}_i \in V_i$ and $\mathrm{v}_j \in V_j$,

$$G_{v\hat{\ }\mathrm{v}_i\hat{\ }\tilde{v}} = G_{v\hat{\ }\mathrm{v}_j\hat{\ }\tilde{v}}.$$

Item 2 is the formalization of the above preliminary discussion of perspectives. It guarantees that, if what any player (including i and j) perceives that i perceives and what that player perceives that j perceives are always the same, then it is redundant to consider i and j as distinct players.

Replace R_I in Definition 2.10 with its new statement and redefine red^p. We write $\bar{I}(i)$ for the equivalence relation under R_I that contains i. We can now define the p-reduced strategic form for EGU.

Definition 3.8 (p-Reduced Strategic Form)**.** Let $\Gamma = \{G_v\}_{v \in \mathcal{V}}$ be an EGU with G its master game. The *strategic form of* Γ is a collection of strategic games $red^p(\Gamma) = \{g_{v'}\}_{v' \in \mathcal{V}'}$ defined as follows:

1. $\bar{I}(\mathbf{i}_1) \ldots \bar{I}(\mathbf{i}_n) \in \mathcal{V}'$ iff $\mathrm{v}_1 \ldots \mathrm{v}_n \in \mathcal{V}$ and $\mathbf{i}_k$ $(1 \leq k \leq n)$ is the player at the viewpoint v_k in Γ^p.

2. Each g_v is the restriction g^* of $red^p(G^p)$ such that $g^* \cong red^p(G^p_v)$.

We need to check that the p-reduced strategic form is well-defined. Indeed let $v, u \in \mathcal{V}$ be distinct views and put $v = \mathrm{v}_1 \dots \mathrm{v}_n$ and $u = \mathrm{u}_1 \dots \mathrm{u}_n$ with $\mathrm{v}_k \in V_{i_k}$ and $\mathrm{u}_k \in V_{j_k}$ $(1 \leq k \leq n)$. If $\bar{I}(\mathbf{i}_1) \dots \bar{I}(\mathbf{i}_n) = \bar{I}(\mathbf{j}_1) \dots \bar{I}(\mathbf{j}_n)$, then (i) v_k, u_k are identical or in the same information set (by p-normality), or (ii) $\mathbf{i}_k R_I \mathbf{j}_k$ (by definition). In each case, we have $G_v = G_u$ by definition of R_I. Therefore, the above definition is well-defined.

Also, it is straightforward to check:

Proposition 4. *$red^p(\Gamma)$ is a strategic game with unawareness.*

Given the definition of p-reduced strategic form, we can now formulate the definition of strategic equivalence on EGU.

Definition 3.9 (Strategic Equivalence on EGU)**.** Γ_1 is equivalent to Γ_2, written as $\Gamma_1 \approx \Gamma_2$, if $red^p(\Gamma_1)$ is isomorphic to $red^p(\Gamma_2)$.

4 Transformations on EGU

Before introducing the transformations for games with unawareness, we need some supplementary definitions. Given an extensive game G, G^{v} is the subgame of G whose root is v. The following definition for $\mathrm{v}|_i\mathrm{u}$ ensures that two viewpoints, v and u, are the product of different actions by the player i.

Definition 4.1 (Inflation Suitability for i)**.** $\mathrm{v}|_i\mathrm{u}$ if there are $\mathrm{v}', \mathrm{u}' \in V_i$, v'', u'' such that $\mathrm{v}' \in f_i(\mathrm{u}')$, $\mathrm{v}'' \leq \mathrm{v}$, and $\mathrm{u}'' \leq \mathrm{u}$; $A_i(\mathrm{v}', \mathrm{v}'') \neq A_i(\mathrm{u}', \mathrm{u}'')$.

We can now define the transformations on EGU which preserve strategic equivalence. We will formulate transformations as equivalence relations on EGU, as Thompson did on EG. Below let $\Gamma = \{G_v\}_{v \in \mathcal{V}}$ and $\Gamma' = \{G'_{v'}\}_{v' \in \mathcal{V}'}$ be EGU's. Also let $G = \langle (W, <), I, \{A_i\}_{i \in I}, \{F_i\}_{i \in I}, \{u_i\}_{i \in I} \rangle$ be the master game of Γ.

Our first transformation takes an EGU into one that has the same p-normal form (as in Definition 3.5). If a player i has distinguishable viewpoints with different perspectives (in the sense of the definition of R_I above), this transformation "splits" player i and relabels the viewpoints with different perspectives as viewpoints of distinct players.

Definition 4.2 (Splitting of Players)**.** $\Gamma \sim_s \Gamma'$ if $\Gamma^p \cong \Gamma'$.

For the p-reduced strategic form of an EGU, we take the p-normal form of the EGU and transform extensive games in it into their p-strategic forms. Therefore, this transformation preserves strategic equivalence on EGU.

The second transformation allows us to consider distinct players i, j equivalent with respect to R_I as the same player.

Definition 4.3 (Coalescing of Players)**.** $\Gamma \sim_c \Gamma'$ if there is an i^* such that $\Gamma \cong \Gamma^* = \{G^*_v\}_{v \in \mathcal{V}}$, where each $G^*_v = \langle (\bigcup_{j \in I^*} V^*_j \cup Z^*, <^*), I^*, \{A^*_i\}_{i \in I^*}, \{F^*_i\}_{i \in I^*}, (u^*_i)_{i \in I^*} \rangle$ in Γ^* is defined by

1. $I^* = I_v - (\bar{I}_v(i^*) - \{i^*\})$
2. $V^*_{i^*} = \bigcup_{j \in \bar{I}(i^*)} (V_j)_v$ and $V^*_i = (V_i)_v$ for $i \notin \bar{I}(i^*)$

3. $Z^* = Z_v$ and $\prec^* = \prec_v$
4. $F^*_{i^*} = \bigcup_{j \in \bar{I}(i^*)} (F_j)_v$ and $F^*_i = (F_i)_v$ for $i \notin \bar{I}(i^*)$.
5. $A^*_{i^*} = \bigcup_{j \in \bar{I}(i^*)} (A_j)_v$ and $A^*_i = (A_i)_v$ for $i \notin \bar{I}(i^*)$.

For the p-reduced strategic form of an EGU, we take the equivalence class under R_I on the set of players I. Therefore, this transformation preserves strategic equivalence on EGU. Intuitively, coalescing of players and splitting of players together ensure that we can take any EGU into one with neither redundant players nor players whose perspective changes over the course of the game, i.e. one with the minimal number of distinct players.

The rest of the transformations are extensions of those in Thompson (1952). The essential idea here is that we transform the master game of Γ in accordance with Thompson's original transformation, then ensure that corresponding changes are made to all $G_v \in \Gamma$. Readers are referred again to Figure 1 for intuitive motivation.

Definition 4.4 (Inflation-Deflation)**.** $\Gamma \sim_1 \Gamma'$ if there are v, u $\in W$, and $i \in I$ such that for any v′ $\in f_i$(v) and u′ $\in f_i$(u), v′$|_i$u′ (in G), and $\Gamma' \cong \Gamma^* = \{G^*_v\}_{v \in \mathcal{V}}$, where each G^*_v in Γ^* is exactly the same as G_v except when both v and u are in W_v, in which case the information partition F^*_i in G^* is $((F_i)_v - \{(f_i)_v(\mathrm{v}), (f_i)_v(\mathrm{u})\}) \cup \{(f_i)_v(\mathrm{v}) \cup (f_i)_v(\mathrm{u})\}$.

Inflation-deflation adds and removes perfect recall. It preserves underlying strategic form because it does not change the set of strategies available to any player, merely the size of his information sets. Of course, in general, enlarging information sets will eliminate potential strategies, but inflation suitability (Definition 4.1) ensures that inflation-deflation can only be applied to information sets which are distinguished by an earlier move, thus ensuring that strategies remain distinct. Therefore the reduced strategic form is unchanged.

Definition 4.5 (Addition of a Superfluous Move)**.** $\Gamma \sim_2 \Gamma'$ if there are v, u$_1$, u$_2$ $\in W$ with $Succ$(v) = {u$_1$, u$_2$} such that

1. for every $v \in \mathcal{V}$ such that u$_1$, u$_2$ $\in W_v$, there exist $\rho_v, \alpha_v, \upsilon_v, \phi_v, \iota_v$, as specified in Definition 3.4, such that
   - they isomorphically map $G^{\mathrm{u}_1}_v$ to $G^{\mathrm{u}_2}_v$
   - for any w $\geq$ u$_1$, $\phi_v(\mathrm{w}) \in (f_i)_v(\mathrm{w})$
2. $\Gamma' \cong \Gamma^* = \{G_v\}_{v \in \mathcal{V}^*}$, where
   - $\mathcal{V}^* = \mathcal{V} - \{v | \exists x \in \{\{\mathrm{v}\} \cup \{\mathrm{w} | \mathrm{u}_2 \leq \mathrm{w}\}\}(x \in v)\}$
   - each G^*_v is exactly the same as G_v except that the tree W_v in G is replaced with $W_v - (\{\mathrm{v}\} \cup \{\mathrm{w} | \mathrm{u}_2 \leq \mathrm{w}\})$ and $\prec_v$ and other items in G_v are restricted to $W_v - (\{\mathrm{v}\} \cup \{\mathrm{w} | \mathrm{u}_2 \leq \mathrm{w}\})$.

Addition of a superfluous move adds a move which has no effect on the strategic structure of the game (in this case, the move at node v). The irrelevance of this move is ensured by the bijection between u$_1$ and u$_2$; since the move produces identical, indistinguishable outcomes, it is irrelevant for decision making purposes. One potential worry in extending this transformation to games with unawareness is the role that node v has in views on the game. However, since the irrelevance of the move at v obtains in all restrictions of the master game, no such view can affect strategic decisions and the corresponding reduced strategic form is preserved.

Definition 4.6 (Coalescing of Moves)**.** $\Gamma \sim_3 \Gamma'$ if there are $\{v_1, \ldots, v_k\} = f_i(v_1)$ and $\{u_1, \ldots, u_k\} = f_i(u_1)$ such that

1. $u_m \in Succ(v_m)$ for $1 \leq m \leq k$
2. $A_i(v_m, u_m) = A_i(v_n, u_n)$ for $1 \leq m, n \leq k$.
3. $v\hat{}v_m\hat{}\tilde{v}, v\hat{}u_m\hat{}\tilde{v} \in \mathcal{V}$ implies $G_{v\hat{}v_m\hat{}\tilde{v}} = G_{v\hat{}u_m\hat{}\tilde{v}}$ for $1 \leq m \leq k$
4. $\Gamma' \cong \Gamma^* = \{G^*_v\}_{v \in \mathcal{V}^*}$, where Γ^* is defined by:
   - $\mathcal{V}^* = \mathcal{V} - \{v \mid \exists x \in \{u_1, \ldots, u_k\}(x \in v)\}$
   - each G^*_v is the same as G_v except that W_v is replaced with $W_v - \{u_1, \ldots u_k\}$ and $<_v$ and other items in G are restricted to $W_v - \{u_1, \ldots u_k\}$.

Coalescing of moves combines into a single decision point moves made by the same player in sequence. Again, the main worry in extending this transformation to games with unawareness is ensuring that the transformation only applies when the player's state of awareness does not change between the information sets. This restriction is taken care of by condition 3, ensuring that the reduced strategic form remains unchanged.

We now define the final transformation: *interchange of moves*. We say an extensive game G is *binary*, if for any viewpoint v in G, $Succ(v) \leq 2$. Similarly EGU $\Gamma = \{G_v\}_{v \in \mathcal{V}}$ is *binary*, if for all $v \in \mathcal{V}$, G_v is binary.

Definition 4.7 (IM Suitability)**.** Let $G = \langle (W, <), I, \{A_i\}_{i \in I}, \{F_i\}_{i \in I}, \{u_i\}_{i \in I} \rangle$ be a binary extensive game. G is *IM suitable* if there are distinct viewpoints, $v, u_1, u_2 \in W$ and $i, j \in I$ such that

1. $v \in V_i$ and $u_1, u_2 \in V_j$
2. $Succ(v) = \{u_1, u_2\}$
3. $u_2 \in f_j(u_1)$

If the master game of Γ is IM suitable, we would like to be able to apply Thompson's interchange of moves transformation. However, we need to guarantee our extension of this transformation preserves conditions **CE1–CE5**, in particular, that all $G_v \in \Gamma$ are still restrictions of the master game. In order to do this, we need to consider several cases which might obtain with respect to the viewpoints, v, u_1, u_2 in any G_v. We define the operation *IM* to produce an extensive game $IM(G_0)$ on restrictions G_0 of the extensive game G in the following way. Below we denote each item of $IM(G)$ by items in G superscripted with ^{IM}.

IM1 If none of v, u_1, u_2 are in G_0, $IM(G_0) = G_0$.

IM2 If all of v, u_1, u_2 are in G_0 and there are distinct viewpoints, w_1, w_2, x_1, x_2 in G_0 such that $Succ(u_1) = \{w_1, w_2\}$, $Succ(u_2) = \{x_1, x_2\}$, then:

(a) $G - G^{v} = IM(G) - (IM(G))^{v}$

(b) $(IM(G))^{\mathrm{w}_k} = G^{\mathrm{w}_k}$ and $IM(G)^{\mathrm{x}_k} = G^{\mathrm{x}_k}$ ($k \in \{1, 2\}$)

(c) $\mathrm{u}_1, \mathrm{u}_2 \in V_i^{IM}$ and $\mathrm{v} \in V_j^{IM}$ in $IM(G)$

(d) $Succ^{IM}(\mathrm{u}_1) = \{\mathrm{w}_1, \mathrm{x}_1\}$ and $Succ^{IM}(\mathrm{u}_2) = \{\mathrm{w}_2, \mathrm{x}_2\}$

(e) $f_i^{IM}(\mathrm{u}_1) = (f_i(\mathrm{v}) - \{\mathrm{v}\}) \cup \{\mathrm{u}_1, \mathrm{u}_2\}$ and $f_j^{IM}(\mathrm{v}) = (f_j(\mathrm{u}_1) - \{\mathrm{u}_1, \mathrm{u}_2\}) \cup \{\mathrm{v}\}$

(f)
- $A_i^{IM}(\mathrm{u}_1, \mathrm{w}_1) = A_i^{IM}(\mathrm{u}_2, \mathrm{x}_1) = A_i(\mathrm{v}, \mathrm{u}_1)$
- $A_i^{IM}(\mathrm{u}_1, \mathrm{w}_2) = A_i^{IM}(\mathrm{u}_2, \mathrm{x}_2) = A_i(\mathrm{v}, \mathrm{u}_2)$
- $A_j^{IM}(\mathrm{v}, \mathrm{u}_1) = A_j(\mathrm{u}_1, \mathrm{w}_1) = A_j(\mathrm{u}_2, \mathrm{x}_1)$
- $A_j^{IM}(\mathrm{v}, \mathrm{u}_2) = A_j(\mathrm{u}_1, \mathrm{w}_2) = A_j(\mathrm{u}_2, \mathrm{x}_2)$

IM3 If all of $\mathrm{v}, \mathrm{u}_1, \mathrm{u}_2$ are in G_0 and there are distinct viewpoints, w, x in G_0 such that $Succ(\mathrm{u}_1) = \{\mathrm{w}\}$, $Succ(\mathrm{u}_2) = \{\mathrm{x}\}$, then:

(a) $G - G^{\mathrm{v}} = IM(G) - (IM(G))^{\mathrm{v}}$

(b) $(IM(G))^{\mathrm{w}} = G^{\mathrm{w}}$ and $IM(G)^{\mathrm{x}} = G^{\mathrm{x}}$

(c) $\mathrm{u}_1 \in V_i^{IM}$ and $\mathrm{v} \in V_j^{IM}$

(d) $Succ^{IM}(\mathrm{u}_1) = \{\mathrm{w}, \mathrm{x}\}$

(e) $f_i^{IM}(\mathrm{u}_1) = (f_i(\mathrm{v}) - \{\mathrm{v}\}) \cup \{\mathrm{u}_1\}$ and $f_j^{IM}(\mathrm{v}) = (f_j(\mathrm{u}_1) - \{\mathrm{u}_1, \mathrm{u}_2\}) \cup \{\mathrm{v}\}$

(f)
- $A_j^{IM}(\mathrm{v}, \mathrm{u}_1) = A_j(\mathrm{u}_1, \mathrm{w}) = A_j(\mathrm{u}_2, \mathrm{x})$
- $A_i^{IM}(\mathrm{u}_1, \mathrm{w}) = A_i(\mathrm{v}, \mathrm{u}_1)$
- $A_i^{IM}(\mathrm{u}_1, \mathrm{x}) = A_i(\mathrm{v}, \mathrm{u}_2)$

IM4 If only v, u_1 are in G_0 and there are distinct viewpoints, $\mathrm{w}_1, \mathrm{w}_2$ in G_0 such that $Succ(\mathrm{u}_1) = \{\mathrm{w}_1, \mathrm{w}_2\}$, then:

(a) $G - G^{\mathrm{v}} = IM(G) - (IM(G))^{\mathrm{v}}$

(b) $(IM(G))^{\mathrm{w}_k} = G^{\mathrm{w}_k}$ ($k \in \{1, 2\}$)

(c) $\mathrm{u}_1, \mathrm{u}_2 \in V_i^{IM}$ and $\mathrm{v} \in V_j^{IM}$ in $IM(G)$

(d) $Succ^{IM}(\mathrm{v}) = \{\mathrm{u}_1, \mathrm{u}_2\}$, $Succ^{IM}(\mathrm{u}_1) = \{\mathrm{w}_1\}$, and $Succ^{IM}(\mathrm{u}_2) = \{\mathrm{w}_2\}$

(e) $f_i^{IM}(\mathrm{u}_1) = (f_i(\mathrm{v}) - \{\mathrm{v}\}) \cup \{\mathrm{u}_1, \mathrm{u}_2\}$ and $f_j^{IM}(\mathrm{v}) = (f_j(\mathrm{u}_1) - \{\mathrm{u}_1\}) \cup \{\mathrm{v}\}$

(f)
- $A_j^{IM}(\mathrm{v}, \mathrm{u}_1) = A_j(\mathrm{u}_1, \mathrm{w}_1)$
- $A_j^{IM}(\mathrm{v}, \mathrm{u}_2) = A_j(\mathrm{u}_1, \mathrm{w}_2)$
- $A_i^{IM}(\mathrm{u}_1, \mathrm{w}_1) = A_i^{IM}(\mathrm{u}_2, \mathrm{w}_2) = A_i(\mathrm{v}, \mathrm{u}_1)$

IM5 If only v, u_2 are in G_0 and there are distinct viewpoints, $\mathrm{x}_1, \mathrm{x}_2$ in G_0 such that $Succ(\mathrm{u}_2) = \{\mathrm{x}_1, \mathrm{x}_2\}$, then $IM(G_0)$ is defined just as in **IM4** except with $\mathrm{u}_1, \mathrm{u}_2, \mathrm{w}_1, \mathrm{w}_2$ replaced by $\mathrm{u}_2, \mathrm{u}_1, \mathrm{x}_1, \mathrm{x}_2$ respectively.

IM6 If only v, u_1 are in G_0 and there is a viewpoint w in G_0 such that $Succ(\mathrm{u}_1) = \{\mathrm{w}\}$, then:

(a) $G - G^{\mathrm{v}} = IM(G) - (IM(G))^{\mathrm{v}}$

(b) $(IM(G))^{\mathrm{w}} = G^{\mathrm{w}}$

(c) $\mathrm{u}_1 \in V_i^{IM}$ and $\mathrm{v} \in V_j^{IM}$

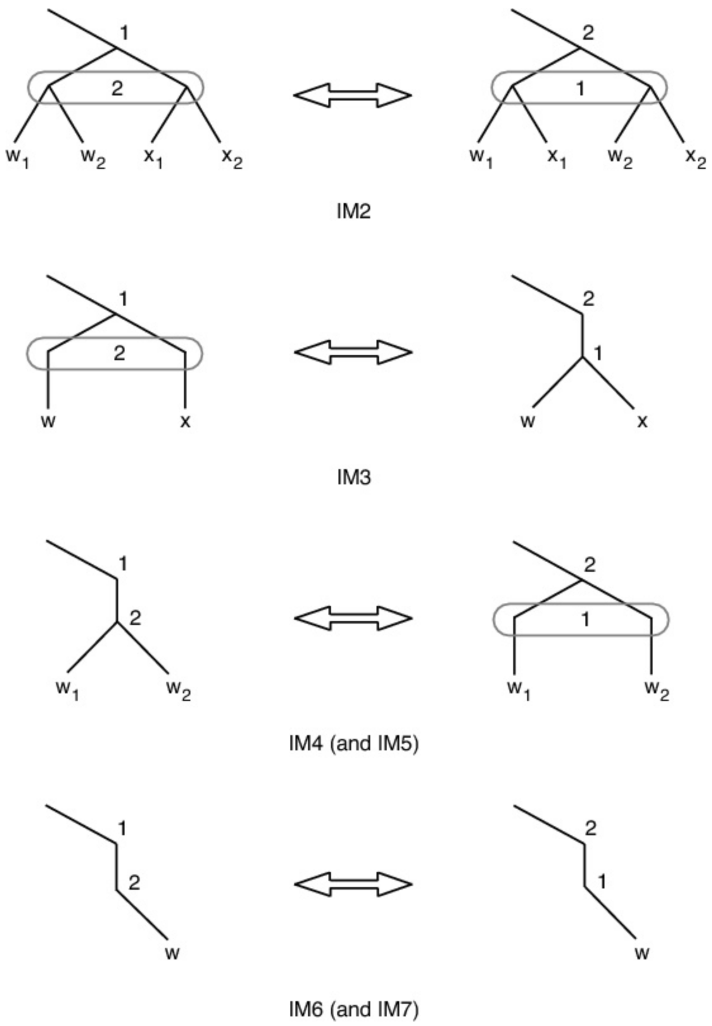

Figure 3: **IM2–IM7**

(d) $f_i^{IM}(\mathrm{u}_1) = (f_i(\mathrm{v}) - \{\mathrm{v}\}) \cup \{\mathrm{u}_1\}$ and $f_j^{IM}(\mathrm{v}) = (f_j(\mathrm{u}_1) - \{\mathrm{u}_1\}) \cup \{\mathrm{v}\}$

(e) • $A_j^{IM}(\mathrm{v}, \mathrm{u}_1) = A_j(\mathrm{u}_1, \mathrm{w})$
- $A_i^{IM}(\mathrm{u}_1, \mathrm{w}) = A_i(\mathrm{v}, \mathrm{u}_1)$

IM7 If only v, u_2 are in G_0 and there is a viewpoint x in G_0 such that $Succ(\mathrm{u}_2) = \{\mathrm{x}\}$, then $IM(G_0)$ is defined just as in **IM6** except with u_1, w replaced with u_2, x respectively.

The operation *IM* "interchanges" the moves of players i and j in the region of G_0 corresponding to $\mathrm{v}, \mathrm{u}_1, \mathrm{u}_2 \in W$ (as in Definition 4.7), leaving the rest of the game tree unchanged. When u_1 and u_2 are each followed by two daughter nodes, it transforms the game as in Thompson's original interchange of moves. *IM* also interchanges the moves of players i and j when the specified part of the tree is restricted in G_0. Since we assume that G is binary, the cases provided in the above definition exhaust all possible restrictions of G affecting the interchange of nodes v, u_1, and u_2 (see Figure 3).

Definition 4.8 (Interchange of Moves)**.** $\Gamma \sim_4 \Gamma'$ if (i) Γ is binary, (ii) there are $\mathrm{v}, \mathrm{u}_1, \mathrm{u}_2 \in W$ and $i, j \in I$ in the master game G of Γ that satisfy *IM Suitability*, and (iii) $\Gamma' \cong \Gamma^* = \{G^*_{v^*}\}_{v^* \in \mathcal{V}^*}$, where $\mathcal{V}^*$ and each $G^*_{v^*}$ are defined as follows:

1. The master game of $\Gamma^* = G^*_\lambda$ is $IM(G)$.
2. $v^*\hat{}\mathrm{v}^* \in \mathcal{V}^*$ iff $v^* \in \mathcal{V}^*$ and v^* is in G_{v^*}
3. $G^*_{v^*} = IM(G_{v^*})$ for $v^* \in \mathcal{V}$
4. $G^*_{v^*\hat{}\mathrm{v}^*}$ is a restriction of $G^*_{v^*}$ that is isomorphic to $IM(G_{v_0})$ for $v^*\hat{}\mathrm{v}^* \notin \mathcal{V}$, where v_0 is in $\mathcal{V}$ and obtained by replacing some occurrences of u_1 (respectively, u_2) in $v^*\hat{}\mathrm{v}^*$ with u_2 (respectively, u_1).

A brief comment on point 4 is in order. It ensures that the node introduced in **IM4** (respectively, **IM5**) agrees in its awareness with the other node in its information set. We have stated the condition slightly imprecisely to suppress irrelevant bookkeeping details.

It is straightforward to check that the Γ^* constructed in Definition 4.8 is an EGU. Given *IM* and the fourth condition in this definition, restrictions G_v of G get transformed to restrictions of $IM(G)$.

Two remarks. First, interchange of moves may only be applied to a binary EGU. However, it is always possible to convert an EGU into a binary EGU through successive applications of coalescing of moves and addition of superfluous moves (as in Thompson's original result). Second, $\mathcal{V}^*$ is different from $\mathcal{V}$. If G_v ($v \in \mathcal{V}$) is of the form specified in **IM4** (or **IM5**), then G_v does not contain u_2 (respectively, u_1), but $IM(G_v)$ will contain u_2 (respectively, u_1). This means that $v\hat{}\mathrm{u}_2 \notin \mathcal{V}$ (since u_2 is not in G_v, by **CE1** in Definition 3.3), yet $v\hat{}\mathrm{u}_2 \in \mathcal{V}^*$. However, the corresponding reduced strategic form is not changed because it collapses all nodes within an information set to a single decision point.

Definition 4.9 (Transformability on EGU)**.** Γ_1 is transformable into Γ_2, written as $\Gamma_1 \sim \Gamma_2$, if there is a sequence of EGU's, $\Gamma^*_1, \ldots, \Gamma^*_n$ such that $\Gamma^*_1 = \Gamma_1$, Γ^*_n is isomorphic to Γ_2, and $\Gamma^*_i \sim_t \Gamma_{i+1}$ ($1 \leq i \leq n-1$) with $t \in \{s, c, 1, 2, 3, 4\}$ (Γ_i is the result of applying one of the rules defined above to Γ_{i+1}).

5 Equivalence and Transformability

Given the definitions for strategic equivalence and transformability on EGU, we now prove the following result on EGU analogous to Thompson's result on extensive games: *For every EGU* Γ_1, Γ_2, $\Gamma_1 \sim \Gamma_2$ *iff* $\Gamma_1 \approx \Gamma_2$.

As we discussed above, it is straightforward to see that the transformations defined in Definitions 4.2–4.6 and 4.8 preserve strategic equivalence. Readers are invited to verify this.

Theorem 3. $\Gamma_1 \sim \Gamma_2$ implies $\Gamma_1 \approx \Gamma_2$.

For the other direction, we need some lemmas. First note the following facts about the transformations defined above. An application of a transformation transforms the master game of a given EGU Γ in the same way as the corresponding rule for extensive games; furthermore, the games $G_v \in \Gamma$ are each transformed similarly. For instance, an application of coalescing of moves to Γ will coalesce moves in the master game G of Γ in exactly the same fashion as Thompson's coalescing of moves would on G as a standard extensive game. Likewise, if they appear, the same moves are coalesced in each $G_v \in \Gamma$. Therefore, the following is a consequence of Lemma 2:

Observation 1. Let Γ be an EGU and G its master game. There is an EGU Γ' with G' its master game such that $\Gamma \sim \Gamma'$ and $sf(G')$ is isomorphic to $red^p(G')$.

Next, note that the extensive game assigned to each view is a restriction of the master game G. Also a brief inspection of Definition 2.10 reveals that any restriction of an extensive game in canonical form is an extensive game in canonical form. Therefore, in the above proposition, if $sf(G')$ is isomorphic to $red^p(G')$, for any restriction G'_v in Γ', $sf(G'_v)$ is isomorphic to $red^p(G'_v)$. Therefore, we have the following result that is analogous to Lemma 2 on extensive games.

Lemma 4. For every EGU Γ, there is an EGU Γ' such that $\Gamma \sim \Gamma'$ and $sf(\Gamma')$ is isomorphic to $red^p(\Gamma')$.

Let us say an EGU Γ is in *canonical form* if $sf(\Gamma')$ is isomorphic to $red^p(\Gamma)$.

Finally, we need a result analogous to Lemma 3.

Lemma 5. Let Γ, Γ' be EGU's in canonical form. $red^p(\Gamma)$ is isomorphic to $red^p(\Gamma')$ iff there is an EGU Γ^* such that Γ is isomorphic to Γ^* with $\Gamma' \sim \Gamma^*$.

Proof. The right-to-left direction is clear by Theorem 3. For the other direction, by Lemma 4, we can assume without loss of generality that Γ, Γ' are in canonical form. Moreover, by splitting of players (Definition 4.2), we can transform Γ, Γ' into p-normal form. Therefore, we can assume that Γ and Γ' are both isomorphic to their p-normal forms.

Next, let G, G' be the master games of Γ, Γ' respectively. By Lemma 3, there is an extensive game G^* such that $G \cong G^*$ and $G' \sim G^*$. Thus there is a function ϕ as specified in Definition 3.4 between the set of viewpoints in G and the set of viewpoints in G^*. Also, given $G' \sim G^*$, there is an EGU Γ^* such that $\Gamma' \sim \Gamma^*$ and G^* is the master game of Γ^*. Set $\Gamma = \{G_v\}_{v \in \mathcal{V}}$ and $\Gamma^* = \{G^*_{v^*}\}_{v^* \in \mathcal{V}^*}$. Define a function Φ from $\mathcal{V}$ to $\mathcal{V}^*$ so that, for all $v = \mathrm{v}_1 \ldots \mathrm{v}_n \in \mathcal{V}$, $\Phi(v) = \phi(\mathrm{v}_1) \ldots \phi(\mathrm{v}_n)$.

Our goal is to show that Γ is isomorphic to Γ^*. Let $\mathcal{V}_k = \{v \mid v \in \mathcal{V} \text{ and } \mathsf{len}(v) = k\}$, where $\mathsf{len}(v)$ is the length of v. By Definition 3.4, it suffices to show that, for all $v \in \mathcal{V}$ with $\mathsf{len}(v) = k$,

1. $\{\Phi(w) \mid w \in \mathcal{V}_{\mathsf{len}(v)}\} = \mathcal{V}^*_k$, and
2. $G_v \cong G_{\Phi(v)}$ with G_v and $G_{\Phi(v)}$ in canonical form.

Since 1 implies Φ is one-to-one, and, from 2 and Definition 3.4, it follows that Γ is isomorphic to Γ^*.

We prove the claim by induction on the length of $v \in \mathcal{V}$. The base case is clear. For the inductive step, first note that the functions that isomorphically map G to G^* induce the functions that isomorphically map $red^p(G)$ to $red^p(G^*)$. Also, given Definition 2.2, if $red^p(\Gamma) \cong red^p(\Gamma^*)$, the functions induce the functions that isomorphically map $red^p(\Gamma)$ to $red^p(\Gamma^*)$. In particular, define a function $\rho : I \to I^*$ such that, for all v_i in G and $\mathrm{v}^*_{i^*}$ in G^* with $\phi(\mathrm{v}_i) = \mathrm{v}^*_{i^*}$, $\rho(i) = i^*$. Then ρ is a function as specified in Definition 3.2. Put $red^p(\Gamma) = \{g_u\}_{u \in \mathcal{U}}$ and $red^p(\Gamma^*) = \{g_{u^*}\}_{u^* \in \mathcal{U}^*}$. Now, by the inductive hypothesis, $\{\Phi(w) \mid w \in \mathcal{V}_{\mathsf{len}(v)}\} = \mathcal{V}^*_k$ for an arbitrary k. Let $v = v_1 \ldots v_k \in \mathcal{V}$ with $\mathsf{len}(v) = k$ and $\Phi(v) = v^*$. Let $\mathbf{i}$=$i_1 \ldots i_k$ be the sequence of players such that $\mathrm{v}_l \in V_{i_l}$ $(1 \leq l \leq k)$. Let $\mathbf{i}\hat{\ }i$ be an arbitrary extension such that $\mathbf{i}\hat{\ }i \in \mathcal{U}$. We need to show that $\{\phi(\mathrm{v}) \in V_i \mid v\hat{\ }\mathrm{v} \in \mathcal{V}\} = \{\mathrm{v}^* \in V^*_{\rho(i)} \mid v^*\hat{\ }\mathrm{v}^* \in \mathcal{V}^*\}$. This follows from the inductive hypothesis that $G_v \cong G_{\Phi(v)}$, which implies by **CE1**: if $v\hat{\ }\mathrm{v} \in \mathcal{V}$, then $v^*\hat{\ }\phi(\mathrm{v}) \in \mathcal{V}^*$; and if $v^*\hat{\ }\mathrm{v}^* \in \mathcal{V}^*$, then there must be $\mathrm{v} \in \mathcal{V}$ such that $\phi(\mathrm{v}) = \mathrm{v}^*$. Since i and v are arbitrary, this proves that $\{\Phi(w) \mid w \in \mathcal{V}_{\mathsf{len}(v)}\} = \mathcal{V}^*_{k+1}$.

For the other part of the claim, by inductive hypothesis, $G_v \cong G_{\Phi(v)}$ for an arbitrary v with $\mathsf{len}(v) = k$, where $G_v, G_{\Phi(v)}$ are in canonical form. Let $v\hat{\ }\mathrm{v} \in \mathcal{V}$. First, it is clear (by Definitions 2.5 and 2.10) that $G_{v\hat{\ }\mathrm{v}}$ and $G^*_{\Phi(v\hat{\ }\mathrm{v})}$ are both in canonical form, since $G_{v\hat{\ }\mathrm{v}}, G^*_{\Phi(v\hat{\ }\mathrm{v})}$ are restrictions of $G_v, G^*_{v^*}$ respectively (by **CE 2**, **3**, and **5**), which are already in canonical form. Therefore, $sf(G_{v\hat{\ }\mathrm{v}}) \cong red^p(G_{v\hat{\ }\mathrm{v}})$ and $sf(G_{\Phi(v\hat{\ }\mathrm{v})}) \cong red^p(G_{\Phi(v\hat{\ }\mathrm{v})})$. Next, by assumption, $red^p(\Gamma) \cong red^p(\Gamma^*)$. By the definition of Φ above, we have $red^p(G_{v\hat{\ }\mathrm{v}}) \cong red^p(G_{\Phi(v\hat{\ }\mathrm{v})})$. This gives us $sf(G_{v\hat{\ }\mathrm{v}}) \cong sf(G_{\Phi(v\hat{\ }\mathrm{v})})$. Also, by assumption, Γ and Γ^* are in p-normal form, and $G_{v\hat{\ }\mathrm{v}}, G_{\Phi(v\hat{\ }\mathrm{v})}$ are restrictions of G_v and $G_{\Phi(v)}$, respectively, which are isomorphic to each other. Therefore, it follows from Definition 2.5 that $G_{v\hat{\ }\mathrm{v}} \cong G_{\Phi(v\hat{\ }\mathrm{v})}$. □

Theorem 4. For every EGU Γ_1, Γ_2, $\Gamma_1 \sim \Gamma_2$ iff $\Gamma_1 \approx \Gamma_2$.

Proof. The desired result follows from Theorem 3 and Lemmas 4 and 5 by an argument analogous to that given for Theorem 1. □

6 Conclusion

We have extended the Thompson transformations to games with unawareness. Along the way, we identified a novel transformation, coalescing of players, based on the insight that players are differentiated by their payoffs and, therefore, players with identical payoffs are strategically equivalent. In the case of games with unawareness, we discovered that players are differentiated by both payoffs and awareness. Consequently, from a strategic standpoint, a single player should be analyzed as two distinct agents if he exhibits two distinct states of awareness at different stages of a temporally extended game. The fact that these two agents share payoffs, i.e. they constitute a team, will ensure their actions are strategically coordinated. The next stage in this project is the investigation of solution concepts: which solution concepts are preserved under each of these transformations and which are not?

Acknowledgements The authors would like to thank Eric Pacuit for extensive comments at early stages of this project. They would like to thank Johan van Benthem not only for his constructive comments, but also his moral and financial support at various stages of this work. Finally, they thank the participants in both the Stanford *Logical Dynamics Workshop* and the ILLC *Seminar on Logics for Dynamics of Information and Preferences* for stimulating comments and discussion at presentations of earlier versions of this material.

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