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Rhetoric and analogies

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ABSTRACT

The art of rhetoric may be defined as changing other people's minds (opinions, beliefs) without providing them new information. One technique heavily used by rhetoric employs analogies. Using analogies, one may draw the listener's attention to similarities between cases and to re-organize existing information in a way that highlights certain regularities. In this paper we offer two models of analogies, discuss their theoretical equivalence, and show that finding good analogies is a computationally hard problem.

1. Motivation[1]

1.1. Example

A “Russia is a dangerous country.”
B “Nonsense.”
A “Don't you think that Russia might initiate a war against a Western country?”
B “Not a chance.”
A “Well, I believe it very well might.”
B “Can you come up with examples of wars that erupted between two democratic countries?”
A “I guess so. Let me see… How about England and the US in 1812?”
B “OK, save colonial wars.”
A “Anyway, you were wrong.”
B “I couldn't be wrong, as I said nothing. I merely posed a question.”
A “Yes, but your implicit claim was obvious.”
B “My implicit claim is that you can barely come up with such examples, apart from colonial wars.”

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[1] We have used the same example in Aragones et al. (2005) in the context of rule-based thinking.

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A “Well, this might be true.”
B “It is called ‘democratic peace' and it's a well-known phenomenon, I didn't make it up.”
A “So what are you saying?”
B “That Russia is not very likely to initiate a war against a Western democracy.”
A “Well, you might have a point, but only if Russia remains a democracy.”

The argument can go on forever. The point we would like to make is that B seems to have managed to change A's views, despite the fact that B has not provided A with any new information. Rather, B has pointed out a certain regularity (the “democratic peace” phenomenon) in the cases that are known to both A and B. This regularity is, apparently, new to A. Yet, A had had all the information needed to observe it before meeting B. It simply did not occur to A to test the accuracy of the generalization suggested to her by B. A's knowledge base contained many cases of wars (and of peaceful resolutions of conflicts as well), but they were not organized by the type of regime of the relevant parties. Indeed, it is likely that, if B were to ask A, “Can you name wars that occurred in the 20th century?” or “Can you name wars in which Germany was involved?”, A would have had no difficulty in coming up with examples. This is due to the fact that most people can access their international conflicts memory by the “indexes” of “period” or “country”. But most people do not generate the index “type of regime” unless they are encouraged to do so. Thus, drawing someone's attention to a certain regularity, or similarity among cases, may cause them to view the world differently without providing them any new factual information.

1.2. Example

> “And the LORD sent Nathan unto David. And he came unto him, and said unto him, There were two men in one city; the one rich, and the other poor. The rich man had exceeding many flocks and herds: But the poor man had nothing, save one little ewe lamb, which he had bought and nourished up: And it grew up together with him, and with his children; it did eat of his own meat, and drank of his own cup, and lay in his bosom, and was unto him as a daughter. And there came a traveller unto the rich man, and he spared to take of his own flock and of his own herd, to dress for the wayfaring man that was come unto him; But took the poor man's lamb, and dressed it for the man that was come to him. And David's anger was greatly kindled against the man; and he said to Nathan, As the LORD liveth, the man that hath done this thing shall surely die: And he shall restore the lamb fourfold, because he did this thing, and because he had no pity. And Nathan said to David, Thou art the man. […] And David said unto Nathan, I have sinned against the LORD.” (2 Samuel 12 :: King James Version)

How did Nathan convince King David that he had sinned? Has he provided any new information, previously unknown to the king? Hardly. King David knew perfectly well all the sordid details of his well-planned crime. He also knew how God would view this crime, were He consulted. Moreover, King David certainly did not need Nathan to tell him what his own moral code would suggest in the case of the two men (“the man… shall surely die”). What, then, did Nathan tell David that the latter did not already know?

We argue that Nathan did not inform David of any new fact. Rather, he pointed out to him the *analogy* between known facts. True, King David did not know the story about the rich and the poor men. But this story is a parable. It need not be factually true to be an effective rhetorical tool. What matters in this story is King David's general moral attitude, which had been known by him before he heard Nathan's parable. Moreover, even after King David heard the story, he only expresses his fury over the rich man's behavior. He seems neither to feel any regret over his own conduct, nor to even think of it. Only after Nathan said “Thou art the man” did the King change his view regarding his crime. Until that point, King David was missing the analogy between his sin and the rich man's wrongdoing.

1.3. Example

In August 1990, A and B argue about Iraq's invasion of Kuwait. A holds that the US should react to the invasion. B believes it should not. A might point out that Iraq's invasion of Kuwait is reminiscent of Nazi Germany's invasion of Czechoslovakia. B might counter that Iraq's invasion of Kuwait is actually analogous to Israel's occupation of the territories it captured in 1967. Such arguments are, for the most part, not about facts. They rarely provide new factual information, or dispute the facts considered known. Rather, it is often the case that all parties know, or at least have access to the same information. Each debater attempts to influence the other's opinion by pointing out analogies that might not have occurred to the other party.

2. Main thesis

We hold that much of the art of rhetoric has to do with changing other people's minds *without providing them new information*. One of the ways in which this can be done is by drawing people's attention to certain analogies between cases that they already know, or to certain regularities that characterize known cases.

The use of cases in rhetoric can be classified according to two criteria:

(i) *The logical structure of the argument* can be *analogical* or *inductive*. Analogies are correspondences between cases. Induction consists of generalization of (presumably) many cases to a rule, abstracting away from the details of these cases. Induction relies on similarity among cases, but it goes beyond similarities to the formulation of general rules. The arguments in Examples 1.2 and 1.3 above are arguments by analogies. By contrast, the argument in Example 1.1 is inductive: it attempts to summarize many similar cases by the regularity "democratic countries do not initiate wars against each other".
(ii) *The type of claim being made* can be *descriptive* or *normative*. A descriptive claim attempts to change the listener's beliefs or predictions about the world. A normative claim, by contrast, aims to change the listener's views about what is the "right", ethically preferred, or morally acceptable course of action.[2] The democratic peace argument is descriptive. It says nothing about the desirability of peace or the horrors of war. It deals only with the empirical truth of certain predictions. Nathan's argument in Example 1.2, on the other hand, is purely normative. It has nothing to do with claims about what the world actually is, but with what it should be. Generally, one often finds a mix of descriptive and normative rhetorical arguments. For instance, the analogy between Iraq's invasion of Kuwait and Germany's invasion of Czechoslovakia in Example 1.3 can be viewed as descriptive, namely, as suggesting that employing a policy, similar to that used by Britain in 1938, would result in a similar outcome. The analogy between the Iraq–Kuwait case and the Israeli occupation of the territories is largely normative: it suggests that just as no action was taken against Israel, none should be taken against Iraq.

The process of induction, and the related concept of abduction have been amply studied by formal models (see Carnap, 1950). It appears that less attention has been devoted to formal modeling of analogies. (Exceptions include Falkenhainer et al., 1989; Gick and Holyoak, 1980, 1983.) This paper attempts to make a contribution to the latter. That is, we do offer here a model of induction, and focus solely on analogies. We will, however, sketch the way in which inductive problems can be described in our models.

Descriptive and normative rhetorical arguments turn out to have a very similar structure.[3] Both descriptive and normative analogies are functions from one case to another. As an argument for or against a certain outcome in a given case, analogies attempt to derive authority from the outcomes of other cases. A descriptive analogy turns to actual events, in an attempt to show that a certain outcome is likely. A normative analogy cites actual or hypothetical ethical judgments and appeals to preferences for consistency of such judgments. Thus, a legal argument cites past court decisions, which are typically viewed as "correct" by definition.[4] But the source of authority has almost no effect on the formal structure of the analogy. Hence, in this paper we deal with both types of analogies.

3. Modeling analogies

In this section we outline two possibilities for formally modeling analogies between cases. The discussion here is informal, and attempts to convey the main intuition in the context of the examples above. We defer the formal definitions of the models and the results to the following section.

3.1. Attributes

Cases are similar if they have similar attributes. For instance, suppose that case c is the Iraqi invasion of Kuwait in 1990, and case d is the German invasion of Czechoslovakia in 1938. One attribute that these cases have in common is that the military might of the invading country. Another is the relative weakness of the invaded country. Yet another is the type of regime: both invaded countries might be viewed as dictatorships. There are, of course, many attributes that the cases do not share, such as the decade in which they occurred, the relative standing of the invading country, and so forth.

Some of the attributes that are relevant to the evaluation of the similarity between cases c and d can be described by the following matrix:

Attribute	a_1 Aggressor is strong	a_2 Victim is weak	a_3 Post-WWII	a_4 Aggressor is a dictatorship	a_5 Aggressor is a superpower
c	1	1	1	1	0
d	1	1	0	1	1

If one were to incorporate all relevant attributes as columns in this matrix, one may identify case c with the vector $(1,1,1,1,0)$, and case d—with $(1,1,0,1,1)$. It would seem natural that the similarity between these cases be determined based on the distance between these vectors. For instance, assume that $w=(w_1,\dots,w_5)$ is a vector of non-negative weights,

[2] Descriptive claims can, of course, be used in the context of a normative argument. For a more detailed discussion of descriptive and normative theories, see Gilboa and Schmeidler (2001).

[3] This is true also of inductive reasoning: rules can be thought of as "rules of nature", namely, descriptive, or as the essence of a moral code, that is, playing a normative role. Indeed, in the latter a rule can be viewed as a description of one's moral intuition, treated as data. (See Gilboa, 1994; Gilboa and Schmeidler, 2001.)

[4] We return to this example in the last section.

adding up to 1, and define, for $x, y \in \mathbb{R}^5$, the w-distance between x and y to be

$$d_w(x,y) = \sqrt{\sum_{i=1}^{5} w_i(x_i - y_i)^2}.$$

Then, one may define the similarity between two cases to be a function of the distance d_w between their attribute vectors. Reasonable conditions on the similarity function $s = s(d_w)$ are that it be non-negative, decreasing, and satisfy $s(0) = 1$ and $s(d) \to 0$ as $d \to \infty$.

For concreteness, assume that we choose

$$s(d) = \frac{1}{1+d}$$

and that $(w_1, \ldots, w_5) = (\frac{1}{5}, \ldots, \frac{1}{5})$. Then the similarity between cases c and d above will be computed as follows:

$$d_w(x,y) = \sqrt{\sum_{i=1}^{5} w_i(x_i - y_i)^2} = \sqrt{\frac{2}{5}} = 0.632$$

$$s(0.632) = \frac{1}{1.632} = 0.612$$

More generally, cases may have attributes as continuous variables. For instance, a country is not simply "strong" or "not strong". Military might is a matter of degree. Even the type of regime a country has varies on a scale between democracy and dictatorship. Since no loss of generality is involved, we will henceforth assume that for each case c and each attribute a there is a real number x_{ca}, between 0 and 1, measuring the degree to which case c had attribute a.

How do people select the weight vector w? Suppose that one suggests that the Iraqi invasion of Kuwait is similar to the German invasion of Czechoslovakia because both Saddam Hussein and Adolph Hitler had moustaches. That is, one offers a new attribute, "leader of aggressor has a moustache", and points out that the two cases have the same value for this attribute. What should be the weights attached to this attribute?

The answer to this question is subjective. Yet, it appears that most people learn the importance of attributes from past data: attributes that have been shown to have a greater explanatory power should be expected to be more heavily weighted. For instance, suppose that we are trying to predict, in case c, whether Saddam Hussein is likely to attack other countries if his invasion of Kuwait left unchallenged. Assume that we have a database of cases, such as d, for which it is known whether they did or did not result in further aggression. Let D be the set of cases that did result in aggression, and E the set of cases that did not. It is then meaningful to ask, what are the weights w that would make the cases within D and the cases within E most similar to each other, while making cases from different sets the most dissimilar? Abusing notation, let us identify cases in D and in E by their attribute vectors $x \in \mathbb{R}^5$. It is then meaningful to solve

$$(\text{Optimal}-\text{Weights}) Max_{w \in \Delta^4} \left[\sum_{x,y \in D} s(d_w(x,y)) + \sum_{x,y \in E} s(d_w(x,y)) - \sum_{x \in D, y \in E} s(d_w(x,y)) \right]$$

where Δ^4 is the 4-dimensional simplex of weights $w = (w_1, \ldots, w_5)$.

Let us consider Example 1.1 again. In this example, A starts the debate with no notion of the democratic peace phenomenon. She may be described as solving the Optimal-Weights problem above, using a restricted matrix, in which the regimes of the fighting countries do not appear as attributes. Once B points out that regimes might matter, A adds this attribute to the matrix. Using her own memory, she can fill up the values of many cases in this column. She then solves the Optimal-Weights problem again. If, indeed, the democratic peace phenomenon gets support based on A's memory, she will start putting more weights on regimes in her similarity judgments, and will use them in specific predictions.

Observe that Optimal-Weights has a somewhat different goal than does induction. The process of induction aims at discovering regularities. As such, the rule "democratic countries do not initiate wars against each other" is a regularity that might suggest itself from the data. Such a rule may indicate that Russia is not a dangerous country as long as it is a democracy. But it does not say how dangerous it is if it ceases to be a democracy. By contrast, the Optimal-Weights problem seeks the attribute weights that best explain the entire database. That is, it should not only distinguish between democracies and dictatorships, but it should also help us distinguish between wars and peaceful resolutions of conflict where at least one party was not a democracy. Thus, even if the democratic peace phenomenon is assumed to be a perfectly accurate description of history, the optimal weight of the attribute "both parties are democracies" will not be 1.

In the language of attributes, Example 1.1 describes a situation in which one debater points out an attribute that the other debater was not aware of, and this new attribute changes that similarity function of the listener. In Example 1.2, by contrast, the listener was not aware that a useful analogy might be used. Indeed, King David walks blindly into Nathan's trap. The King might have been more careful in passing judgment on the rich man, had he had some suspicion that his own conduct was about to be discussed. Finally, in Example 1.3 both debaters are aware of the issue they are debating, but they might not be aware of all the relevant cases, or of all the relevant attributes.

3.2. Predicates

Further insight might be gained if we endow cases with additional structure. Assume that each case consists of one or more entities, and certain relations between them. For instance, case c, of the Iraqi invasion of Kuwait, will deal with the entities "Iraq" and "Kuwait", and possibly also with the entity "Saddam". A two-place predicate $invade(Iraq, Kuwait)$ will reflect the main fact in the story. Other predicates may supply more details. For instance $powerful\ (Iraq)$ will state that the invading country was indeed strong, $leader(Saddam, Iraq)$ and $moustache(Saddam)$ will reflect the additional piece of information about the leader of the invading country, and so forth.

As in the case of attributes, certain predicates are fuzzy. Countries cannot be simply classified as "powerful" or "not powerful". Rather, they are powerful to a certain degree. Even a moustache can continuously be transformed into non-moustache. Thus, we do not model predicates as relations but as functions.

It will be convenient to define a case as a set of entities, and assume that, given these entities, all possible predicates can be computed for all possible arguments from a given case. Thus, we assume that the function *moustache* is defined for all entities. The fact that it is meaningless to ask whether Iraq has a moustache will simply be denoted by $moustache(Iraq) = 0$.

In this model, case c ("Iraq invaded Kuwait") is similar to case d ("Germany invaded Czechoslovakia") because there is a *mapping* between the entities involved in these cases. That is, let ϕ be a function from a subset of entities in case c to entities, such that $\phi(Iraq) = Germany$, $\phi(Kuwait) = Czechoslovakia$, and $\phi(Saddam) = Hitler$. The similarity between cases c and d rests, presumably, on the fact that predicates take similar values when they are applied to entities in c as when they are applied to their ϕ-images in d. For instance

$$\begin{aligned} invade(Iraq, Kuwait) &= invade(\phi(Iraq), \phi(Kuwait)) \\ &= invade(Germany, Czechoslovakia) = 1; \\ leader(Saddam, Iraq) &= leader(\phi(Saddam), \phi(Iraq)) \\ &= leader(Hitler, Germany) = 1; \\ moustache(Saddam) &= moustache(\phi(Saddam)) \\ &= moustache(Hitler) = 1 \end{aligned}$$

but there are also predicates that would distinguish between the cases

$$\begin{aligned} post-WWII(Iraq, Kuwait) = 1 &\neq post-WWII(\phi(Iraq), \phi(Kuwait)) \\ &= post-WWII(Germany, Czechoslovakia) = 0; \\ superpower(Iraq) = 0 &\neq superpower(\phi(Iraq)) \\ &= superpower(Germany) = 1. \end{aligned}$$

Fig. 1 depicts the predicate model in Example 1.2. In this figure, entities are written in lower-case (unboxed) and predicates—in upper case (boxed). The dashed lines describe the mapping ϕ. Curved lines represent non-zero values of predicates. For instance, the line going through "David", "TOOK", and "Bathsheba" means that $TOOK(David, Bathsheba) = 1$.

Predicates in this model are roughly analogous to attributes in the previous one.[5] Yet, several distinctions exist. Predicates are formally defined as functions of entities. A predicate can have any number of arguments (with one or two arguments used in the examples above). A case may involve many entities. Thus, a predicate can be applied to many subsequences of entities. For instance, given the predicate *invade*, it is meaningful to ask the meaning of $invade(Kuwait, Iraq)$, namely whether Kuwait also invaded Iraq. Indeed, the analogy between cases c and d has to do with the fact that in both cases the invaded country was not aggressive:

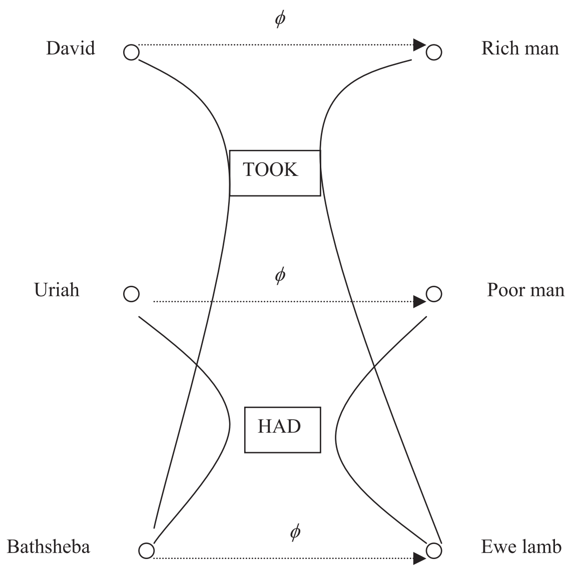
$$invade(Kuwait, Iraq) = invade(Czechoslovakia, Germany) = 0.$$

Moreover, given two cases, the analogy between them is not uniquely defined. An analogy, which is a mapping from one case into another, determines which entity in the range is the equivalent of each entity in the domain. In the examples above only one reasonable mapping was possible. But consider, for instance, the claim that "The situation in Israel brings to mind WWII". Some would agree with this statement because they draw an analogy between Israel's treatment of the Palestinian people and Germany's treatment of Jews in WWII. Others would subscribe to this very statement, because they feel that Jews are always victims of violence. That is, one mapping would be $\phi(Israel) = Germany$; $\phi(Palestinians) = Jews$, whereas another would be $\phi(Israel) = Jews$; $\phi(Palestinians) = Germany$. Thus, even if one knows what are the cases under discussion, it might not be obvious which analogy is being drawn.[6] The attribute model is not rich enough to describe a situation in which the cases are known, but the analogy between them is not.

How similar are two cases c and d? In light of the discussion above, this question cannot be answered unless we know what is the appropriate mapping between them. We will therefore judge the similarity of two cases given an analogy, or simply discuss the *strength* of this analogy.

[5] A formal equivalence theorem is provided in the next section.

[6] In the next section we prove that, for large cases, finding the "best" analogy might be computationally hard.



Entities are denoted by names (without boxes)

Predicates are denoted by boxed names

The dashed lines represent the analogy

Curved lines represent non-zero values of predicates

Fig. 1.

The strength of an analogy ϕ between case c and its image $\phi(c) \subset d$ will be defined based on the proximity of the values of all predicates, given all possible arguments from c and the same predicates applied to their images from d. Formally, for a case c define a vector x, such that each entry in x specifies a certain predicate and a sequence of arguments for it from case c. Similarly, let y be the corresponding vector generated for the $\phi-$image of c. The evaluation of similarity between cases c and $\phi(c)$ may then proceed as in the attribute model.

It follows that the problem of optimal weights for the similarity function can be re-defined in the predicate model. Indeed, the weights one attaches to predicates depend on their relative success in explaining past data. As in the attribute model, a person who is not aware of a certain predicate will use a similarity function that makes no use of this predicate. Should the person become aware of the predicate, she might realize that it suggests an important distinction between known cases, and she might consequently change her similarity function.

Thus, the predicate model provides an account of Example 1.1 that parallels that of the attribute model. In Example 1.2 the predicate model might have additional explanatory power. In particular, it is only within this model that we can formally prove that finding analogies is a difficult computational task. Yet, it is obvious from the story that King David's problem was not solving a hard problem, but realizing that there is a problem to be solved in the first place. Finally, in Example 1.3 the predicate model can embed the account provided by the attribute model, but it can also provide different analogies to the same "case".

4. Formal models and results

4.1. The attribute model

Let $\mathbb{C}$ be a set of *cases*, and $\mathbb{A}$ be a set of *attributes*. We assume that there is a matrix $(x_{ca})_{c \in \mathbb{C}, a \in \mathbb{A}}$ of real numbers in $[0,1]$, such that, for all $c \in \mathbb{C}, a \in \mathbb{A}$, x_{ca} measures the degree to which case c has attribute a. (Equivalently, we may view x as a function $x : C \times A \to [0,1]$.) It will be assumed that if a decision maker (or predictor) is aware of a (finite) set of cases $C \subset \mathbb{C}$ and a (finite) set of attributes $A \subset \mathbb{A}$, she knows the values of x_{ca} for all $c \in C$ and $a \in A$.

How does the decision maker judge similarity between cases $c, d \in C$? We will assume that there is a function $\Psi : [0,1]^A \times [0,1]^A \to [0,1]$ such that the similarity of c to d is $s(c,d) = \Psi((x_{ca})_{a \in A}, (x_{da})_{a \in A})$. We assume that $\Psi(x,x) = 1$ for all $x \in [0,1]^A$.

We refer to the quadruple (C, A, x, Ψ) as an *attribute model*.

4.2. The predicate model

Let there be given a set of *entities* $\mathcal{E}$ and, for each $n \geq 1$, a set of n-place *predicates* $\mathcal{P}_n$. For each $p \in \mathcal{P}_n$ there is a function $f_p : \mathcal{E}^n \to [0,1]$. The value $f_p(e_1, \ldots, e_n)$ measures the degree to which predicate p is true of the n-tuple $(e_1, \ldots, e_n)$. The set of all predicates is denoted $\mathcal{P} \equiv \bigcup_{n \geq 1} \mathcal{P}_n$. We will assume that, at a given time, the decision maker is aware of a finite set of entities $E \subset \mathcal{E}$ and a finite set of predicates $P \subset \mathcal{P}$. It will be useful to denote $P_n \equiv P \cap \mathcal{P}_n$ and $F_n = \{f_p | p \in P_n\}$.

A *case* is modeled as a finite set of entities of which the decision maker is aware $c \subset E$. Given two cases c, d, an *analogy* between c and d is a function $\phi : c \to d$. Thus, any function $\phi : c \to E$ defines an analogy between c and its image $\phi(c)$.

We assume that the *strength* of the analogy $\phi : c \to E$, or the $\phi-$similarity of case c to case $\phi(c)$, is a function of the values of known predicates, when applied to all possible arguments in the case. Specifically, consider a case $c \subset E$ and let $B(c)$ be the set consisting of all pairs of a sequence of entities from the case c up to length $|c|$, and a corresponding predicate

$$B = B(c) = \bigcup_{i=1}^{|c|} [c^i \times P_i].$$

Assume that the decision maker has a function $\Theta : [0,1]^B \times [0,1]^B \to [0,1]$ such that the strength of an analogy ϕ is

$$S(\phi) = \Theta((f_p(e))_{i \leq |c|, e \in c^i, p \in P_i}, (f_p(\phi(e)))_{i \leq |c|, e \in c^i, p \in P_i},)$$

Thus, for an analogy ϕ with domain c, $S(\phi)$ measures the similarity of c to $\phi(c)$. We assume that $\Theta(x,x) = 1$ for all $x \in [0,1]^B$.

We refer to the quintuple $(E, P, (f_p)_{p \in P}, c, \Theta)$ as a *predicate model*. Observe that the case c is part of the definition of the predicate model. Indeed, the domain of the evaluation function Θ depends on c (or, to be precise, on $|c|$).[7]

4.3. An equivalence result

Proposition 1. *Let there be given an attribute model (C, A, x, Ψ) and a case in it $c \in C$. There exists a predicate model $(E, P, (f_p)_{p \in P}, \hat{c}, \Theta)$ and, for each case $d \in C$, an analogy $\phi_d : \hat{c} \to E$, such that $S(\phi_d) = s(c, d)$.*

Conversely, let there be given a predicate model $(E, P, (f_p)_{p \in P}, \hat{c}, \Theta)$. There exists an attribute model (C, A, x, Ψ), a case in it $c \in C$, and, for each analogy $\phi : \hat{c} \to E$ a case $d_\phi \in C$, such that $s(c, d_\phi) = S(\phi)$.

The above result suggests that theoretical models can use either the attribute model or the predicate model. The predicate model deals more explicitly with various relations that make up a story. It may therefore be more insightful for the analysis of actual cases. By contrast, the attribute model is mathematically simpler and may render itself more easily to theoretical analysis. The two models might be compared to models of non-cooperative games in the extensive or in the normal form. The predicate model, like the extensive form, is perhaps closer to the way people actually think about specific stories. The attribute model, like the normal form, allows a unified mathematical analysis using standard mathematical and statistical techniques.

It would appear that theoretical models should always prefer the attribute model. This may not hold when complexity considerations are introduced. Again, in a way that parallels the distinction between normal and extensive form games, the attribute model may be exponentially large relative to the predicate model. Moreover, it abstracts away from computational problems. The following sub-section shows that the problem of finding a “best” analogy may be computationally hard when viewed in the predicate model. The same problem is easy in the attribute model. It should thus be emphasized that the above result only deals with the theoretical expressive power of the two models.

4.4. A note on complexity

There are several reasons for which finding useful, insightful, or compelling analogies may be a difficult cognitive task. First, it may be hard to think of all the cases that might be relevant to a case at hand, simply because memory contains a vast collection of cases. Second, conjuring up all the relevant attributes, or predicates, is also a non-trivial task, as many attributes/predicates can potentially prove relevant. Third, even if the predicates or attributes are given, and so are the cases, there may be more than one way to draw analogies between two given cases. This issue cannot be captured in the attribute model, in which cases have no structure. But in the predicate model cases are sets of entities, and two such sets can be mapped to each other in more than one way. In this section we prove that the problem of finding a “best” analogy is computationally hard. This formal result captures only the last source of difficulty in finding “correct” analogies. It shows that drawing analogies is a hard problem *even if* cases and predicates are given. It should be interpreted as suggesting that finding analogies is a hard problem, *a fortiori*, if the relevant cases and predicates are not given in the problem.

Formally, let there be given a predicate model $(E, P, (f_p)_{p \in P}, c, \Theta)$. The function Θ may be represented in several ways, not all of which will be equivalent for complexity calculations. For instance, a function that can be described by a short PASCAL program may take exponential space if described by enumeration.

[7] One may define a function Θ_i for all the cases c with $|c| = i$, and omit c from the definition of the predicate model.

To simplify matters, we will focus on the following problem. Let there be given $(E, P, (f_p)_{p \in P}, c)$, where the functions $(f_p)_{p \in P}$ assume only rational numbers as values, and are described by enumeration. An analogy $\phi : c \to E$ is *perfect* if, for each $1 \le i \le n$, each ordered i-tuple of entities in c, $e = (e_{j_1}, \ldots, e_{j_i}) \in c^i$, and each i-place predicate $f \in F_i$, $f(e) = f(\phi(e))$. The trivial analogy, where ϕ is the identity function, is clearly perfect. Other analogies are *non-trivial*. We can now state the following.[8]

Proposition 2. *The following problem is NP-Complete: Given* $(E, P, (f_p)_{p \in P}, c)$, *is there a non-trivial perfect analogy for* c?

Observe that the problem of existence of a perfect analogy can also be described as a problem of maximizing the function Θ. Thus, finding the strongest analogy (based on evaluation by Θ) is a hard problem. Indeed, if we allow the function Θ to be described in an algorithmic way (say, as a PASCAL program), then it can be shown that $\Theta-$maximization is an NP-Hard problem.

We do not intend to argue that the main difficulty is drawing analogies is the combinatorial complexity of checking all possible permutations of entities in cases. Indeed, as mentioned above, much of the difficulty is in thinking of cases and of predicates. Yet, finding convincing analogies shares an important aspect with solving NP-Complete problems: it is typically very difficult to come up with a solution, while it is much easier to check that a suggested solution is, indeed, a solution.

5. Modeling issues and future directions

Most of the formal literature in economic theory and in related fields adheres to the Bayesian model of information processing. In such a model, a decision maker starts out with a prior probability, and she updates it in face of new information by Bayes rule. Hence, this model can easily capture changes in opinion that result from new information. But it does not deal very graciously with changes of opinion that are not driven by new information. In fact, in a Bayesian model with perfect rationality people cannot change their opinions unless new information has arrived. It follows that the examples we started out with cannot be explained by such models.

It is not only stylized examples such as those above that cannot be accommodated in the standard model. Suppose one was interested in modeling appellate courts (or courts of appeal) and the adjudication process in those courts. Courts of appeals do not hold trials, but rather they review decisions of trial courts for errors of law. Accordingly, an appeal court considers only the record (that is, the papers the parties filed and the transcripts and any exhibits from any trial) from the trial court, and the legal arguments of the parties. These arguments, which are presented in written form, and can range in length from dozens to hundreds of pages, are known as “briefs”. Sometimes lawyers are permitted to add to their written briefs with oral arguments before the appeals judges. At such hearings, only the parties' lawyers speak to the court.

It is somewhat difficult to understand the role of the lawyers in this process. The only relevant information for the appellate court is public—the trial transcript and all previous court decisions, which are in the public domain. Since all information that can be used in making the decision is public, why isn't the question of whether or not the relevant law for a particular case has been applied correctly “ simply” a matter of seeing how it has been applied in previous relevant cases, and deciding the case at hand on that basis? More precisely, what is it that the lawyers do that the judge could not do himself?

From the perspective of this paper, we might model the adjudication process as having lawyers presenting briefs to a judge containing their arguments, buttressed by references to past cases and the decisions reached in those cases. With this perspective, lawyers present briefs that contain cases that they think are good analogs to the case at hand, and for which the decision was the one the lawyer is arguing for in the current case. The judge then weighs the strength of the analogies of the current case with each of the previous cases presented in the two briefs. After aggregating the weighted strengths of the analogies, the judge makes the decision that coincides with the decision taken in the cases that are more similar to the current case.

If analogies were easy to find, one could dispense with the lawyers. For any given case, all practical relevant cases are on computers. The judge need only a computer program that will search for the relevant analogs to the current case, and weigh the strengths; these will serve the judge at least as well as a pair of lawyers' briefs. The proposition above points to one serious difficulty in computerizing the adjudication process: the computational complexity of finding the best analogies. The process instead is put in the hands of competing lawyers to identify analogs to the case at hand; highly regarded lawyers are those who can find ingenious ways to identify particular cases that are good analogs.[9]

Relaxing the perfect rationality assumption, one may attempt to provide a pseudo-Bayesian account of the phenomena discussed here. We believe, however, that the models offered in this paper may provide further insight into the mechanism by which rhetoric works.

The predicate model allows considerable freedom in the definition of a “case”. The German invasion of Czechoslovakia might be described as a case that ended in the invasion of Poland in 1939. It can also be expanded to include WWII, or the history of Czechoslovakia. Indeed, histories of cases are relevant, if not crucial to the judgment of similarity between them. Future events that followed a given case are similarly relevant to the generation of predictions based on it. Thus, there is a certain degree of arbitrariness in delimiting a case in time. Moreover, it is also not always clear what is the scope of a case.

[8] The following result was quoted in Gilboa et al. (forthcoming).

[9] These lawyers, of course, may use computers to identify potential analogous cases.

For instance, should the case "the invasion of Kuwait by Iraq" tell us something about oil prices during this period? Thus, even within a given time frame, a case may be described in varying degrees of detail.

Because cases are defined as subsets of entities, they can also be thought of as *stories*. Like stories in natural communication, cases may overlap, one case may be a subset of another, and, indeed, all cases may be viewed as subsets of a single, "universal" case.

Observe that the predicate model describes similarity only between entities, measured by the degrees to which they satisfy certain predicates. It does not allow one to describe similarity between predicates. For instance, one might find the predicate "annex" similar to the predicate "invade", and suggest an analogy between Germany's annexation of Austria to Iraq's invasion of Kuwait. Our model can only describe such an analogy by suggesting that both "invade" and "annex" are different degrees to which a single predicate holds. One may wish to augment the model by the introduction of similarity relations between predicate.

There are other relations between predicates that one might wish to model formally. For instance, thinking of a case as a story, it is natural to suggest that one predicate, denoting a temporal fact (such as "King David took Bathsheba") occurred *after* another (say, after Bathsheba had been married to Uriah). In any given story, predicates may be indexed by time, so that temporal precedence will be implicitly defined by the predicate symbols. But in order to evaluate similarity between stories that occurred in different periods, an explicit temporal precedence relation might be indispensable.

In this paper we only model case-to-case inductive inferences. Modeling case-to-rule inferences involves the use of logic. Future work might seek to describe both case-to-case and case-to-rule inductive inference in a unified formal framework.

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Appendix A

Proof of Proposition 1. Let there be given an attribute model (C, A, x, Ψ) and a case $c \in C$. Define $E = C$. Define the set of predicates P as follows: for each attribute $a \in A$, introduce a one-place predicate p_a such that $f_{p_a}(d) = x_{a,d}$ for all $d \in C$. Let $\widehat{c} \subset E$ be defined by $\widehat{c} = \{c\}$. We now turn to define the function Θ. Since $|\widehat{c}| = 1$ and all predicates in P are one-place predicates, $B(\widehat{c}) = \bigcup_{i=1}^{|c|}[\widehat{c}^i \times P_i]$ equals $\widehat{c} \times P_1 = \widehat{c} \times P$. Thus, there is a natural bijection between the domain of Θ, namely, $[0,1]^B \times [0,1]^B$, to that of Ψ, namely, $[0,1]^A \times [0,1]^A$. Abusing notation we can therefore define $\Theta = \Psi$. Given a case $d \in C$, define $\phi_d : \widehat{c} \to E$ by $\phi_d(c) = d$. With this definition, $S(\phi_d) = s(c, d)$ clearly holds.

Conversely, let there be given a predicate model $(E, P, (f_p)_{p \in P}, \widehat{c}, \Theta)$. Let $n = |\widehat{c}|$ and denote the entities in $\widehat{c}$ by $\{e_1, \ldots, e_n\}$. Define the attribute model as follows. Let $C = E^n$. Choose $c = (e_1, \ldots, e_n)$ to be the case in C corresponding to $\widehat{c}$. We now turn to define the set of attributes A and the matrix x. For each $1 \le i \le n$, each ordered i-tuple of entities in $\widehat{c}$, $e = (e_{j_1}, \ldots, e_{j_i}) \in \widehat{c}^i$, and each i-place predicate $f \in F_i$, define an attribute $a_{i,e,f}$. Let A be the set of all attributes so obtained. Given a case $d = (g_1, \ldots, g_n) \in E^n$ and attribute $a_{i,e,f}$, define $x_{d,a_{i,e,f}} = f(g_{j_1}, \ldots, g_{j_i})$. That is, the attribute defined by the triple (i, e, f) is the value of the predicate f on the i elements of the case at hand, defined by the ordered selection e out of the case c. By definition of A, there is a natural bijection between A and $B(\widehat{c}) = \bigcup_{i=1}^{|c|}[\widehat{c}^i \times P_i]$. Thus we may again abuse notation and define $\Psi = \Theta$. Let there be given an analogy $\phi : \widehat{c} \to E$, and define $d_\phi = (\phi(e_1), \ldots, \phi(e_n)) \in C$. Clearly, $s(c, d_\phi) = S(\phi)$. □

Proof of Proposition 2. It is easy to see that the problem, "given $(E, P, (f_p)_{p \in P}, c)$, is there a non-trivial perfect analogy for c?" is in NP. Indeed, checking whether an analogy ϕ is perfect or not can be done in a linear number of operations (as a function of the input). To show that it is NP-Complete, we reduce the CLIQUE problem to this one.

PROBLEM CLIQUE: Given an undirected graph $Gr(V, A)$ and a number $k \ge 1$, are there k vertices in V each two of which are connected by an arc in A?

The CLIQUE problem is known to be NP-Complete (see Gary and Johnson, 1979). The reduction is as follows: given a graph $Gr(V, A)$ (where A is a set of subsets of V of size 2) and a number $k \ge 1$, define $(E, P, (f_p)_{p \in P}, c)$ as follows. Define the entities to be the vertices in the graph, as well as k new entities. That is, $E = V \cup \{e_1, \ldots, e_k\}$ where $\{e_1, \ldots, e_k\} \cap V = \varnothing$. Let there be a single two-place predicate p (i.e., $P = \{p\}$) which is the incidence matrix of the graph Gr, coupled with the incidence matrix of a clique on $\{e_1, \ldots, e_k\}$. Formally: $f_p(e_i, e_j) = 1$ for every $1 \le i \ne j \le k$. For $v, w \in V$, $f_p(v, w) = 1$ iff $\{v, w\} \in A$. All other values of f_p are set to zero. (In particular, $f_p(e, e) = 0$ for every $e \in E$, and for $i \le k$ and $v \in V$, $f_p(e_i, v) = f_p(v, e_i) = 0$.) The case c is $\{e_1, \ldots, e_k\}$. It is easy to see that c has a non-trivial perfect analogy if and only if Gr contains a clique. Further, writing down the enumeration of $(E, P, (f_p)_{p \in P}, c)$ can be done in polynomial time complexity (relative to the size of $Gr(V, A)$). □

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