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APPROXIMATE EFFICIENCY OF NON-WALRASIAN NASH EQUILIBRIA[1]

BY A. POSTLEWAITE AND D. SCHMEIDLER[2]

1. INTRODUCTION

M. SHUBIK HAS RECENTLY introduced a model in which a good is exchanged for money. Each seller offers a quantity of the good for sale and in exchange receives a proportion of the total amount of money bid for the commodity equal to the proportion he offered to sell. Similarly, each buyer bids an amount of money and receives a proportion of the good sold equal to the proportion of his bid to the total money bid by all buyers. This model has been extended by L. Shapley [**6**] and M. Shubik [**7**] to include the standard Arrow–Debreu pure exchange economy with one of the goods being used as money.

The model used here is that introduced by E. Pazner and D. Schmeidler which eliminates the need for commodity money by using units of account. This technique eliminates an inefficiency arising from a fixed initial amount of money postulated by previous models. This inefficiency which prevails in even large economies is analysed by J. Jaynes, H. Okuno, and D. Schmeidler [**3**].

This model has several interesting features when compared to the Walrasian model. First, the rules of the market yield a well-defined outcome regardless of whether or not the traders behave intelligently. Secondly, there is no exogenously imposed distinction made between the optimizing criteria of large and small traders. In particular, the extent to which an agent takes advantage of his ability to influence prices is endogenously determined by his economic environment, i.e., his preferences and endowment relative to the aggregate behavior of the others. Thirdly, it is a price-formation model.

The Nash equilibria of this model, which are the counterparts of the Walrasian equilibria, are generally not Pareto efficient. However, this is to be expected if there are only a few traders active on some market. The inefficiency should be negligible when each agent's trades are negligible relative to the aggregate trade if this is to be a plausible model. The object of this paper is to show such a result.

Since in any model with a finite number of traders, a specific trader cannot be completely negligible, we should not expect to get complete efficiency in such a case. Rather, when each agent is nearly negligible, the allocation resulting from a Nash equilibrium should be nearly efficient. Roughly speaking an allocation is said to be ε-efficient if it cannot be Pareto dominated when aggregate endowment is reduced to $(1-\varepsilon)$ time the original endowment. Intuitively, this means that at most an ε proportion of the aggregate endowment is being wasted.

A precise presentation of the Shapley–Shubik models as well as a more detailed motivation appear in Shapley [**6**]. This paper contains a partial bibliography of earlier work.

[1] This research was partially supported by NSF Grant #SOC75-05317. This paper was presented at the NBER Conference-Seminar on General Equilibrium Theory at Berkeley in February, 1976. Thanks are due the class of Econ. 467, Fall, 1975, for debugging an earlier version of this paper.

[2] D. Schmeidler was on leave from Tel Aviv University when the work was done.

2. DESCRIPTION OF THE MODEL AND THE MAIN RESULT

A (pure exchange) economy is an order triple $(T, (w_t)_{t\in T}, (\succsim_t)_{t\in T})$ where T is a finite nonempty set, the set of traders in the economy, $(w_t)_{t\in T}$ is a T-list of vectors in R^l_+, the nonnegative orthant of an Euclidean space of dimension l, and $(\succsim_t)_{t\in T}$ is a T-list of binary relations on R^l_+. A vector in R^l_+ is a commodity bundle; the different commodities in the economy are indexed by elements of the set $L=\{1, 2, \ldots, l\}$. The number of commodities l is fixed throughout this paper. The T-list $(w_t)_{t\in T}$ is the initial endowments of the traders in T and $(\succsim_t)_{t\in T}$ is the list of their preferences. It is assumed, throughout the paper, that for each t in T the binary relation $\succsim_t$ is total, transitive, continuous, monotonic and convex. To state it formally, the binary relation $\succsim_t$ satisfies for all x, y, z in R^l_+: $x \succsim_t y$ or $y \succsim_t x$, $x \succsim_t y$ and $y \succsim_t z$ imply $x \succsim_t z$, the sets $\{x' \in R^l_+ | x' \succsim_t x\}$ and $\{x' \in R^l_+ | x \succsim_t x'\}$ are closed in R^l_+, $x \geqslant y$ implies $x \succsim_t y$ and $x > y$ implies $x \succ_t y$, and the set $\{x' \in R^l_+ | x' \succsim_t x\}$ is convex. (Inequalities between vectors in R^l hold coordinatewise, by definition, and the relations $\succ_t$ and $\sim_t$ are induced by $\succsim_t$ in the usual way.)

The trade in the economy is institutionalized by the rules of a T-person game in strategic form defined as follows: For each t in T let $S_t = \{(b, s) \in R^l_+ \times R^l_+ | s \leqslant w_t\}$ be the set of strategies for trader t. Given a T-list of strategies $(b_t, s_t)_{t\in T}$, the payoff to trader h (h in T) denoted by $c_h((b_t, s_t)_{t\in T})$ (or c_h for short), is a commodity bundle defined as follows: $c_h = (c^1_h, c^2_h, \ldots, c^l_h)$ in R^l_+ and for each k in L:

$$c^k_h = \begin{cases} w^k_h - s^k_h + \dfrac{b^k_h}{\sum_{t\in T} b^k_t} \sum_{t\in T} s^k_t & \text{if } (*), \\[2ex] w^k_h - s^k_h & \text{otherwise}, \end{cases}$$

where * is a balance condition:

$$(*) \qquad \sum_{k\in L} b^k_h \leqslant \sum_{k\in L} \left(\frac{s^k_h}{\sum_{t\in T} s^k_t} \sum_{t\in T} b^k_t \right)$$

and it is agreed that whenever the denominator is zero ($\Sigma_{t\in T} s^k_t = 0$ or $\Sigma_{t\in T} b^k_t = 0$) the fraction is also equal to zero ($0/0 = 0$). In our interpretation trader h offers for sale on market k the quantity s^k_h and offers b^k_h units of account to buy goods on this market. The condition (*) means that he is not bankrupt. If he is, the commodities that he bought are confiscated. We denote by $C_h\big((b_t, s_t)_{t\in T, t\neq h}\big)$ the set $\{c_h\big((b_s, s_t)_{t\in T, t\neq h}, (b, s)\big) | (b, s) \text{ in } S_h\}$, the set of attainable consumption bundles for trader h given some fixed strategies of all other traders.

A T-list of strategies $\{b^*_t, s^*_t)_{t\in T}$ is a Nash-equilibrium (N.E. for short) if for all h in T, $c_h\big((b^*_t, s^*_t)_{t\in T}\big) \succsim_h c_h\big((b^*_t, s^*_t)_{t\in T \text{ and } t\neq h}, (b, s)\big)$ for all (b, s) in S_h. There is one trivial N.E. in which for all t in T, $b^*_t = 0 = s^*_t$. An N.E. is full if each market is activated, i.e., for all k in L, $\Sigma_{t\in T} b^{*k}_t > 0$ and $\Sigma_{t\in T} s^{*k}_t > 0$.

An allocation in the economy is a T-list of commodity bundles, $(x_t)_{t\in T}$ such that $\Sigma_{t\in T} x_t \leqslant \Sigma_{t\in T} w_t$. It is obvious that for any T-list of strategies, the resulting T-list of payoffs $(c_t)_{t\in T}$ is an allocation. For an $\varepsilon>0$, an allocation $(x_t)_{t\in T}$ is ε-efficient (or ε-Pareto-efficient) if for any T-list of commodity bundles $(y_t)_{t\in T}$: $\forall t\in T$, $y_t \succsim_t x_t$) implies (not $\Sigma_{t\in T}\, y_t \leqq (1-\varepsilon)\,\Sigma_{t\in T}\, w_t$).[3]

APPROXIMATE EFFICIENCY THEOREM: *For any positive numbers α, β, and ε, any allocation resulting from a full Nash equilibrium in an economy $(T, (w_t)_{t\in T}, (\succsim t)_{t\in T})$ with $w_t<\beta(1,1,\ldots,1)$ for all t in T, $\Sigma_{t\in T}\, w_t > \# T\alpha(1,1,\ldots,1)$ and $\# T > 16l\beta/\alpha\varepsilon^2$ is ε-efficient.*

3. THE PROOF

First, several lemmata are proved that are of independent interest. All of them deal with possible responses of an arbitrary (but fixed throughout the discussion) trader, h, given the strategies of others $(b_t, s_t)_{t\neq h, t\in T}$ for some economy as defined in the previous section.

The following notation will be used: $\hat{b}=\Sigma_{t\neq h}\, b_t$, $\hat{s}=\Sigma_{t\neq h}\, s_t$, $\bar{b}=\Sigma_{t\in T}\, b_t$, and $\bar{s}=\Sigma_{t\in T}\, s_t$. The last two notations will be used when it is clear what the strategy of h is. We define also $p^k=\bar{b}^k/\bar{s}^k$, the average price of commodity k, when $\bar{s}^k>0$. For a vector of prices $(p^1, p^2, \ldots, p^l)=p\in R^l_+$, the corresponding Walrasian budget at x in R^l_+, $\{y\in R^l_+ | p\cdot y\leq p\cdot x\}$, is denoted by $B(p, x)$. The first two lemmata appear also in [**4**].

LEMMA 1: *Given a T-list of strategies with $\hat{b}>0$ and $\hat{s}>0$, any x in $C_h((b_t, s_t)_{t\in T, t\neq h})$ can be achieved by some (b, s) in S_h with $b\cdot s=0$.*

PROOF: A somewhat stronger result is proved: For any k in L and a strategy for trader h, (b^k_h, s^k_h), on the market for this commodity there is a strategy (b^k, s^k) which yields the same net trade in the kth commodity and the same contribution to the balance condition (*) and satisfies $b^k s^k=0$.

Suppose, first, that trader h is a net seller of commodity k, i.e., $w^k_h \geq w^k_h - s^k_h + b^k_h(\hat{s}^k+s^k_h)/(\hat{b}^k+b^k_h)$. Consider the strategy $(b^k, s^k)=(0, s^k_h - b^k_h(\hat{s}^k+s^k_h)/(\hat{b}^k+b^k_h))$. It obviously yields the required net sale. The net gain in units of account, which previously was $s^k_h(\hat{b}^k+b^k_h)/(\hat{s}^k+s^k_h)-b^k_h$, is now:

$$\frac{[s^k_h - b^k_h(\hat{s}^k+s^k_h)/(\hat{b}^k+b^k_h)]\hat{b}^k}{\hat{s}^k+[s^k_h - b^k_h(\hat{s}^k+s^k_h)/(\hat{b}^k+b^k_h)]}.$$

The last expression can be reduced to the previous one.

In case that h is a net buyer of commodity k consider the strategy $(b^k, s^k)=(b^k_h - s^k_h(\hat{b}^k+b^k_h)/(\hat{s}^k+s^k_h), 0)$. It is clear that this strategy yields the same net expenditure in units of account on the market for commodity k as (b^k_h, s^k_h). The appropriate computation shows that the net quantities bought via (b^k_h, s^k_h) and (b^k, s^k) are identical also in this case. *Q.E.D.*

LEMMA 2: *Given a T-list of strategies and a trader h such that $\hat{b}>0$ and $\hat{s}>0$, there is a strictly convex function with continuous partial derivatives F_h: $\{x\in R^l_+ | x< w_h+\hat{s}\}\to R$ such that $C_h=\{x\in R^l_+ | F_h(x)\leq 0\}$.*

[3] This concept is reminiscent of Debreu's coefficeint of resource utilization [**1**].

PROOF: For $x \in R^l_+$ and $x < w_h + \hat{s}$ define

$$F_h(x) = \sum_{k \in L} \left(\frac{x^k - w^k_h}{\hat{s}^k - x^k + w^k_h} \right) \hat{b}^k.$$

This is a well defined function and at least twice continuously differentiable. The kth partial derivative of F_h at x, $(\partial F_h(x)/\partial x^k)$, is equal to $\hat{b}^k \hat{s}^k/(\hat{s}^k - x^k + w^k_h)^2$. Successive differentiation leads to: $(\partial^2 F_h(x)/(\partial x^k)^2) = 2\hat{b}^k \hat{s}^k (\hat{s}^k - x^k + w^k_h)/(\hat{s}^k - x^k + w^k_h)^4 > 0$ and $(\partial^2 F_h(x)/\partial x^k \partial x^j) = 0$ for $j \neq k$. So F_h is a strictly convex function.

With $x \in R^l_+$, $x < w_h + \hat{s}$ we associate the strategy (b, s) in S_h where $b^k = 0$ and $s^k = w^k_h - x^k$ for k in L such that $w^k_h - x^k \geqslant 0$, and $b^k = \hat{b}^k(x^k - w^k_h)/\hat{s}^k - x^k + w^k_h$ and $s^k = 0$ for k in L such that $x^k - w^k_h > 0$. By the rules of the game the final bundle of trader h is precisely x if the balance condition (*) is satisfied for this strategy. From the definition of this strategy it is obvious that the condition (*) is satisfied if and only if $F_h(x) \leqslant 0$. It is also immediate that the final bundle of commodity k in L with $w^k_h - x^k \geqslant 0$ is x^k. For commodity k in L with $x^k - w^k_h > 0$ the final consumption is $w^k_h + b^k \hat{s}^k/(\hat{b}^k + b^k)$ which also yields x^k when b^k is replaced by $\hat{b}^k(x^k - w^k_h)/\hat{s}^k - x^k + w^k_h$. Thus $\{x \in R^l_+ | F_h(x) \leqslant 0\} \subset C_h$. In order to see the reverse inclusion, first note that any strategy (b, s) in S_h with $b \cdot s = 0$ and condition (*) satisfied results in the final bundle $x \leqslant w_h + \hat{s}$ with $F_h(x) \leqslant 0$. Next, by Lemma 1 any strategy in S_t (satisfying (*)) can be reduced to a strategy (b, s) with $b \cdot s = 0$ as above. *Q.E.D.*

COROLLARY: (i) *The choice set C_h is strictly convex (relative to R^l_+) and if x is on the boundary of C_h (relative to R^l_+ i.e., $F_h(x) = 0$), then there is a unique hyperplane supporting C_h at x defined by the partial derivatives of F_h at x.* (ii) *If some x on the boundary of C_h is obtained by h via (b_h, s_h) in S_h, then the kth partial derivative of F_h at x, to be denoted by p^k_h, is equal to $[(\hat{b}^k + b^k_h)/(\hat{s}^k + s^k_h)]^2 \hat{s}^k/\hat{b}^k$.*

PROOF: The first part of the corollary is an immediate conclusion of Lemma 2 and only part (ii) will be proved here. The expression for the kth partial derivative of $F_h(x)$ appears in the proof of Lemma 2 and is equal to $\hat{b}^k \hat{s}^k/(\hat{s}^k - x^k + w^k_h)^2$. Suppose that trader h is a net seller of commodity k, i.e., $w^k_h - x^k \geqslant 0$. By Lemma 1 the consumption is not affected if trader h replaced his strategy on the market for commodity k, (b^k_h, s^k_h), with $(0, s^k_h - b^k_h(\hat{s}^k + s^k_h)/(\hat{b}^k + b^k_h))$. Replacing $w^k_h - x^k$ in the expression for the kth derivative with its strategic representation, we get the following expression for p^k_h:

$$\hat{b}^k \hat{s}^k \Big/ \left(\hat{s}^k + s^k_h - b^k_h \frac{\hat{s}^k + s^k_h}{\hat{b}^k + b^k_h} \right)$$

which, in turn, can be reduced to the desired $[(\hat{b}^k + b^k_h)/(\hat{s}^k + s^k_h)]^2 \hat{s}^k/\hat{b}^k$.

If trader h is a net buyer of commodity k, the amount of his net trade of this commodity $(x^k - w^k_h)$ is equal to $(b^k_h/(\hat{b}^k + b^k_h))(\hat{s}^k + s^k_h) - s^k_h$. Inserting this in the expression for $(\partial F_h(x)/\partial x^k)$, i.e., p^k_h which is $\hat{b}^k \hat{s}^k/(\hat{s}^k - x^k + w^k_h)^2$ is also reducible to the desired form. *Q.E.D.*

LEMMA 3: *Given a T-list of strategies and a trader h and a good k with* $b_h^k/\bar{b}^k<\delta$ *and* $s_h^k/\bar{s}^k<\delta$, $(1-\delta)p^k<p_h^k<[1/(1-\delta)]p^k$.

PROOF: From the Corollary we have $p_h^k=(\bar{b}^k/\bar{s}^k)^2(\hat{s}^k/\hat{b}^k)$; by definition $p^k=\bar{b}^k/\bar{s}^k$. Therefore $p_h^k/p^k=(\bar{b}^k/\bar{s}^k)(\hat{s}^k/\hat{b}^k)=[\bar{b}^k/(\bar{b}^k-b_h^k)][(\bar{s}^k-s_h^k)/\bar{s}^k]=[1/(1-b_h^k/\bar{b}^k)][1-s_h^k/\bar{s}^k]$. From the hypothesis, we see then that $1-\delta<p_h^k/p^k<1/(1-\delta)$ and the lemma is proved. *Q.E.D.*

PROPOSITION: *If* $\forall k\in L$, $b_h^k/\bar{b}^k<\delta$ *and* $s_h^k/\bar{s}^k<\delta$, *then* $B(p,(1-\delta)^2c_h)\subset B(p_h,c_h)$.

PROOF: We will show that the maximal amount of each good trader h could purchase in $B(p,(1-\delta)^2c_h)$ is less than the maximal amount he could purchase in $B(p_h,c_h)$.

The maximal amount of good k which trader h can purchase in $B(p,(1-\delta)^2c_h)$ is $p\cdot c_h(1-\delta)^2/p^k$; the maximal amount of good k which he can purchase in $B(p_h,c_h)$ is $p_h\cdot c_h/p_h^k$. (Recall p_h defines the supporting hyperplane of the Corollary.) Now, $p_h\cdot c_h/p_h^k>p\cdot c_h(1-\delta)^2/p^k$ is true if and only if $p_h\cdot c_hp^k/p\cdot c_hp_h^k>(1-\delta)^2$. But

$$\begin{aligned}p_h\cdot c_hp^k/p\cdot c_hp_h^k&=\frac{(p_h^1c_h^1+\cdots+p_h^lc_h^l)p^k}{(p^1c_h^1+\cdots+p^lc_h^l)p_h^k}\\&\geqslant\frac{[(1-\delta)p^1c_h^1+\cdots+(1-\delta)p^lc_h^l]p^k}{(p^1c_h^1+\cdots+p^lc_h^l)p^k[1/(1-\delta)]}\\&=(1-\delta)^2,\end{aligned}$$

where the inequality follows from Lemma 3. Thus the proposition follows.

Q.E.D.

PROOF OF THEOREM: Consider an economy as in the statement of the theorem with $\# T>16l\beta/\varepsilon^2\alpha$, $(b_t^*,s_t^*)_{t\in T}$ N.E. strategies, and $(c_t)_{t\in T}$ the resulting allocation. If $R=\{t\in T|b_t^{*k}/\bar{b}^k<\varepsilon/4$ and $s^{*k}/\bar{s}^k<\varepsilon/4, \forall k\in L\}$, there are at most $8l/\varepsilon$ traders in $T\backslash R$, the complement of R in T. Therefore $\Sigma_{t\in T\backslash R}w_t<8l\beta(1,\dots,1)/\varepsilon$ or $(\varepsilon/8l\beta)\Sigma_{t\in T\backslash R}\,w(t)<(1,\dots,1)$. We also have by hypothesis that $(1/\# T\alpha)\Sigma_{t\in T\backslash R}\,w_t>(1,\dots,1)$, and thus from these two inequalities and the fact that $\# T>16l\beta/\varepsilon^2\alpha$, we get $(\varepsilon/2)\Sigma_{t\in T}\,w_t>\Sigma_{t\in T\backslash R}\,w_t$ which in turn implies $\Sigma_{t\in R}\,w_t>(1-\varepsilon/2)\Sigma_{t\in T}\,w_t$.

If $(c_t)_{t\in T}$ is not ε-efficient there exists an allocation $(y_t)_{t\in T}$ with $y_t\succsim_t c_t\forall t$ and $\Sigma_{t\in T}\,y_t\leqslant\Sigma_{t\in T}(1-\varepsilon)w_t$. But then

$$(**)\qquad\begin{aligned}\sum_{t\in R}y_t\leqslant\sum_{t\in T}y_t\leqslant\sum_{t\in T}(1-\varepsilon)w_t&<[(1-\varepsilon)/(1-\varepsilon/2)]\sum_{t\in R}w_t\\&<(1-\varepsilon/2)\sum_{t\in R}w_t.\end{aligned}$$

The third inequality follows from the last inequality of the previous paragraph.

The hyperplane of the Corollary supporting C_t at c_t supports also $\{x \in R^l_+ | x \succsim_t c_t\}$ since $\succsim_t$ is convex and c_t maximizes $\succsim_t$ over C_t. Therefore, $y_t \succsim_t c_t \Rightarrow p_t \cdot y_t \geqslant p_t \cdot c_t$, $\forall t \in T$. Also, for $t \in R$ it follows from the Proposition that $[p_t \cdot y_t \geqslant p_t \cdot c_t \Rightarrow p \cdot y_t \geqslant p \cdot (1-\varepsilon/4)^2 c_t = p \cdot (1-\varepsilon/4)^2 w_t > p \cdot (1-\varepsilon/2) w_t]$ (recall p is the vector of average prices). Thus $p \cdot \Sigma_{t \in R}\, y_t \geqslant p \cdot (1-\varepsilon/2)\, \Sigma_{t \in R}\, w_t$, contradicting (**). *Q.E.D.*

4. NASH EQUILIBRIUM ALLOCATIONS AS APPROXIMATE WALRAS ALLOCATIONS

An examination of the proof of the theorem reveals that what has been shown is in fact stronger than the statement of the theorem. Not only is the Nash equilibrium in the large economy ε-efficient, but also a T can be found such that except for an arbitrarily small proportion of the traders ($T \backslash R$ in the proof), each trader receives a consumption bundle which is preference maximizing over a set which is almost his Walrasian budget set. It fails to be the normal Walrasian budget set in two respects; first traders in R can be facing prices which differ slightly from each other and, secondly given these prices, the value of the consumption bundle will generally be slightly higher than the value of the trader's initial endowment. More formally we introduce the following definition.

For $\varepsilon > 0$, an allocation $(x_t)_{t \in T}$ is an ε-Walras allocation if there exists $p \neq 0$ in R^l_+ (prices) such that for all t in T, $p \cdot x_t = p \cdot w_t$ and $\#\{t \in T | \forall x \in R^l_+ : x \succ_t x_t \Rightarrow p \cdot x > (1-\varepsilon) p \cdot w_t\} > (1-\varepsilon) \# T$. The second condition means that except for an ε proportion of the traders, the consumption bundle of any trader is as good as any bundle in the ε "trimmed" budget set.

APPROXIMATE WALRAS ALLOCATION THEOREM: *Under the conditions of the Approximate Efficiency Theorem, a Nash equilibrium allocation is an ε Walras allocation.*

PROOF: As in the proof of the Theorem we choose δ of the Proposition to be $\varepsilon/4$; then the number of traders who do satisfy the conditions of the Proposition is at most $8l/\varepsilon$. Hence the relative number of these traders is at most $(8l/e)/(16 l\beta/\alpha\varepsilon^2) = \varepsilon\alpha/2\beta < \varepsilon$. The last inequality holds since $\alpha \leqslant \beta$. The Corollary tells us that for each trader we have: $x > x_t \Rightarrow p_h \cdot x > p_h \cdot x_t$. By the Proposition we have, for a trader whose proportional trade is less than δ in each market, $p_h \cdot x > p_h \cdot x_t \Rightarrow p \cdot x > p \cdot x_t (1-\delta)^2$ which in turn is greater than $(1-\varepsilon) p \cdot x_t$.
Q.E.D.

5. EXTENSION TO VIRTUALLY FULL NASH EQUILIBRIA

The theorem dealt only with full Nash equilibria, that is, it said nothing about a Nash equilibrium in which there were one or more commodities which were not traded.

If a Nash equilibrium is not full, we could restrict our attention to those goods which are traded. In the subspace corresponding to those commodities traded, the theorem is still true. However, the allocation resulting from a Nash equilibrium might be ε-efficient with respect to the entire set of commodities. The commodities which are not traded may be not traded due to their being efficiently distributed already given trades in other commodities. In particular, if for each good not traded there is a price such that no trader would want to buy or sell that commodity (given the other trades as fixed), then a commodity's not being traded does not affect the ε-efficiency of the theorem.

More formally, for a given economy and Nash equilibrium (and using the notation of previous sections), suppose that some commodities are not traded; set $L' = \{k \in L | \bar{b}^k = 0$ (and of course $\bar{S}^k = 0)\}$. The list $(\tilde{p}^k)_{k \in L'}$ is said to be a *virtual price vector* (for the given N.E.) if $p^k > 0$ for all k in L' and all traders' best responses are their original N.E. strategies when faced with the new aggregate strategy vector $(\hat{b}, \hat{s})$ where $\hat{b}^k = \tilde{p}^k$ and $\hat{s}^k = 1$ for k in L' and $(\hat{b}^k, \hat{s}^k) = (\hat{b}^k, \hat{s}^k)$ for k in $L \backslash L'$. This notion originated with Shapley and Shubik (see [6]). A Nash equilibrium is said to be *virtually full* if there exists a virtual price vector for it.

EXTENDED APPROXIMATE EFFICIENCY THEOREM: *The Approximate Efficiency Theorem holds when "full Nash equilibrium" is replaced by "virtually full Nash equilibrium."*

PROOF: As noted above, all the lemmata, in particular Lemma 3, hold when one restricts his attention to the subspace of commodities traded. For the commodities not traded, $k \in L'$, we define the new average price $p^k = \tilde{p}^k$, the virtual price. It is obvious that the corresponding p_h^k (for $k \in L'$) is also equal to $\tilde{p}^k$ since trader h, $h \in T$, maximizes his preferences over $B(p_h, c_h) = \{x \in R_+^l | \Sigma_{k \in L \backslash L'} p_h^k x^k + \Sigma_{k \in L'} \tilde{p}^k x^k \leqslant \Sigma_{k \in L \backslash L'} p_h^k c_h^k + \Sigma_{k \in L'} \tilde{p}^k c_h^k\}$ at c_h. Hence Lemma 3, and consequently the proposition, hold for the full dimensional commodity space. *Q.E.D.*

This result can be further extended by allowing different traders to have slightly different virtual prices such that: for each h, $t \in T$, $(1/(1-\varepsilon/4))\tilde{p}_h^k \geqslant \tilde{p}_t^k \geqslant (1-\varepsilon/4)\tilde{p}_h^k$. We refer to a N.E. where each trader has different virtual prices restricted as above as an *ε-virtually full N.E.* It goes without saying that the conclusion of the main theorem holds when "full N.E." is replaced by "ε-virtually full N.E."

6. CONCLUDING REMARKS

We note first that ε Walras allocation is not necessarily ε efficient. Without the restrictions on initial distribution of the resources, the relatively few traders who prefer a consumption bundle of the ε "trimmed" budget set to the final allocation may control most of the resources. If the distribution among these agents is very inefficient, so is the entire allocation.

However an ε Walras allocation and in particular a Nash equilibrium allocation in a sufficiently large economy is a Walras equilibrium for some economy close to it (with an appropriate definition of "close"). This will be the subject of a later paper.

We also note that the bankruptcy rule which confiscates the purchases of a trader who is bankrupt, is not essential. It is clear that it is sufficient that a bankruptcy rule prevents a strategy which results in bankruptcy from being an optimal response. In particular, less stringent penalties for bankruptcy could be devised. As an example we quote from Pazner–Schmeidler [**4**]:

BANKRUPTCY LEMMA: *The set of full Nash equilibria is not affected if the payoff function is modified as follows*:

$$c_h^k = w_h^k - s_h^k + \rho \frac{b_h^k}{\sum_{t\in T} b_t^k} \sum_{t\in T} s_t^k \qquad (\forall k, h)$$

where

$$\rho = min \left(1, \frac{\sum_{k\in L} \left(\frac{s_h^k}{\sum_{t\in T} s_t^k} \sum_{k\in T} b_t^k \right)}{\sum_{k\in L} b_h^k} \right)$$

(*with our usual agreement*, $0/0 = 0$).

These remarks suggest that for sufficiently large economies, a Nash equilibrium would in fact be a competitive equilibrium for some economy close to it (with an appropriate definition of "close"). This will be the subject of a later paper.

Hurwicz [**2**] showed that there is a basic incompatibility between efficiency of an economic system and its manipulability. In particular he showed that any non-coercive system which guaranteed Pareto-efficient outcomes could, at least in some cases, be manipulated by some trader to his own advantage. Roberts–Postlewaite [**5**] showed that for a system which leads to a competitive equilibrium, the advantage of such manipulation asymptotically disappears. Thus in sufficiently large economies the competitive equilibrium maintains its efficiency properties and is almost non-manipulable.

The main theorem in this note is essentially a dual to this result. Nash equilibria of a game are by definition non-manipulable. Moreover, the theorem tells us that in sufficiently large economies, it is nearly efficient. Hence, although non-manipulability and efficiency may be incompatible, in large economies either property may be weakened only slightly and preserve the other.

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