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APPROXIMATE WALRASIAN EQUILIBRIA AND NEARBY ECONOMIES*

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1. INTRODUCTION

If one considers an economy it is sometimes useful to have a notion of an allocation being "nearly" Walrasian. Notions of an allocation being approximately Walrasian generally refer to relaxing the definition of Walrasian equilibrium. For example, an allocation might be called nearly Walrasian if there is a price vector p such that each agent is nearly maximizing his utility subject to his budget constraint with respect to p, or if he is maximizing his utility with respect to some price which is close to p.

There is a second reason for examining the notion of nearly Walras allocations, however. Suppose we were to ask whether an allocation is Walrasian for a particular economy. If we know precisely the characteristics of the agents in an economy, we can determine whether the allocation is Walrasian or not. If we don't know the characteristics, but rather, we must estimate them, it is clearly too much to hope that the allocation would be Walrasian with respect to the estimated characteristics even if it were Walrasian with respect to the true characteristics. Thus we might think of an allocation as being nearly Walrasian for an (estimated) economy if there is some economy quite similar for which the allocation is truly Walras. If we have a notion of ε-Walras allocation which has this property and if we see for some economy an ε-Walras allocation, it is possible that it is truly Walrasian and the "non-Walrasianness" comes from estimation errors rather than from a non-Walrasian process.

In Postlewaite and Schmeidler [1978] we introduced a notion of an approximate Walras allocation. Roughly speaking, an allocation is approximate Walras if there exists a price such that "nearly" all people are "almost" as well off as if they received their Walrasian demand at the given price. It was shown that the non-trivial Nash equilibria of a particular non-cooperative market game were ε-Walras under certain regularity conditions when there are many traders. It was remarked there that for a sufficiently large economy, a Nash equilibrium would in fact probably be a competitive equilibrium for some economy close to it (with an appropriate definition of "close"). This paper will provide a proof of this assertion. A metric is defined on economies and it is shown that if an allocation is ε-Walras for some economy, then this is Walras for another economy within ε distance of the given economy.

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[1] This work was done while D. Schmeidler was visiting the University of Minnesota.

In light of the discussion of ε-Walras allocations above, we can interpret our results as follows. If the process which determines trades were the non-cooperative game considered in Postlewaite and Schmeidler [1978], the Nash equilibria when there are many agents are ε-Walras. Given the result that an ε-Walras allocation is truly Walras for a nearby economy one could not easily pronounce that the procedure generating the allocation was not Walrasian by examining the allocation unless one is certain that there have been no errors in determining the agents' characteristics.

The proof of the theorem is constructive: new preference relations are defined for each trader so that his net trade under the ε-Walras allocation is preference maximal with respect to the price in the definition of the ε-Walras allocation. This guarantees that in the new economy, the allocation is Walras. It is then shown that the perturbations necessary to alter the preferences leave the new economy within ε of the original economy. It is worthwhile noting that if an allocation is a Nash equilibrium for the non-cooperative game mentioned above, then the alterations made to the preferences do not disturb this status. Thus in the altered economy the allocation is both a Nash equilibrium and a Walras allocation.

There is a corollary to this result that an ε-Walras allocation in an economy is a Walras allocation in a nearby economy. We say that an allocation is ε-efficient if it is impossible to Pareto improve upon it with an aggregate endowment which is $1-\varepsilon$ times the original endowment.[2] We then show that if an allocation is ε-Walras, then it is ε-efficient. That is an ε-Walras allocation wastes not more than ε proportion of the endowment. We also prove a second corollary: that an ε-efficient allocation is fully efficient in a nearby economy.

The corollary which states that an ε-Walras allocation is ε-efficient is interesting in that this contrasts with another notion of approximate equilibrium due to Scarf. We say that a price p is an approximate equilibrium price if the aggregate excess demand for an economy at price p is less than ε times the aggregate endowment. There must of course be rationing of some sort if the excess demand is not zero. When this rationing takes place however, much inefficiency in the allocation may be introduced. For an arbitrary positive ε, an example is constructed such that a particular price p is an ε equilibrium price in the sense of Scarf. Yet after rationing, the final allocation is seen to be not ε-efficient for any $\varepsilon<1/2$. Despite the fact that at the given price excess demand is small, the rationed allocation wastes at least one-half of the aggregate endowment. This is true regardless of the rationing scheme which is used so long as the rationing scheme cannot confiscate money nor force people to sell more than they desire at the stated price. This means that you may find prices with arbitrarily small excess demands for an economy and ration the remaining excess demand in any manner you please (within the constraints mentioned above) and the resulting allocation may be extremely inefficient. Given the result that an ε-Walras allocation is

[2] $1-\varepsilon$ is the coefficient of resource utilization in Debreu [1951].

ε-efficient, this means that the allocation which you arrive at after rationing is not ε-Walras.

2. DESCRIPTION OF THE MODEL AND RESULTS

A (pure exchange) economy is an ordered triple $(T, (w_t)_{t\in T}, (\succsim_t)_{t\in T})$ where T is a finite nonempty set, the set of traders in the economy, $(w_t)_{t\in T}$ is a T-list of vectors in R^l_+, the nonnegative orthant of a Euclidean space of dimension l, and $(\succsim_t)_{t\in T}$ is a T-list of binary relations on R^l_+. A vector in R^l_+ is a commodity bundle; the different commodities in the economy are indexed by elements of the set $L=\{1, 2,\ldots, l\}$. The number of commodities l is fixed throughout this paper. The T-list $(w_t)_{t\in T}$ is the initial endowments of the traders in T and $(\succsim_t)_{t\in T}$ is the list of their preferences. It is assumed throughout the paper that for each t in T the binary relation $\succsim_t$ is total, transitive, continuous, monotonic and convex. To state it formally, the binary relation $\succsim_t$ satisfies for all x, y, z in R^l_+: $x\succsim_t y$ or $y\succsim_t x$; $x\succsim_t y$ and $y\succsim_t z$ imply $x\succsim_t z$; the sets $\{x'\in R^l_+ \mid x'\succsim_t x\}$ and $\{x'\in R^l_+ \mid x\succsim_t x'\}$ are closed in R^l_+; $x\geqq y$ implies $x\succsim_t y$ and $x>y$ implies $x\succ_t y$; and the set $\{x'\in R^l_+ \mid x'\succsim_t x\}$ is convex. (Inequalities between vectors in R^l hold coordinatewise, by definition, and the relations $\succ_t$ and $\sim_t$ are induced by $\succsim_t$ in the usual way.)

An allocation in the economy is a T-list of commodity bundles, $(x_t)_{t\in T}$ s.t. $\sum_{t\in T} x_t \leqq \sum_{t\in T} w_t$. For an $\varepsilon>0$, an allocation $(x_t)_{t\in T}$ is ε-efficient (or ε-Pareto-efficient) if for any T-list of commodity bundles $(y_t)_{t\in T}$: $(\forall t\in T,\ y_t\succsim_t x_t)$ implies (not $\sum_{t\in T} y_t \leqq (1-\varepsilon)\sum_{t\in T} w_t$).

For $\varepsilon>0$, an allocation $(x_t)_{t\in T}$ is an ε Walras allocation if there exists $p\neq 0$ in R^l_+, $\sum_{i\in L} p^i=1$ such that for all t in T, $p\cdot x_t = p\cdot w_t$ and $\sum_{t\in T}(p\cdot x_t - v_p(x_t)) < \varepsilon\cdot \sum_{t\in T} p\cdot w_t$ where $v_p(x_t)=\inf(p\cdot x \mid x\in R^l_+$ and $x\succsim_t x_t\}$. If a trader consumes a bundle satisfying his budget equality which is not utility maximal over his budget set, there is a bundle which costs less yielding the trader the same utility. The above definition states that an allocation is ε-Walras if the average proportion of wasted income is less than or equal to ε.

If an allocation is ε efficient for every positive ε, it is efficient in the usual sense. Similarly an ε Walras allocation for ε equals 0 is a Walras allocation in the usual sense.

We want to define a metric on economies. Since we are interested in the simple case where there is a fixed finite number of traders we will use a variation of the metrics used by Kannai [1970], Debreu [1969], and Hildenbrand [1970]. We start by defining a metric on preferences. Given two preference relations $\succsim$ and $\succsim'$ define $d(\succsim, \succsim')$ to be the Hausdorff distance between their graphs. We omit the proof that d is a metric. Given two economies $e=\{T, (w_t)_{t\in T}, (\succsim_t)_{t\in T}\}$, $e'=\{T, (w'_t)_{t\in T}, (\succsim'_t)_{t\in T}\}$, we define

$$m(e, e') = (1/\# T)\textstyle\sum_{t\in T}(d(\succsim_t, \succsim'_t) + 2\|w_t - w'_t\|/\|\sum_{t\in T} w_t + w'_t\|).$$

THEOREM. *If $(x_t)_{t\in T}$ is an ε Walras allocation in an economy e, there exists an* *economy* e' *with* $m(e, e')<\varepsilon$ *and* $(x_t)_{t\in T}$ *a Walras allocation in* e'.

COROLLARY 1. *If* $(x_t)_{t\in T}$ *is an* ε *Walras allocation in an economy* e, *it is* ε *efficient.*

COROLLARY 2. *If* $(x_t)_{t\in T}$ *is an* ε *efficient allocation in an economy* e, *there exists an economy* e' *with* $m(e, e')<\varepsilon$ *and* (x_t) *an efficient allocation in* e'.

The above results can be generalized to include the possibility of non-varying production. In order to avoid ambiguity, we state the definitions of ε efficient and ε Walras allocations in economies with production. As usual the production possibilities will be given by a subset Y of R^l. An allocation $(x_t)_{t\in T}$ is a T-list of commodity bundles such that $\sum_{t\in T}(x_t)\in(Y+\sum_{t\in T}w_t)$. For $\varepsilon>0$, an allocation $(x_t)_{t\in T}$ is ε efficient if for any T-list of commodity bundles $(y_t)_{t\in T}$: $(\forall_{t\in T}, y_t\succsim_t x_t)\rightarrow \sum_{t\in T}y_t\notin(Y+(1-\varepsilon)\sum_{t\in T}w_t)$. In order that this definition coincide with the previous definition in the pure exchange case, free disposal is assumed. Similarly an allocation $(x_t)_{t\in T}$ is an ε Walras allocation if there exists $p\neq 0$ in R^l_+ such that for all $t\in T$: $p\cdot x_t\geqq p\cdot w_t$ and $\sum_{t\in T}(p\cdot x_t-v_p(x_t))\leqq\varepsilon V_p(Y+\sum_{t\in T}w_t)$, where $V_p(Y+\sum_{t\in T}w_t)=\sup\{p\cdot z\mid z\in Y+\sum_{t\in T}w_t\}$. The manner in which production possibilities are described does not preclude the possibility of several firms. Under the usual conditions on the aggregate production possibility set, the appropriate analogues of the Theorem and the Corollaries hold.

Another concept of approximate Walrasian equilibrium is that which is obtained via Scarf's algorithm. A price p is said to be an ε excess demand price for an economy e if

$$|\textstyle\sum_{t\in T}y_t - w_t| < \varepsilon\sum_{t\in T}w_t$$

where y_t is agent t's Walrasian demand at price p. Since in the aggregate these demands are in general not feasible, some form of rationing must take place before an allocation is reached. We will only consider rationing schemes which maintain agents' budget restrictions and which give an agent his stated demand of a commodity if he is on the short side of the market for that commodity. In the next section an example is presented that shows for arbitrary positive ε there exists an economy with three people and three commodities which has an ε excess demand price such that no allocation arrived at by a rationing scheme as described above is δ efficient for any $\delta<1/2$.

3. PROOFS AND EXAMPLE

PROOF OF THE THEOREM. We consider an arbitrary but fixed trader h in T. Let δ_h denote $[p\cdot x_t-v_p(x_t)][\# T/p\cdot\sum_{t\in T}w_t]$, which is trader h's wasted income at the ε Walrasian allocation $(x_t)_{t\in T}$ multiplied by a fixed coefficient. We will construct a preference relation $\succsim'_h$ such that $d(\succsim_h, \succsim'_h)<\delta_h$ and x_h is the Walrasian demand for a trader characterized by $\succsim'_h$ and w_h given price p. A way to do this is to construct $\succsim'_h$ so that the indifference surface through x_h coincides with the indifference surface of $\succsim_h$ through x_h outside the budget set. However, inside the budget set the indifference surface of $\succsim_h$ is flattened onto the budget surface so that there is no y in the budget set with $y \succsim'_h x$. Naturally this correction must be extended continuously to neighboring surfaces.

For purposes of rigorous exposition we use the following utility function representing $\succsim_h$:

$$u(x) = \min \{p \cdot y \mid y \sim_h x\}.$$

The preference relation $\succsim'_h$ is derived from the utility function $u'(x) = \min [u(x), g(p \cdot x)]$ where the function $g: R_+ \to R_+$ is defined as follows:

$$g(r) = \begin{cases} r & \text{if} \quad r \leq p \cdot w_h - 2\delta_h \quad \text{or} \quad r \geq p \cdot w_h + \delta_h \\ [r - (p \cdot w_h - 2\delta_h)]/2 + p \cdot w_h - 2\delta_h & \text{if} \quad p \cdot w_h - 2\delta_h \leq r \leq p \cdot w_h \\ 2(r - p \cdot w_h) + p \cdot w_h - \delta_h & \text{if} \quad p \cdot w_h \leq r \leq p \cdot w_h + \delta_h \end{cases}$$

g is a continuous strictly monotonically increasing function. Hence u', the minimum of two quasi-concave continuous functions is itself continuous and quasi-concave.

Clearly, x_h is a maximizer of u'_h over the budget set $\{x \mid p \cdot x < p \cdot w_h\}$. Thus $(x_h)_{h \in T}$ is a Walras allocation for the economy e'.

We now show $d(\succsim_h, \succsim'_h) \leq \delta_h$. Let $x, y \in R^l_+$ with $x \succsim_h y$. We will find $\bar{x}, \bar{y} \in R^l_+$ with $\bar{x} \succsim'_h \bar{y}$ and $\|\bar{x} - x\| + \|\bar{y} - y\| < \delta_h$. If $x \succsim'_h y$ we are done. If not, $u(x) \geq u(y)$ and $u'(x) < u'(y)$ which implies $u'(x) < u(x)$. Thus $u'(x) = g(p \cdot x) \leq u(x) \leq p \cdot x$. Since $r - g(r) \leq \delta_h$, $g(p \cdot x) \geq p \cdot x - \delta_h$.

Because $u(x) \geq u(y) \geq u'(y)$ it suffices to find $\bar{x}$ such that $\|\bar{x} - x\| \leq \delta_h$ and $u'(x) = u(x)$, with $\bar{y} \equiv y$. Define $\bar{x} = x + \delta_h p / \|p\|$; $\|\bar{x} - x\| = \delta_h \cdot p / \|p\| \leq \delta_h$. Since $p \cdot \bar{x} = p \cdot x + \delta_h$, $g(p, \bar{x}) = g(p \cdot x + \delta_h) \geq (p \cdot x + \delta_h) - \delta_h = p \cdot x \geq u(x)$. By monotonicity, $u(\bar{x}) \geq u(x)$. Therefore $u'(\bar{x}) = \min (u(\bar{x}), g(p \cdot \bar{x})) \geq u(x)$. Hence, the distance of (x, y) to the graph of $\succsim'_h$ is bounded by δ_h.

For the other direction suppose $x \succsim'_h y$. We now need $\bar{x}, \bar{y} \in R^l_+$ with $\|\bar{x} - x\| + \|\bar{y} - y\| < \delta_h$ and $\bar{x} \succsim_h \bar{y}$.

If $x \succsim_h y$ we are done. If not, $u(x) < u(y)$ and $u'(x) \geq u'(y)$. Since $u'(x) \leq u(x)$, we have $u'(y) < u(y)$. We let $\bar{x} = x$ and $\bar{y} = y - \delta_h \, p / \|p\|$. As above $u(\bar{y}) \leq p \cdot \bar{y} = p \cdot y - \delta_h$. But then $u'(y) = g(p \cdot y) \geq p \cdot y - \delta_h \geq u(\bar{y})$. Since $u(\bar{x}) = u(x) \geq u'(x) \geq u'(y) \geq u(\bar{y})$, we have that $\bar{x} \succsim_h \bar{y}$, hence the distance of (x, y) to the graph of $\succsim_h$ is also bounded by δ_h. Thus $d(\succsim_h, \succsim'_h) \leq \delta_h$ as desired.

By the definition of the metric on economies, if for each trader h, $d(\succsim_h, \succsim'_h) \leq \delta_h$, and $w_h = w'_h$, then $m(e, e') \leq (1/\#T) \sum_{h \in T} \delta_h$. By the definition of ε-Walras allocation and the numbers δ_h, $(\sum_{h \in T} \delta_h)(p \cdot \sum_{h \in T} w_h / \#T) < \varepsilon p \cdot \sum_{t \in T} w_t$. Thus $m(e, e') < \varepsilon$. Q.E.D.

Proof of Corollary 1. If $(x_t)_{t \in T}$ is an ε Walras allocation, $\sum_{t \in T} [p \cdot x_t - v_p(x_t)] < \varepsilon \sum_{t \in T} p \cdot w_t$. Let $y_t \succsim_t x_t$; then $p \cdot y_t \geq v_p(x_t)$ and $p \cdot \sum_{t \in T} y_t \geq \sum_{t \in T} v_p(x_t)$. Thus $p \cdot \sum_{t \in T} (y_t) \geq \sum_{t \in T} v_p(x_t) > p \cdot [\sum_{t \in T} x_t - \varepsilon \sum_{t \in T} w_t] = p(1 - \varepsilon) \sum_{t \in T} w_t$. This implies $\sum_{t\in T} y_t \leqq (1-\varepsilon) \sum_{t\in T} w_t$ is impossible. Q. E. D.

PROOF OF COROLLARY 2. Let $(x_t)_{t\in T}$ be ε efficient allocation in the economy e. If $(x_t)_{t\in T}$ is not efficient, there is an allocation $(y_t)_{t\in T}$ with $\sum_{t\in T} y_t = \lambda \sum x_t$, $\lambda < 1$. Let $(y_t)_{t\in T}$ be the allocation which minimizes λ. Since $(y_t)_{t\in T}$ is an efficient distribution of $\sum_{t\in T} y_t$, $\exists\, p \neq 0$ with $p\cdot x > p\cdot y_t$ for $x \succ_t y_t$. We will show that $(x_t)_{t\in T}$ is ε Walras at price p. If not, $\sum_{t\in T} p\cdot x_t - v_p(t) \geq \varepsilon \sum_{t\in T} p\cdot w_t$. But $v_p(t) = p\cdot y_t$ and thus $p\cdot \sum_{t\in T} y_t \leq (1-\varepsilon)p\cdot \sum_{t\in T} x_t = (1-\varepsilon)p\cdot \sum_{t\in T} w_t$. Since $\sum_{t\in T} y_t = \lambda \sum_{t\in T} w_t$ we then have $\sum_{t\in T} y_t \leq (1-\varepsilon) \sum_{t\in T} w_t$ contrary to $(x_t)_{t\in T}$ being ε-efficient.

By the theorem, $(x_t)_{t\in T}$ ε Walras implies that there is an economy e' with $(x_t)_{t\in T}$ Walras and hence efficient in e'. Q. E. D.

EXAMPLE. For any $\varepsilon > 0$, we will construct an economy and a price p such that for the given economy the absolute value of aggregate excess demand at price p for each commodity is less than ε (i.e., that p is an ε-equilibrium price in Scarf's sense). However, when individual demands are rationed to make them feasible, the resulting allocation is not "approximately" efficient.

Given $\varepsilon > 0$, the economy will consist of three traders and three commodities. The initial endowments are $w_1 = (1-\varepsilon/2, 0, 0)$, $w_2 = (0, 1, 0)$, $w_3 = (0, 0, 1+\varepsilon/2)$ and the price which will be the ε-equilibrium price is $p = (1, 1, 1)$. Each agent has linear indifference surfaces. For trader 1, one surface has intercepts on the axes of (1, 0, 0) (0, 1, 0) and $(0, 0, 1-\varepsilon/2)$. Thus his utility maximizing excess demand is $e_1 = (-1+\varepsilon/2, 0, 1-\varepsilon/2)$. For trader 2 the intercepts of one surface are (1, 0, 0) (0, 2, 0) and (0, 0, 2) which gives rise to an excess demand for him of $e_2 = (1, -1, 0)$. Similarly, trader 3 has intercepts of an indifference surface of (2, 0, 0) $(0, 1+\varepsilon/2, 0)$ and (0, 0, 2) and excess demand at price p of $e_3 = (0, 1+\varepsilon/2, -1-\varepsilon/2)$. The aggregate excess demand at price p is then $e = e_1 + e_2 + e_3 = (\varepsilon/2, \varepsilon/2, -\varepsilon)$, which makes p an ε equilibrium price as desired.

Since trader 1 wants to purchase less of good 3 than trader 3 wishes to sell, trader 3 will not have the income to purchase the desired quantity of good 2. Hence the final allocations after this rationing takes place will be $x_1 = (0, 0, 1-\varepsilon/2)$, $x_2 = (1-\varepsilon/2, \varepsilon/2, 0)$ and $x_3 = (0, 1-\varepsilon/2, \varepsilon)$. For agent 1, his bundle is less preferred than 1/3(1, 1, 1) by the above definition of the indifference surface for him. It is possible that the final allocation for trader 2 lies on a linear indifference surface with intercepts $(1-\varepsilon/4, 0, 0)$ $(0, 2-\varepsilon/2, 0)$ and $(0, 0, \delta)$, $\delta > 0$. For δ sufficiently small x_2 would be less preferred than $\varepsilon(1-\varepsilon/2, 1, 1+\varepsilon/2)$. Similarly, it is possible that trader 3's final allocation x_3 may be less preferred than $\varepsilon(1-\varepsilon/2, 1, 1+\varepsilon/2)$. This implies that the final allocation resulting from the ε-equilibrium after rationing may not be ε efficient for any $\varepsilon > 1/2$.

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