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A Group Incentive Compatible Mechanism Yielding Core Allocations*

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INTRODUCTION

The possibilities for achieving desirable standards of economic performance in a decentralized, incentive compatible manner have recently received widespread research attention. Perhaps the most notable contribution in this area has been the remarkable paper by Groves and Ledyard [7]. They devised a mechanism for allocating resources in economies with public goods which gives agents broad opportunities for strategic behavior but under which Nash equilibrium results in Pareto optimal allocations. These allocations resulting from the Groves–Ledyard rules are not, however, guaranteed to be individually rational, so some agents may end up worse off under the operation of the mechanism than at their initial endowment points. Further, the mechanism is not group incentive compatible: if even two agents can coordinate their choices of messages, then the Nash equilibria are no longer stable and inefficiency may result [3].

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Several more recent papers have investigated the achievability via Nash equilibria of Pareto optima which are also individually rational, with particular attention being paid to designing mechanisms under which Nash equilibria yield precisely the Walrasian and Lindahl allocations [9, 18, 21]. Indeed, Hurwicz has shown (under mild continuity but rather stringent convexity assumptions) that the only individually rational Pareto optimal allocations that can be obtained in this way are the Walras–Lindahl ones [8].

In this paper we continue the study of the possibilities of implementation of different standards of performance in economies with public goods in an incentive compatible fashion. However, rather than asking simply for individual incentive compatibility as formalized by Nash equilibrium, we demand that the procedure should be group incentive compatible as well. This leads us to consideration of strong equilibria (see [1]), which are Nash equilibria which meet the further condition that no group can alter their strategies in concert and thereby gain.[1]

Although strong equilibrium involves the actions of groups of agents, it does not involve the use of binding agreements at the equilibrium point and thus may be considered a non-cooperative concept (even though the actual act of improving by a group at a nonequilibrium point may be cooperative). This makes it an attractive notion for modeling incentives, especially if one does not insist that all logically possible coalitions may actually form. The difficulty with the strong equilibrium concept is that equilibrium fails to exist in a broad class of situations. However, in the game induced by the outcome function (the mapping from individual's messages to allocations) which is our central focus here, the strong equilibrium allocations coincide with the core of the economy.[2] Thus, well-known conditions for non-emptiness of the core (see [15] or any of the proofs of existence of Lindahl equilibria) imply existence of strong Nash equilibria. Further, since core allocations are individually rational, the strong Nash equilibria under this mechanism also enjoy this property (provided that the allowable coalitions include the singletons).

The mechanism described here has a further important property not shared by some other mechanisms (see, e.g., [9] and [18]). In order that a game in strategic (normal) form be well-defined, an outcome must be specified for each possible strategy choice. In an economic context, this implies that no matter what messages are chosen, the resulting allocation must be implementable, i.e., must satisfy the material balance conditions and must at least not allocate negative quantities to any individual and, preferably,

[1] The first use of the strong equilibrium notion in the context of an incentive problem was in the pathbreaking study by Drèze and de la Vallée Poussin [4]. More recently, it has been used in [10, 11, 12].

[2] Although the strong equilibrium and core notions are obviously closely associated, they do differ: see section IV of this paper.

should provide each individual with a bundle lying in his consumption set. Assuming that the initial endowments lie in the individual consumption sets and that no one offers to trade more than he can supply,[3] this property obtains under the mechanism presented here.

On the other hand, this mechanism has the unfortunate property that the Nash equilibrium allocations are the whole set of individually rational ones. Thus, if the coalition formation envisioned in the strong equilibrium does not occur, serious inefficiency may result.

In the following sections we describe the mechanism and establish its properties. We then briefly discuss the possibilities for restricting the set of allowable coalitions. The next section contains some game-theoretic observations based on viewing the mechanism as a method for associating a game in strategic form with a market. We also mention two other mechanisms. The strong equilibria of the first of these yield the set of individually rational Pareto optima, while those of the second yield the Lindahl equilibria. We close with a brief concluding section.

THE MECHANISM AND THE GAME[4]

We consider an economy described by the characteristics of the n traders in the economy and by a set Y which describes the production possibilities in this society. We let m denote the number of private goods and l denote the number of public goods. The consumption set of individual i is X^i, u^i denotes his utility function over X^i, and ω^i is his endowment of goods.

Thus formally the economy $E = ((\omega^i, X^i, u^i)_{i \in N}, Y)$, where $N = \{1, 2,..., n\}$, $\omega^i \in R_+{}^m$, $X^i \subseteq R_+^{m+l}$, $u^i\colon X^i \to R$ and $Y \subseteq R^{m+l}$. We follow the usual convention: positive components of elements of Y are outputs, while inputs are negative. We also assume throughout the paper that

(a) $(\omega^i, 0) \in X^i$,

(b) for $x \in R^m$, $y, \bar{y} \in R^l$, if $(x, y) \in X^i$ and $\bar{y} \geqslant y$ then $(x, \bar{y}) \in X^i$,

and

(c) Y is additive [i.e., $z \in Y$, $z' \in Y \Rightarrow (z + z') \in Y$].

To define the mechanism, we must specify the language S^i in terms of which agent i communicates and the outcome function p which associates an allo-

[3] Vernon Smith has suggested that this condition is not, in fact, always met away from equilibrium in tatonnement games in an experimental context. Whether this is the result of mistakes by the players or of sophisticated strategic behavior is unknown.

[4] Lloyd Shapley has pointed out to us that the game defined here is similar in some ways to the method used by Von Neumann and Morgenstern [20, p. 243–244] to show that any real-valued set function meeting certain conditions is the characteristic function of some game.

cation with each n-tuple of messages. The message space is $S = S^1 x \cdots x S^n$, where S^i consists of all triples of the form $S^i = (T^i, x^i, y^i)$ with $i \in T^i \subseteq N$ and $(x^i, y^i) \in X^i$. The natural interpretation of this message is that T^i consists of those agents with whom i proposes to cooperate in trade and production, and (x^i, y^i) is his proposal as to the consumption he should receive from the agreement, with x^i being his proposed private goods consumption and y^i his proposal for the level of public goods to be produced.

To define the outcome function p, we must first introduce a concept of consistency of a group's plans.

A non-empty coalition C is called *consistent* (at s) if:

(a) $T^i = C$ for every $i \in C$;

(b) $y^i = y^j = y$ for every $i, j \in C$, and

(c) $(\sum_{i \in C} x^i - \sum_{i \in C} \omega^i, y) \in Y$.

Condition (a) means that the members of C have agreed to cooperate with each other and not to cooperate with anyone outside of C. Condition (b) indicates that they agree on the public goods level to produce. Condition (c) means that the production plan implicit in their agreement is feasible. Let C^0 be the (possibly empty) coalition of players that do not belong to any consistent set. It is easy to see that $\{C^0, C^1, C^2,..., C^K\}$ is a partition of N, where $C^1, C^2,..., C^K$ are the consistent coalitions at s, and that the partition is unique for any message n-tuple s. Finally, we define the payoff to i at the message n-tuple s as follows. For every k, $1 \leqslant k \leqslant K$, let y_k be the agreed upon level of public goods in C^k. Let $y = \sum_{k=1}^{K} y_k$. For $i \in C^0$ define the payoff to i at s to be $p^i(s) = (\omega^i, y)$. For $i \in C^k$, $1 \leqslant k \leqslant n$, define $p^i(s) = (x^i, y)$.

Given this mechanism, a game in strategic form is induced in which the players are the n traders, the S^i are the strategy sets and the $u^i \circ p^i$ are the outcome functions. In this game, agents determine the messages they will send and thereby, through p, jointly determine the final allocation.

THE RELATION BETWEEN THE GAME AND THE ECONOMY

An allocation in the economy E is a vector $a = (x^1, x^2,..., x^n, y)$ where $x^i \in R^m$ and $y \in R^l$. An allocation a is *feasible for a coalition* C if

(a) $(x^i, y) \in X^i$ for every $i \in C$, and

(b) $(\sum_{i \in C} (x^i - \omega^i), y) \in Y$.

An allocation is *feasible* if it is feasible for the grand coalition N.

PROPOSITION 1. (a) *For every $s \in S$, $p(s)$ is a feasible allocation.*

(b) *If each* u^i *is non-decreasing in the public goods then for any coalition* C *and for any* C*-feasible allocation* $(x^1,..., x^n, y)$ *there exists* s^i, $i \in C$, *such that for any* n*-tuple* $t = (t^1,..., t^n)$ *of strategies with* $t^i = s^i$, $i \in C$, $u^i(p^i(t)) \geqslant u^i(x^i, y)$ *for all* $i \in C$.

Proof. Part (a) is obvious under the assumption that Y is additive. For part (b) define the strategies for the members of C by $s^i = (C, x^i, y)$. Thus, any message n-tuple results in a well-defined, implementable allocation and any economically relevant outcome can be achieved via the mechanism.

We next investigate the relationship between the core of the economy and the strong Nash equilibria of the corresponding game. We say that the coalition C can block the allocation $(x^1,..., x^n, y)$ if there exists a C-feasible allocation $(\bar{x}^1,..., \bar{x}^n, \bar{y})$ with $u^i(\bar{x}^i, \bar{y}) > u^i(x^i, y)$ for every $i \in C$. Now let $\mathscr{T}$ be a set of coalitions: we define the core($\mathscr{T}$) of the economy E to be all the feasible allocations that cannot be blocked by some coalition in $\mathscr{T}$. Notice when $\mathscr{T}$ consists of all the subsets of N then core($\mathscr{T}$) equals the core as it is more usually defined. We will discuss other specifications and the interpretation of $\mathscr{T}$ below, but it is worth noting here that the introduction of a collection of allowable coalitions affords a great richness in modeling.

For a strategic form game and a strategy s we say that a coalition C can improve upon s if there exist strategies $(t^i)_{i \in C} = t^C$ such that $u^i(p^i(s)) < u^i(p^i(s \mid t^C))$ for every $i \in C$ where $(s \mid t^C)^j = s^j$ if $j \notin C$ and $(s \mid t^C)^j = t^j$ if $j \in C$. Let $\mathscr{T}$ be a set of coalitions. We say that the strategy s is a strong equilibrium relative to $\mathscr{T}(SE(\mathscr{T}))$ if s cannot be improved upon by any coalition in $\mathscr{T}$. Notice that when $\mathscr{T} = \{\{i\}: i \in N\}$ then s is a Nash equilibrium if and only if it is a $SE(\mathscr{T})$. When $\mathscr{T}$ consists of all coalitions then s is a $SE(\mathscr{T})$ if and only if it is a strong equilibrium as defined in [1].

Note that consistency of a coalition C does not require that C belong to $\mathscr{T}$. Instead, we allow individuals to propose trade among the members of any group. The structure $\mathscr{T}$ captures possibilities for varying the "strength" of the strong equilibrium and of the core as stability requirements.

PROPOSITION 2. *Suppose that the* u^i*'s are monotonically increasing in all goods. Let* $\mathscr{T}$ *be a collection of coalitions*:

(a) *If* s *is a* $SE(\mathscr{T})$ *then* $p(s) \in$ Core($\mathscr{T}$).

(b) *If* $a \in$ Core($\mathscr{T}$) *and* a *is Pareto optimal then there exists an* $SE(\mathscr{T})$ s *with* $p(s) = a$.

Proof. For part (a), suppose that s is a $SE(\mathscr{T})$ but that it is blocked by some coalition $C \in \mathscr{T}$. Then there exists a C-feasible allocation $(\bar{x}^1,..., \bar{x}^n, \bar{y})$ with $\bar{u}^i(x^i, \bar{y}) > u^i(p^i(s))$ for all $i \in C$. Consider the strategy $(s \mid \bar{s}^C)$ defined by $(s \mid \bar{s}^C) = (C, \bar{x}^i, \bar{y})$ for $i \in C$. Since $(\bar{x}^1,..., \bar{x}^n, \bar{y})$ is feasible for C, C is a consistent coalition at $(s \mid \tilde{s}^C)$. Thus its members receive $p^i(s \mid \tilde{s}^C) = (\bar{x}^1, \tilde{y})$, where $\tilde{y} \geqslant \bar{y}$. This implies that s was not a $SE(\mathscr{T})$.

For part (b), suppose $a = (\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^n, \bar{y}) \in \text{Core}(\mathscr{T})$. Define s^i for every $i \in N$ as follows: $T^i = N$, $x^i = \bar{x}^i$ and $y^i = \bar{y}$. Then $s = (s^1, s^2, \ldots, s^n)$ is a permissible strategy n-tuple with $p(s) = a$. Suppose s is not a $SE(\mathscr{T})$. Then it can be improved upon by a coalition $C \in \mathscr{T}$ via the strategies $\tilde{s}^C$ where $\tilde{s}^i = (\tilde{T}^i, \tilde{x}^i, \tilde{y})$.

Case 1. $\tilde{T}^i = N$ for all $i \in C$. In this case the only possible consistent coalition is N. If N is not consistent then every player ends up with his initial endowment $(\omega^i, 0)$. Since $C \in \mathscr{T}$ and $a \in \text{Core}(\mathscr{T})$, some member of C must be at least as well off at a as he is at $(\omega^i, 0)$. Then, since C has improved upon s, this member of C is strictly better off at $p^i(s \mid \tilde{s}^C)$ than at $(\omega^i, 0)$, so $p^i(s \mid \tilde{s}^C) \neq (\omega^i, 0)$. Then N must be consistent. If $N = C$ then we get a contradiction because the fact that N can improve implies that N can block a from being in the core. So we assume that $C \neq N$. Since N is consistent, $\tilde{y}^i = y^i = y^j$ for every $i \in C$ and $j \in N$. Thus the same levels of public goods will be produced as before. In fact, players outside of C will have the same final allocation as before. Then, given that the members of C have improved, monotonicity implies that all members of N could have been made better off, i.e., a was not Pareto optimal, which is a contradiction.

Case 2. $\tilde{T}^i \neq N$ for some $i \in C$. In this case the only possible consistent coalitions are subsets of C. Then for every $i \notin C$ the final allocation of private goods is ω^i. Thus the coalition C will produce all the public goods and will trade only among its own members. Thus C could block a from the Core($\mathscr{T}$), which is a contradiction.

Remark. If $N \in \mathscr{T}$ and preferences are monotone, the Core($\mathscr{T}$) and $SE(\mathscr{T})$ allocations coincide.

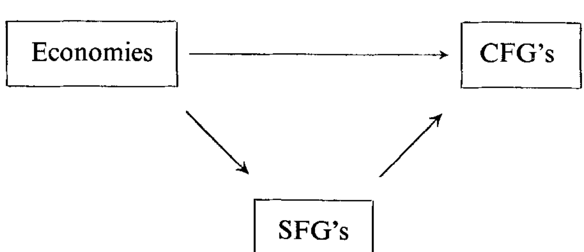
III. The Set of Permissible Coalitions

Specifying a set $\mathscr{T}$ of coalitions which may form in order to block or improve, instead of assuming that all logically possible groups may actually cooperate, offers interesting possibilities for modeling different notions of stability and yields a variety of solution concepts of differing strengths. As noted, if $\mathscr{T}$ is the set of all non-empty subsets of N, then $SE(\mathscr{T})$ is simply the set of strong equilibria, while if $\mathscr{T}$ consists only of the singletons, then it becomes the usual Nash equilibria. In [10] and [11], we adopted an approach due to Myerson [13] in which the set of players is endowed with a graph structure, with two players being linked only if they are able to communicate and/or exchange goods with one another. In this set-up, it is natural to take $\mathscr{T}$ as the set of coalitions which are connected in the graph. Models obtained in this way are appropriate for examining issues of market organization and the structure of trade. However, the framework in these papers differ somewhat from that used here in that there trade as well as strategic cooperation was limited by the structure $\mathscr{T}$, while here any trade is allowed and $\mathscr{T}$ affects only the possibilities for cooperation. A further example is given by the literature on syndicates of traders, where it is assumed that certain coalitions cannot be broken up. In this case, $\mathscr{T}$ would consist of all sets of agents which contain either all or none of the members of a syndicate. A further possibility would be the situation in which $\mathscr{T}$ consists of all groups of less than a certain size. Such small coalitions have been studied in the context of large economies [6, 17, 19] and in models of bilateral trade [14]. Hurwicz has suggested that they would be especially interesting to investigate in the framework of this model, since it would be a way of recognizing costs of coalition formation, at least in an *ad hoc* fashion. One would then wish to determine such issues as the efficiency of the resulting allocations.

IV. Game-Theoretic Considerations

The mechanism we have presented in this paper may be interpreted as a mapping associating a game in strategic form with any economy. In this section we explore this interpretation and some related issues.

There are standard methods already in the literature for associating characteristic function games (CFG's) with economies and with normal or strategic form games (SFG's). Then, given a proposed method of assigning SFG's to economies, it becomes important to consider the commutativity of the following diagram:



In this same context, the correspondence between various solution concepts for the economy and the associated games is also of interest.

For economies with only private goods there is a well-accepted procedure for assigning a CFG to each economy [15]: one specifies utility functions for the agents, then defines the characteristic function evaluated at a coalition to be all those utility vectors which are less than or equal to the utility image of an allocation which is feasible for the coalition using only its own endowments and production possibilities. There is less general acceptance of the appropriate definition in the presence of public goods, but the most widely used one is that introduced by Foley [5]. This parallels that for private goods in that a coalition can use only its own resources in producing public goods (i.e., the complementary coalition is assumed to produce no public goods).

There are two methods of associating CFG's with SFG's which are discussed in the game theory literature. These were suggested by Aumann and Peleg [2] and are known as the α and β methods. The essential element in these is that a coalition and its complement are treated as playing a two-person game. The difference between the notions is that to say that a coalition can achieve a particular utility vector under the α method, it has to have a strategy which guarantees at least this outcome independent of the complement's strategy, while under the β method, an outcome can be attained by the coalition if the complementary coalition does not have a strategy which, independent of what the coalition does, can prevent it from getting an outcome at least as good as the specified one.

It is trivial to show, but still of interest, that whether one uses the α or β method, the diagram above commutes when one uses the method suggested in this paper for mapping economies to SFG's and the standard method to go from economies to CFG's.[5] Further, our results show that the core of the CFG and the strong equilibria of the SFG coincide, with both being the image of the core of the economy (which is defined in allocation space).

Of course, these results are not surprising in that the strategic form game we associate with the economy simply formalizes the story one has in mind when defining the standard CFG for an economy. Yet there are reasonable strategic form games which could be associated with economies for which parallel claims are not true.

[5] It should be observed that the strategic form game presented above is, in the private goods case, what Shapley and Shubik [16] call a "game of orthogonal coalitions," i.e. it is one in which the outcome to a coalition is independent of the strategy of the complementary coalition. Such games are of special significance since the loss of strategic information that is typically incurred in going from strategic to characteristic function form is not suffered for these games. Shapley and Shubik commented that "models of pure competition with out externalities" are of this form. For such game, the α and β notions coincide. Yet, the game proposed by Schmeidler [18] is one of pure competition without externalities, since the Nash and strong equilibria both coincide with the competitive outcomes, but the game is not, in fact, one of orthogonal coalitions.

For instance, consider a situation with one private good and one public good, where a unit of public good can be produced from a unit of private. Let Q be the unit simplex in R^n, let $\mathscr{T}$ be the set of non-empty subsets of N and let $S^i = \mathscr{T} \times Q \times R$. Thus, a generic element of S^i is $s^i = (T^i, q_1{}^i,..., q_n{}^i, y^i)$. Define $C \subset N$ to be consistent if (i) $s^i = s^j$ for all i and j in C, (ii) $T^i = C$, $i \in C$; and (iii) $\sum_{j \in C} q_j{}^j = 1$, $i \in C$. If C is a consistent set at s, let $q^i = q^C$ and $y^i = y^C$, $i \in C$, and let y be the sum of the y^C over all consistent sets.

We now consider three games using $S^1 \times \cdots \times S^n$ as their strategy spaces. The first of these is isomorphic to the one discussed above and is defined by $p^i(s) = (\omega^i - q_i{}^C y^C, y)$ for i belonging to the consistent set C and $p^i(s) = (\omega^i, y)$ otherwise. For this game the diagram commutes and the strong equilibria and core coincide.

The second game is given by $p^i(s) = (\omega^i - q_i{}^N y^N, y^N)$ if N is consistent and by $(\omega^i, 0)$ otherwise. This is a version of a game considered by Maskin [12]. For it, the strong equilibria correspond to the individually rational Pareto optimal allocations for the economy and, clearly, the diagram does not commute.

To define the third game, call C quasi-consistent at a if (i) $T^i = C$, $i \in C$ (ii) $q^i = q^j$, $i, j \in C$, and (iii) $\sum_C q_j{}^i = 1$, $i \in C$. Given a quasi-consistent C, let $q^i = q^C$, $i \in C$ and $\bar{y}^C = \sum_{i \in C} y^i / \#C$, and let $\bar{y}$ be the sum of the $\bar{y}^C$ over the quasi-consistent sets. Then define $p^i(s)$ to be $(\omega^i - q_i{}^N \bar{y}^N, \bar{y}^N)$ if N is quasi-consistent, and $(\omega^i, 0)$ otherwise. The strong equilibria of this game are easily seen to correspond to the Lindahl equilibrium allocations of the economy (which are a subset of the core), yet the characteristic function derived from this strategic form game coincides with that of the second game (whether one uses the α or β method).

V. Conclusions

We have investigated the question of imposing group incentive compatibility on allocation mechanisms and shown that there is at least one mechanism which performs well by this standard in that the strong equilibria in the induced strategic form game coincide with the core of the economy. This mechanism has the further desirable property that the outcome resulting from any messages is feasible. On the other hand, it should be noted that we do not get the properties for free, since the Nash Equilibria of this game correspond to the whole set of individually rational allocations. Thus, if the coalitions actually will form, the mechanism in this paper works well, but if coalition formation cannot be expected, it may perform very badly. For this reason, it may be best to consider this mechanism as one geared particularly to situations with small numbers of agents.

REFERENCES

1. R. AUMANN, A survey of cooperative games without side payments, *in* "Essays in Mathematical Economics in Honor of Oscar Morgenstern" (M. Shubik, Ed.), Princeton Press, Princeton, N.J. 1964.
2. R. AUMANN AND B. PELEG, von Neumann–Morgenstern solutions to cooperative games without side payments, *Trans. Amer. Math. Soc.* **98** (1961), 539–552.
3. E. BENNETT AND D. CONN, The group incentive properties of mechanisms for the provision of public goods, *Public Choice* **29** (1977), 95–102.
4. J. DRÈZE AND D. DE LA VALLÉE POUSSIN, A tatonnement process for public goods, *Rev. Econ. Studies* **38** (1971), 133–150.
5. D. FOLEY, Lindahl's solution and the core of an economy with public goods, *Econometrica* **38** (1970), 66–72.
6. B. GRODAL, A second remark on the core of an atomless economy, *Econometrica* **40** (1972), 581–583.
7. T. GROVES AND J. LEDYARD, Optimal allocation of public goods: a solution to the "free rider" problem, *Econometrica* **45** (1977), 783–810.
8. L. HURWICZ, On allocations attainable through Nash equilibria, *J. Econ. Theory*, in press.
9. L. HURWICZ, Outcome functions yielding Walrasian and Lindahl allocations at Nash equilibrium points, mimeograph, University of Minnesota, 1976.
10. E. KALAI, A. POSTLEWAITE, AND J. ROBERTS, Barriers to trade and disadvantageous middlemen: Non-monotonicity of the core, *J. Econ. Theory*, **19** (1978), 200–209.
11. E. KALAI, A. POSTLEWAITE, AND J. ROBERTS, "A Simple Game of Exchange," Discussion Paper No. 280, The Center for Mathematical Studies in Economics and Management Science, Northwestern University, 1977.
12. E. MASKIN, Implementation and strong Nash equilibrium, mimeograph, Massachusetts Institute of Technology, 1978.
13. R. MYERSON, Graphs and cooperation in games, *Math. Operations Res.* **2** (1977), 225–229.
14. T. RADER, "Theory of General Economic Equilibrium," pp. 80–90, Academic Press, New York, 1972.
15. H. SCARF, The core of an N person game, *Econometrica* **35** (1968), 50–69.
16. L. SHAPLEY AND M. SHUBIK, "Game Theory in Economics," Chap. 6, RAND Report R-904/6, 1973.
17. D. SCHMEIDLER, A remark on the core of an atomless economy, *Econometrica* **40** (1972), 579–580.
18. D. SCHMEIDLER, A remark on microeconomic models of an economy and a game theoretic interpretation of Walras equilibria, mimeograph, University of Minnesota, 1976.
19. K. VIND, A third remark on the core of an atomless economy, *Econometrica* **40** (1972), 585–586.
20. J. VON NEUMANN AND O. MORGENSTERN, "Theory of Games and Economic Behavior," Princeton Univ. Press, Princeton, N.J. 1953.
21. R. WILSON, "A Competitive Model of Exchange," Technical Report 221, Institute for Mathematical Studies in the Social Sciences, Stanford University, 1976.