Econometrica Supplementary Material

SUPPLEMENT TO "STABLE MATCHING WITH INCOMPLETE INFORMATION": ONLINE APPENDIX

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O.1. PROOFS FOR SECTION 4.1.3

O.1.1. *Information-Revealing Prices*

THE FOLLOWING LEMMA IDENTIFIES conditions under which a firm entertaining a deviation to match with a worker of unknown type can be certain of a lower bound on the worker's type.

LEMMA O.1: *Suppose Assumptions* 1 *and* 3 *hold, and* $(\mu, \mathbf{p}, \mathbf{w}, \mathbf{f})$ *is individually rational. If a type* w^* *worker is matched with a type* f^* *firm at a payment* p^*, *then, for any firm with type* $f < f^*$, *there exists* $\varepsilon > 0$ *such that, for any* $p \in (\nu_{w^*f^*} + p^* - \nu_{w^*f}, \nu_{w^*f^*} + p^* - \nu_{w^*f} + \varepsilon]$,

$$\nu_{wf} + p > \nu_{wf^*} + p^*, \quad \textit{for all } w \geq w^*, \tag{O.1}$$

$$\nu_{wf} + p \geq 0, \quad \textit{for all } w \geq w^*, \quad \textit{and} \tag{O.2}$$

$$\nu_{wf} + p \leq \nu_{wf^*} + p^*, \quad \textit{for all } w < w^*. \tag{O.3}$$

If w^* *is unmatched in an individually rational matching outcome, then, for any firm type* f, *there exists* $\varepsilon > 0$ *such that, for any* $p \in (-\nu_{w^*f}, -\nu_{w^*f} + \varepsilon]$,

$$\nu_{wf} + p > 0, \quad \textit{for all } w \geq w^*, \quad \textit{and}$$

$$\nu_{wf} + p \leq 0, \quad \textit{for all } w < w^*.$$

PROOF: Define

$$p^\varepsilon := \nu_{w^*f^*} + p^* - \nu_{w^*f} + \varepsilon, \tag{O.4}$$

where $\varepsilon > 0$ will be determined later. The first required inequality (O.1) with $p = p^\varepsilon$ is

$$\nu_{wf} + \nu_{w^*f^*} + \varepsilon > \nu_{wf^*} + \nu_{w^*f} \quad \text{for any } w \geq w^*,$$

which is immediate when $w = w^*$. When $w > w^*$, it follows from the assumption of strict submodularity (since $f < f^*$). Since $(\mu, \mathbf{p})$ is an individually rational matching, $\nu_{w^*f^*} + p^* \geq 0$. Hence, for any $w \geq w^*$, $f > f^*$, and p^ε defined in (O.4),

$$\nu_{wf} + p^\varepsilon \geq \nu_{w^*f} + p^\varepsilon > \nu_{w^*f^*} + p^*,$$

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proving (O.2).

After substituting for $p = p^\varepsilon$ defined in (O.4), the inequality (O.3) becomes

$$\nu_{wf} + \nu_{w^*f^*} + \varepsilon \le \nu_{wf^*} + \nu_{w^*f} \quad \text{for any } w < w^*.$$

For ε sufficiently small, this inequality follows from the assumption of strict submodularity (since $f < f^*$). Inequalities (O.1)–(O.3) immediately hold for $p \in (\nu_{w^*f^*} + p^* - \nu_{w^*f}, p^\varepsilon]$. The proof for the case that w^* is unmatched is similar. *Q.E.D.*

O.1.2. *Proof of Proposition 4*

O.1.2.1. *Preliminaries*

Constrained Efficiency. We begin by formulating an inductive notion of efficiency. As before, we write the finite set of possible worker and firm types as $W = \{w^1, w^2, \dots, w^K\}$ and $F = \{f^1, f^2, \dots, f^L\}$, with both w^k and f^ℓ increasing in their indices. To deal with unmatched agents, we introduce the notation $\mathbf{f}(\emptyset) = \mathbf{w}(\emptyset) = \emptyset$, with the conventions $\emptyset < w^k$ and $\emptyset < f^\ell$ for any k and ℓ. For any matching function μ, denote by I_μ the set of matched workers and by J_μ $(= \mu(I_\mu))$ the set of matched firms. By definition, $|I_\mu| = |J_\mu|$.

DEFINITION O.1: A matching outcome $(\mu, \mathbf{p}, \mathbf{w}, \mathbf{f})$ is *constrained efficient on* $W' \subset W$ if

$$\sum_{i \in \mathbf{w}^{-1}(W') \cap I_\mu} [\nu_{\mathbf{w}(i)\mathbf{f}(\mu(i))} + \phi_{\mathbf{w}(i)\mathbf{f}(\mu(i))}] = \max_{\mu' \in M} \sum_{i \in \mathbf{w}^{-1}(W') \cap I_\mu} [\nu_{\mathbf{w}(i)\mathbf{f}(\mu'(i))} + \phi_{\mathbf{w}(i)\mathbf{f}(\mu'(i))}],$$

where M is the set of one-to-one functions from $\mathbf{w}^{-1}(W') \cap I_\mu$ onto $\mu(\mathbf{w}^{-1}(W')) \cap J_\mu$.[1]

In this definition, M consists of all possible matching functions between $\mathbf{w}^{-1}(W') \cap I_\mu$ and $\mu(\mathbf{w}^{-1}(W') \cap I_\mu) = \mu(\mathbf{w}^{-1}(W')) \cap J_\mu$ with no agent in these two sets left unmatched. Hence, constrained efficiency might violate individual rationality; for example, it could be that a matched worker–firm pair generates a negative surplus. The following observation follows immediately from the definition of submodularity.

[1]We adopt the convention that M is empty if $\mathbf{w}^{-1}(W')$ is empty, and that a summation over a empty set equals to 0.

LEMMA O.2: *A matching outcome* $(\mu, \mathbf{p}, \mathbf{w}, \mathbf{f})$ *is constrained efficient on* $W' \subset W$ *if and only if the matching outcome is negative assortative* (*i.e., for all* $i, i' \in I$ *such that* $\mu(i), \mu(i') \in J$, *if* $\mathbf{w}(i) < \mathbf{w}(i')$, *then* $\mathbf{f}(\mu(i)) \geq \mathbf{f}(\mu(i'))$).

DEFINITION O.2: For $1 \leq k < K$, a matching outcome $(\mu, \mathbf{p}, \mathbf{w}, \mathbf{f})$ is *kth-order constrained efficient* if, for all $w > w^k$ and $w \in W$, $(\mu, \mathbf{p}, \mathbf{w}, \mathbf{f})$ *is constrained efficient on* $\{w^1, \dots, w^k, w\}$.

The following observation explores the submodularity assumption and is useful in our inductive proofs.

LEMMA O.3: *A matching outcome* $(\mu, \mathbf{p}, \mathbf{w}, \mathbf{f})$ *is* $(k+1)$*th-order constrained efficient if and only if it is kth-order constrained efficient and, for all* $w > w^{k+1}$, *it is constrained efficient on* $\{w^{k+1}, w\}$.

PROOF: The "only if" parts are immediate by definition. "If": suppose $(\mu, \mathbf{p}, \mathbf{w}, \mathbf{f})$ is kth-order constrained efficient. Consider any $w > w^{k+1}$. If $\mathbf{w}(i) \neq w^{k+1}$ and $\mathbf{w}(i) \neq w$ for all $i \in I_\mu$, then trivially, $(\mu, \mathbf{p}, \mathbf{w}, \mathbf{f})$ is constrained efficient on $\{w^1, \dots, w^k, w^{k+1}, w\}$. Suppose $\mathbf{w}(i) = w^{k+1}$ and $\mathbf{w}(i') = w$ for some $i, i' \in I_\mu$. By assumption, $(\mu, \mathbf{p}, \mathbf{w}, \mathbf{f})$ is constrained efficient on $\{w^{k+1}, w\}$, and hence it follows from Lemma O.2 that $\mathbf{f}(\mu(i)) \geq \mathbf{f}(\mu(i'))$. By Lemma O.2 again, $(\mu, \mathbf{p}, \mathbf{w}, \mathbf{f})$ is constrained efficient on $\{w^1, \dots, w^k, w^{k+1}, w\}$. Hence, $(\mu, \mathbf{p}, \mathbf{w}, \mathbf{f})$ is $(k+1)$th-order constrained efficient. *Q.E.D.*

Unmatched Agents.

LEMMA O.4: *Suppose* $(\mu, \mathbf{p}, \mathbf{w}, \mathbf{f})$ *is incomplete-information stable*:

1. *there does not exist* $i, i' \in I$ *such that* $\mu(i) = \emptyset \neq \mu(i')$ *and* $\mathbf{w}(i) > \mathbf{w}(i')$,
2. *there does not exist* $j, j' \in J$ *such that* $\mu^{-1}(j) = \emptyset \neq \mu^{-1}(j')$ *and* $\mathbf{f}(j) > \mathbf{f}(j')$, *and*
3. *there does not exist* $i \in I$ *and* $j \in J$ *such that* $\mu(i) = \emptyset = \mu^{-1}(j)$ *and* $\nu_{\mathbf{w}(i)\mathbf{f}(j)} + \phi_{\mathbf{w}(i)\mathbf{f}(j)} > 0$.

PROOF: Statement 1: Suppose there exist $i, i' \in I$ such that $\mu(i) = \emptyset \neq \mu(i')$ and $\mathbf{w}(i) > \mathbf{w}(i')$. We claim that $(i, \mu(i'))$ can form a blocking pair with a payment $p = -\nu_{\mathbf{w}(i)\mathbf{f}(\mu(i'))} + \varepsilon$ for some small enough $\varepsilon > 0$. To see this, note that, by Lemma O.1, worker i receives a positive payoff and reveals that his type is at least $\mathbf{w}(i)$ if ε is small enough. The expected payoff of firm $\mu(i')$ from this deviation is at least

$$\phi_{\mathbf{w}(i)\mathbf{f}(\mu(i'))} - p = \phi_{\mathbf{w}(i)\mathbf{f}(\mu(i'))} + \nu_{\mathbf{w}(i)\mathbf{f}(\mu(i'))} - \varepsilon.$$

Since $\mathbf{w}(i) > \mathbf{w}(i')$, Assumption 1 implies that $\phi_{\mathbf{w}(i)\mathbf{f}(\mu(i'))} + \nu_{\mathbf{w}(i)\mathbf{f}(\mu(i'))} - \varepsilon > \phi_{\mathbf{w}(i')\mathbf{f}(\mu(i'))} + \nu_{\mathbf{w}(i')\mathbf{f}(\mu(i'))}$ for a small ε. But $\phi_{\mathbf{w}(i')\mathbf{f}(\mu(i'))} + \nu_{\mathbf{w}(i')\mathbf{f}(\mu(i'))}$ is the total surplus in a match $(i', \mu(i'))$. So firm $\mu(i')$ finds this deviation profitable.

Statement 2: Suppose there exist $j, j' \in J$ such that $\mu^{-1}(j) = \emptyset \neq \mu^{-1}(j')$ and $\mathbf{f}(j) > \mathbf{f}(j')$. We claim $(\mu^{-1}(j'), j)$ form a blocking pair with payment $p = \mathbf{p}_{\mu^{-1}(j'),j'} + \varepsilon$ for some $\varepsilon > 0$. Observe first that worker $\mu^{-1}(j')$ receives a strictly higher payoff in this block, since ν is weakly monotonic in f, and so

$$\nu_{\mathbf{w}(\mu^{-1}(j'))\mathbf{f}(j)} + \mathbf{p}_{\mu^{-1}(j'),j'} + \varepsilon > \nu_{\mathbf{w}(\mu^{-1}(j'))\mathbf{f}(j')} + \mathbf{p}_{\mu^{-1}(j'),j'}.$$

Although firm j may be uncertain about the type of worker $\mu^{-1}(j')$, the firm knows that the individual rationality for firm j' and the strict monotonicity of ϕ in f imply

$$0 \le \phi_{\mathbf{w}(\mu^{-1}(j'))\mathbf{f}(\mu(j'))} - \mathbf{p}_{\mu^{-1}(j'),j'} < \phi_{\mathbf{w}(\mu^{-1}(j'))\mathbf{f}(j)} - \mathbf{p}_{\mu^{-1}(j'),j'}.$$

Therefore, for ε sufficiently small, firm j gets a strictly positive payoff in the matching with worker $\mu^{-1}(j')$ at price $p = \mathbf{p}_{\mu^{-1}(j'),j'} + \varepsilon$.

Statement 3: Suppose there exist $i \in I$ and $j \in J$ such that $\mu(i) = \emptyset \neq \mu^{-1}(j)$ and $\nu_{\mathbf{w}(i)\mathbf{f}(j)} + \phi_{\mathbf{w}(i)\mathbf{f}(j)} > 0$. We claim (i, j) form a blocking pair with payment $p = -\nu_{\mathbf{w}(i)\mathbf{f}(j)} + \varepsilon$ for small enough ε. Worker i's payoff is positive from this deviation, and, by Lemma O.1, firm j knows the block is only possible if worker i's type is at least $\mathbf{w}(i)$. Assumption 1 implies that firm j's payoff is at least $\nu_{\mathbf{w}(i)\mathbf{f}(j)} + \phi_{\mathbf{w}(i)\mathbf{f}(j)} - \varepsilon$, which is strictly positive for ε sufficiently small. *Q.E.D.*

O.1.2.2. *Completion of the Proof of Proposition 4*

Constrained Efficiency. Lemmas O.5 and O.6 inductively show that every incomplete-information stable matching outcome is constrained efficient.

LEMMA O.5: *If* $(\mu, \mathbf{p}, \mathbf{w}, \mathbf{f}) \in \Sigma^1$, *then* $(\mu, \mathbf{p}, \mathbf{w}, \mathbf{f})$ *is first-order constrained efficient*.

PROOF: Suppose, to the contrary, that some $(\mu, \mathbf{p}, \mathbf{w}, \mathbf{f}) \in \Sigma^1$ is not first-order constrained efficient. Then, by Lemma O.2, there exist two workers, say 1 and 2, such that $\mathbf{w}(2) > \mathbf{w}(1) = w^1$ and $\mathbf{f}(\mu(2)) > \mathbf{f}(\mu(1)) \neq \emptyset$.

CLAIM O.1: *If* $(\mu, \mathbf{p}, \mathbf{w}, \mathbf{f}) \in \Sigma^1$, $\mathbf{w}(2) > \mathbf{w}(1) = w^1$, *and* $\mathbf{f}(\mu(2)) > \mathbf{f}(\mu(1)) \neq \emptyset$, *then*

$$\begin{aligned}(\text{O.5}) \qquad &\phi_{\mathbf{w}(2),\mathbf{f}(\mu(1))} + \nu_{\mathbf{w}(2),\mathbf{f}(\mu(1))} \\ &\quad \le \nu_{\mathbf{w}(2),\mathbf{f}(\mu(2))} + \mathbf{p}_{2,\mu(2)} + \phi_{\mathbf{w}(1),\mathbf{f}(\mu(1))} - \mathbf{p}_{1,\mu(1)}.\end{aligned}$$

PROOF: Consider a match by worker 2 and firm $\mu(1)$ with payment

$$p := \nu_{\mathbf{w}(2),\mathbf{f}(\mu(2))} + \mathbf{p}_{2,\mu(2)} - \nu_{\mathbf{w}(2),\mathbf{f}(\mu(1))} + \varepsilon,$$

for small $\varepsilon > 0$. By Lemma O.1, this match is only attractive to worker 2 if his type is $\mathbf{w}(2)$ or higher. Since $(\mu, \mathbf{p}, \mathbf{w}, \mathbf{f}) \in \Sigma^1$, firm $\mu(1)$ must not be better off in this match. Hence,

$$\phi_{\mathbf{w}(2),\mathbf{f}(\mu(1))} - p \leq \phi_{\mathbf{w}(1),\mathbf{f}(\mu(1))} - \mathbf{p}_{1,\mu(1)}.$$

Substituting for p,

$$\begin{aligned}&\phi_{\mathbf{w}(2),\mathbf{f}(\mu(1))} - (\nu_{\mathbf{w}(2),\mathbf{f}(\mu(2))} + \mathbf{p}_{2,\mu(2)} - \nu_{\mathbf{w}(2),\mathbf{f}(\mu(1))} + \varepsilon) \\ &\quad \leq \phi_{\mathbf{w}(1),\mathbf{f}(\mu(1))} - \mathbf{p}_{1,\mu(1)},\end{aligned}$$

implying (O.5). *Q.E.D.*

CLAIM O.2: *If* $(\mu, \mathbf{p}, \mathbf{w}, \mathbf{f}) \in \Sigma^1$, $\mathbf{w}(2) > \mathbf{w}(1) = w^1$, *and* $\mathbf{f}(\mu(2)) > \mathbf{f}(\mu(1)) \neq \emptyset$, *then*

$$\begin{aligned}&\nu_{\mathbf{w}(1),\mathbf{f}(\mu(2))} + \phi_{\mathbf{w}(1),\mathbf{f}(\mu(2))} \\ &\quad \leq \nu_{\mathbf{w}(1),\mathbf{f}(\mu(1))} + \mathbf{p}_{1,\mu(1)} + \phi_{\mathbf{w}(2),\mathbf{f}(\mu(2))} - \mathbf{p}_{2,\mu(2)}.\end{aligned}$$

PROOF: If the inequality in (B.6) did not hold, we can find $q \in \mathbb{R}$ such that

$$\nu_{\mathbf{w}(1),\mathbf{f}(\mu(2))} + q > \nu_{\mathbf{w}(1),\mathbf{f}(\mu(1))} + \mathbf{p}_{1,\mu(1)} \quad \text{and} \tag{O.6}$$

$$\phi_{\mathbf{w}(1),\mathbf{f}(\mu(2))} - q > \phi_{\mathbf{w}(2),\mathbf{f}(\mu(2))} - \mathbf{p}_{2,\mu(2)}. \tag{O.7}$$

Since ϕ is weakly increasing and $\mathbf{w}(1) = w^1$ is the smallest type, (O.7) implies

$$\min_{w \in W} \phi_{w,\mathbf{f}(\mu(2))} - q > \phi_{\mathbf{w}(2),\mathbf{f}(\mu(2))} - \mathbf{p}_{2,\mu(2)}. \tag{O.8}$$

Hence, (O.6) and (O.8) imply $(1, \mu(2))$ is a blocking pair, contradicting $(\mu, \mathbf{p}, \mathbf{w}, \mathbf{f}) \in \Sigma^1$. *Q.E.D.*

Finally, we combine Claims O.1 and O.2. Adding the two inequalities, we obtain

$$\begin{aligned}&(\nu_{\mathbf{w}(1),\mathbf{f}(\mu(2))} + \nu_{\mathbf{w}(2),\mathbf{f}(\mu(1))}) + (\phi_{\mathbf{w}(1),\mathbf{f}(\mu(2))} + \phi_{\mathbf{w}(2),\mathbf{f}(\mu(1))}) \\ &\quad \leq (\nu_{\mathbf{w}(1),\mathbf{f}(\mu(1))} + \nu_{\mathbf{w}(2),\mathbf{f}(\mu(2))}) + (\phi_{\mathbf{w}(1),\mathbf{f}(\mu(1))} + \phi_{\mathbf{w}(2),\mathbf{f}(\mu(2))}).\end{aligned}$$

Since $\mathbf{w}(1) < \mathbf{w}(2)$ and $\mathbf{f}(\mu(2)) > \mathbf{f}(\mu(1))$, this inequality contradicts the strict submodularity of $\nu + \phi$. This completes the proof of Lemma O.5. *Q.E.D.*

LEMMA O.6: *For any* $k \geq 1$, *if* $(\mu, \mathbf{p}, \mathbf{w}, \mathbf{f}) \in \Sigma^k$, *then* $(\mu, \mathbf{p}, \mathbf{w}, \mathbf{f})$ *is kth-order constrained efficient*.

PROOF: We proceed by induction. Suppose the claim holds for some $k \geq 1$ (from Lemma O.5, the claim holds for $k = 1$). Suppose, to the contrary, that $(\mu, \mathbf{p}, \mathbf{w}, \mathbf{f}) \in \Sigma^{k+1}$, and $(\mu, \mathbf{p}, \mathbf{w}, \mathbf{f})$ is not $(k+1)$th-order constrained efficient. There then exist two workers i and i' such that worker i's type is $\mathbf{w}(i) = w^{k+1} < \mathbf{w}(i')$ and $\emptyset \neq \mathbf{f}(\mu(i)) < \mathbf{f}(\mu(i'))$.

CLAIM O.3: *If* $(\mu, \mathbf{p}, \mathbf{w}, \mathbf{f}) \in \Sigma^{k+1}$, $\mathbf{w}(i) = w^{k+1} < \mathbf{w}(i')$, *and* $\emptyset \neq \mathbf{f}(\mu(i)) < \mathbf{f}(\mu(i'))$, *then*

$$\text{(O.9)}\qquad \nu_{\mathbf{w}(i'),\mathbf{f}(\mu(i))} + \phi_{\mathbf{w}(i'),\mathbf{f}(\mu(i))} \leq \nu_{\mathbf{w}(i'),\mathbf{f}(\mu(i'))} + \mathbf{p}_{i',\mu(i')} + \phi_{\mathbf{w}(i),\mathbf{f}(\mu(i))} - \mathbf{p}_{i,\mu(i)}.$$

PROOF: Consider a match by worker i' and firm $\mu(i)$ with payment

$$p := \nu_{\mathbf{w}(i'),\mathbf{f}(\mu(i'))} + \mathbf{p}_{i',\mu(i')} - \nu_{\mathbf{w}(i'),\mathbf{f}(\mu(i))} + \varepsilon,$$

for small $\varepsilon > 0$. By Lemma O.1, this match is only attractive to worker i' if his type is at least $\mathbf{w}(i')$. Since $(\mu, \mathbf{p}, \mathbf{w}, \mathbf{f}) \in \Sigma^{k+1}$, the match $(i', \mu(i))$ with p cannot make firm $\mu(i)$ better off for any consistent belief. Hence, there exists $w \geq \mathbf{w}(i')$ such that

$$\phi_{w,\mathbf{f}(\mu(i))} - p \leq \phi_{\mathbf{w}(i),\mathbf{f}(\mu(i))} - \mathbf{p}_{i,\mu(i)}.$$

By monotonicity of ϕ and $\mathbf{w}(i') \leq w$, we have

$$\phi_{\mathbf{w}(i'),\mathbf{f}(\mu(i))} - p \leq \phi_{\mathbf{w}(i),\mathbf{f}(\mu(i))} - \mathbf{p}_{i,\mu(i)}.$$

Substituting for p, we get (O.9). *Q.E.D.*

CLAIM O.4: *If* $(\mu, \mathbf{p}, \mathbf{w}, \mathbf{f}) \in \Sigma^{k+1}$, $\mathbf{w}(i) = w^{k+1} < \mathbf{w}(i')$, *and* $\emptyset \neq \mathbf{f}(\mu(i)) < \mathbf{f}(\mu(i'))$, *then*

$$\text{(O.10)}\qquad \nu_{\mathbf{w}(i),\mathbf{f}(\mu(i'))} + \phi_{\mathbf{w}(i),\mathbf{f}(\mu(i'))} \leq \nu_{\mathbf{w}(i),\mathbf{f}(\mu(i))} + \mathbf{p}_{i,\mu(i)} + \phi_{\mathbf{w}(i'),\mathbf{f}(\mu(i'))} - \mathbf{p}_{i',\mu(i')}.$$

PROOF: Suppose, to the contrary, that the claimed inequality does not hold. We can then find $q \in \mathbb{R}$ such that

$$\text{(O.11)}\qquad \nu_{\mathbf{w}(i),\mathbf{f}(\mu(i'))} + q > \nu_{\mathbf{w}(i),\mathbf{f}(\mu(i))} + \mathbf{p}_{i,\mu(i)} \quad \text{and}$$

$$\text{(O.12)}\qquad \phi_{\mathbf{w}(i),\mathbf{f}(\mu(i'))} - q > \phi_{\mathbf{w}(i'),\mathbf{f}(\mu(i'))} - \mathbf{p}_{i',\mu(i')}.$$

By monotonicity of ϕ, (O.12) implies

$$\text{(O.13)}\qquad \phi_{w,\mathbf{f}(\mu(i'))} - q > \phi_{\mathbf{w}(i'),\mathbf{f}(\mu(i'))} - \mathbf{p}_{i',\mu(i')} \quad \text{for all } w \geq \mathbf{w}(i) = w^{k+1}.$$

By the induction hypothesis, Σ^k only contains outcomes that are kth-order constrained efficient. Consider the following set of worker type assignments:

$$\Omega' = \{\mathbf{w}' \in \Omega : (\mu, \mathbf{p}, \mathbf{w}', \mathbf{f}) \in \Sigma^k, \mathbf{w}'(i') = \mathbf{w}(i'),$$
$$\nu_{\mathbf{w}'(i),\mathbf{f}(\mu(i'))} + q > \nu_{\mathbf{w}'(i),\mathbf{f}(\mu(i))} + \mathbf{p}_{i,\mu(i)}\}.$$

We claim that, for any $\mathbf{w}' \in \Omega'$, $\mathbf{w}'(i) \geq w^{k+1}$. To see this, suppose, to the contrary, that $\mathbf{w}'(i) \leq w^k$. By assumption, $\mathbf{w}'(i') = \mathbf{w}(i') > w^{k+1} > w^k$ and $\mathbf{f}(\mu(i)) < \mathbf{f}(\mu(i'))$. But then $\mathbf{w}'(i) < \mathbf{w}'(i')$, while $\mathbf{f}(\mu(i)) < \mathbf{f}(\mu(i'))$, and so $(\mu, \mathbf{p}, \mathbf{w}', \mathbf{f})$ is not kth-order constrained efficient, contradicting the assumption that $(\mu, \mathbf{p}, \mathbf{w}', \mathbf{f}) \in \Sigma^k$.

It then follows from (O.13) that

$$\text{(O.14)} \quad \min_{\mathbf{w}' \in \Omega'} \phi_{\mathbf{w}'(i),\mathbf{f}(\mu(i'))} - q > \phi_{\mathbf{w}(i'),\mathbf{f}(\mu(i'))} - \mathbf{p}_{i',\mu(i')}.$$

Hence, from (O.11) and (O.14), the unmatched pair $(i, \mu(i'))$ at payment q can form a blocking pair. A contradiction. *Q.E.D.*

Summing (O.9) and (O.10), we have

$$(\nu_{\mathbf{w}(i'),\mathbf{f}(\mu(i))} + \nu_{\mathbf{w}(i),\mathbf{f}(\mu(i'))}) + (\phi_{\mathbf{w}(i'),\mathbf{f}(\mu(i))} + \phi_{\mathbf{w}(i),\mathbf{f}(\mu(i'))})$$
$$\leq (\nu_{\mathbf{w}(i'),\mathbf{f}(\mu(i'))} + \nu_{\mathbf{w}(i),\mathbf{f}(\mu(i))}) + (\phi_{\mathbf{w}(i),\mathbf{f}(\mu(i))} + \phi_{\mathbf{w}(i'),\mathbf{f}(\mu(i'))}),$$

contradicting strict submodularity of $\nu + \phi$. This completes the proof of Lemma O.6. *Q.E.D.*

Efficiency and Constrained Efficiency. By definition, efficiency implies constrained efficiency, but the converse is not true without further assumptions. As shown in the example, leaving some agents unmatched, or creating more matched pairs, could improve efficiency. It follows from Lemma O.4 that in a stable matching outcome, all the unmatched workers (firms, resp.) must have lower realized types than the matched workers (firms, resp.) and no ex post surplus can be generated by unmatched agents.

Without loss of generality, we assume $\mathbf{w}(i)$ is increasing in i. Suppose the incomplete-information stable matching outcome $(\mu, \mathbf{p}, \mathbf{w}, \mathbf{f})$ is constrained efficient but not efficient. Lemma O.4 implies that all unmatched agents have types lower than those in I_μ and J_μ. Let μ^* be an efficient matching. Since efficiency implies constrained efficiency, Lemma O.4 implies that all unmatched agents under μ^* have types lower than those of I_{μ^*} and J_{μ^*}. Therefore, without loss of generality, we can assume either $I_\mu \subsetneq I_{\mu^*}$ and $J_\mu \subsetneq J_{\mu^*}$, or $I_{\mu^*} \subsetneq I_\mu$ and $J_{\mu^*} \subsetneq J_\mu$. In addition, it follows from Lemma O.2 that we can assume that, without loss of generality, $\mu^*(i) \neq \mu(i)$ for each worker $i \in I_{\mu^*}$.

Under the hypothesis that all matches yield a positive surplus, it is immediate that there are no unmatched pairs of workers and firms under μ^*, and so we must have $I_\mu \subsetneq I_{\mu^*}$ and $J_\mu \subsetneq J_{\mu^*}$.

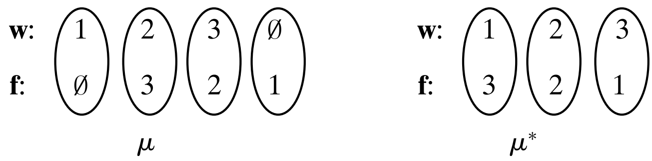

FIGURE O.1.—An illustration of the derivation of (O.15).

LEMMA O.7: *Suppose* $(\mu, \mathbf{p}, \mathbf{w}, \mathbf{f})$ *is a constrained efficient incomplete-information stable matching outcome. Suppose* $I_\mu \subsetneq I_{\mu^*}$ *and* $J_\mu \subsetneq J_{\mu^*}$. *Then* $(\mu, \mathbf{p}, \mathbf{w}, \mathbf{f})$ *is efficient.*

PROOF: We claim that, for each $i \in I_\mu$, it must be that

$$\mathbf{f}\big(\mu^*(i)\big) \le \mathbf{f}\big(\mu(i)\big). \tag{O.15}$$

To see that (O.15) holds, note that, by assumption, I_{μ^*} is obtained from I_μ by adding some lower worker types. Lemma O.2 implies that those added low types must match with high type firms under μ^*, and hence $i \in I_\mu$ will be rematched to lower firms under μ^* (see Figure O.1).

In the statement of the next claim, (O.16) is well-defined for workers $i \in I_{\mu^*} \setminus I_\mu$ using the convention $\nu_{\mathbf{w}(i),\emptyset} = \mathbf{p}_{i,\emptyset} = 0$.

CLAIM O.5: *Suppose the matching outcome* $(\mu, \mathbf{p}, \mathbf{w}, \mathbf{f})$ *is incomplete-information stable. If* $I_\mu \subsetneq I_{\mu^*}$, $J_\mu \subsetneq J_{\mu^*}$, *and* $\mathbf{f}(\mu(i)) \neq \mathbf{f}(\mu^*(i))$ *for all* $i \in I_\mu$, *then, for each* $i \in I_{\mu^*}$,

$$\begin{aligned} &\nu_{\mathbf{w}(i),\mathbf{f}(\mu^*(i))} + \phi_{\mathbf{w}(i),\mathbf{f}(\mu^*(i))} \\ &\quad \le \nu_{\mathbf{w}(i),\mathbf{f}(\mu(i))} + \mathbf{p}_{i,\mu(i)} + \phi_{\mathbf{w}(\mu^{-1}(\mu^*(i))),\mathbf{f}(\mu^*(i))} - \mathbf{p}_{\mu^{-1}(\mu^*(i)),\mu^*(i)}. \end{aligned} \tag{O.16}$$

PROOF: Since $\mathbf{f}(\mu(i)) \neq \mathbf{f}(\mu^*(i))$ for all $i \in I_\mu$, (O.15) implies $\mathbf{f}(\mu^*(i)) < \mathbf{f}(\mu(i))$ for all $i \in I_\mu$. By Lemma O.1, under μ, for each worker $i \in I_\mu$, the match with firm $\mu^*(i)$ with payment

$$p := \nu_{\mathbf{w}(i),\mathbf{f}(\mu(i))} + \mathbf{p}_{i,\mu(i)} - \nu_{\mathbf{w}(i),\mathbf{f}(\mu^*(i))} + \varepsilon, \tag{O.17}$$

for $\varepsilon > 0$ small, is only profitable if his type is at least $\mathbf{w}(i)$.

In addition, under μ, each worker $i \in I_{\mu^*} \setminus I_\mu$ is unmatched, and hence (by Lemma O.1 again) there are matches for such workers i with firm $\mu^*(i) \neq \emptyset$ with payment (O.17) that are only profitably if his type is at least $\mathbf{w}(i)$.

Since $(\mu, \mathbf{p}, \mathbf{w}, \mathbf{f})$ is stable, for each $i \in I_{\mu^*}$, firm $\mu^*(i)$ must not find any such match profitable, that is,

$$\phi_{\mathbf{w}(i),\mathbf{f}(\mu^*(i))} - p \le \phi_{\mathbf{w}(\mu^{-1}(\mu^*(i))),\mathbf{f}(\mu^*(i))} - \mathbf{p}_{\mu^{-1}(\mu^*(i)),\mu^*(i)}.$$

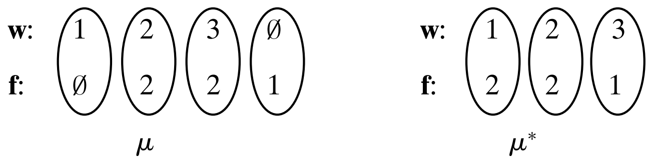

FIGURE O.2.—An illustration of the case ruled out in Claim O.5.

Substituting p and taking $\varepsilon \to 0$, we obtain (O.16). *Q.E.D.*

Summing (O.16) over all $i \in I_{\mu^*}$, the payments cancel, yielding

$$\sum_{i \in I_{\mu^*}} (\nu_{\mathbf{w}(i),\mathbf{f}(\mu^*(i))} + \phi_{\mathbf{w}(i),\mathbf{f}(\mu^*(i))}) \leq \sum_{i \in I_{\mu^*}} (\nu_{\mathbf{w}(i),\mathbf{f}(\mu(i))} + \phi_{\mathbf{w}(\mu^{-1}(\mu^*(i))),\mathbf{f}(\mu^*(i))}),$$

contradicting the hypothesized inefficiency of $(\mu, \mathbf{p}, \mathbf{w}, \mathbf{f})$.

Claim O.5 requires $\mathbf{f}(\mu(i)) \neq \mathbf{f}(\mu^*(i))$ for all $i \in I_\mu$ in order to apply Lemma O.1 for each such i. To illustrate the issue that arises if $\mathbf{f}(\mu(i)) = \mathbf{f}(\mu^*(i))$ for some $i \in I_\mu$, consider the matchings in Figure O.2. In this example, the argument in the proof of Claim O.5 cannot be used because, under μ, there is no match for the type 2 worker with another firm whose type is the same as his currently matched firm that is only profitable if his type is at least 2. However, in this case, to compare the efficiency of μ and μ^*, we only need to consider where they differ, that is, we only need to look at the two matchings illustrated in Figure O.3.

Claim O.5 applies to $\hat{\mu}$ and $\hat{\mu}^*$, and the same conclusion holds. We omit the obvious formal argument that requires additional notation. This completes the proof of Lemma O.7. *Q.E.D.*

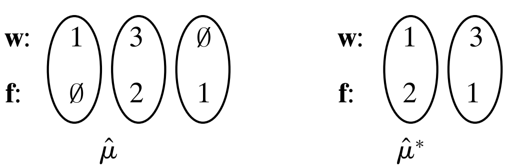

FIGURE O.3.—The relevant part of the matching from Figure O.2.

O.2. EXAMPLE AND PROOFS FOR SECTION 5

O.2.1. *The Example Illustrating Multiple Rounds*

Consider n firms and n workers: $W = F = \{1, \dots, n\}$. Worker types are drawn from the set of all permutations. Worker premuneration values are identically zero, $\nu_{wf} \equiv 0$. Firm premuneration values are given by

$$\phi_{wf} = \begin{cases} wf, & \text{if } w \leq f, \\ f^2, & \text{if } w > f. \end{cases}$$

Consider the price matrix $\mathbf{P} = \mathbf{0}$, which assigns a price of zero to every match. Any price-taking matching outcome is then individually rational. In the matching given in Figure O.4, each firm is matched with the worker of the same index, except for an inversion in the match of the top two firms and workers. We show that this outcome is not price-sustainable. The only potentially profitable deviation in this matching outcome is for the type n firm to buy the type n worker; all workers and all other firms are getting the most they could obtain under the constant price matrix $\mathbf{P} = \mathbf{0}$. However, under incomplete information, the type n firm does not know which worker is type n, and hence this matching outcome is Ψ^0-sustainable. In fact, for the same reason, every matching outcome that matches the type 1 worker with the type 1 firm is Ψ^0-sustainable. Moreover, Ψ^1, the set of Ψ^0-sustainable outcomes, coincides with the set of matching outcomes that match the type 1 worker with the type 1 firm. To see this, it is enough to notice that any type $j > 1$ firm, if matched with the type 1 worker, would profitably deviate to purchase any other worker, whose type (given that types are drawn from the set of permutations) must be larger than 1. Now restricting attention to the set Ψ^1, it must be that the set of Ψ^1-sustainable outcomes coincides with the set of matching outcomes that match type i workers with type i firms for $i = 1, 2$, a subset of Ψ^1. To see this, note that any firm $j > 2$, if matched with the type 2 worker, would profitably deviate to any worker other than the one matched with the type 1 firm (since the latter must be type 1, given that we are looking at Ψ^1). Iterating this argument, we conclude that Ψ^{k+1}, the set of Ψ^k-competitive outcomes, coincides with the set of matching outcomes that match type i workers with type i firms for $i = 1, 2, \dots, k+1$. Hence, the candidate matching outcome we constructed above is in Ψ^{n-2}, but not in Ψ^{n-1}.

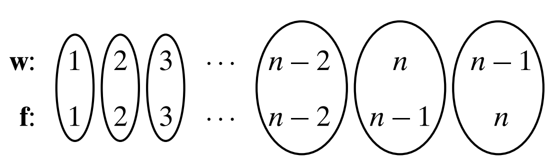

FIGURE O.4.—The matching illustrating the need for many rounds of iteration in Definition 10.

O.2.2. *Proof of Lemma 3*

Take any set of price-taking matching outcomes, C, that is self-sustaining. Then $C \subset \Psi^0$, where Ψ^0 is the set of individually rational outcomes. Since each $(\mu, \mathbf{P}, \mathbf{w}, \mathbf{f}) \in C$ is C-competitive, it follows from Definition 9 that such $(\mu, \mathbf{P}, \mathbf{w}, \mathbf{f})$ is Ψ^0-sustainable, that is, $(\mu, \mathbf{P}, \mathbf{w}, \mathbf{f}) \in \Psi^1$. Hence, $C \subset \Psi^1$. Using again the fact that each $(\mu, \mathbf{P}, \mathbf{w}, \mathbf{f}) \in C$ is C-sustainable and $C \subset \Psi^1$, we obtain that $(\mu, \mathbf{P}, \mathbf{w}, \mathbf{f})$ is Ψ^1-sustainable and hence $C \subset \Psi^2$. We conclude that $C \subset \Psi^\infty$ by iterating this argument. That is, Ψ^∞ contains any set that is self-sustainable. Moreover, by definition, Ψ^∞ is Ψ^∞-sustainable and hence is self-sustainable.

O.2.3. *Proof of Proposition 7*

The proof uses the fixed-point characterizations of stability and price sustainability. Fix an incomplete-information stable outcome $(\mu, \mathbf{p}, \mathbf{w}, \mathbf{f})$ and let E be a self-stabilizing set that contains it. In particular, by part 4 of Lemma 1, we can take E such that it contains matching outcomes with the same allocation $(\mu, \mathbf{p})$. Our goal is to extend $\mathbf{p}$ to $\mathbf{P}$ in an appropriate way and then show that $(\mu, \mathbf{P}, \mathbf{w}, \mathbf{f})$ is price-sustainable. To do so, we extend the entire self-stabilizing set E to a set of price-taking outcomes C and then show C is self-sustainable.

O.2.3.1. *Step 1. Constructing C*

Let E be a self-stabilizing set. For each element of E, $(\mu, \mathbf{p}, \tilde{\mathbf{w}}, \mathbf{f})$, we extend $(\mu, \mathbf{p}, \tilde{\mathbf{w}}, \mathbf{f})$ to $(\mu, \tilde{\mathbf{P}}, \tilde{\mathbf{w}}, \mathbf{f})$, and define C as the resulting set of price-taking matching outcomes.

Consider a worker–firm pair $(i, j) \in I \times J$ such that $j \neq \mu(i)$. Since E is self-stabilizing, there does *not* exist $p \in \mathbb{R}$ such that

$$\nu_{\tilde{\mathbf{w}}(i),\mathbf{f}(j)} + p > \nu_{\tilde{\mathbf{w}}(i),\mathbf{f}(\mu(i))} + \mathbf{p}_{i,\mu(i)} \tag{O.18}$$

and

$$\Big(\min_{\mathbf{w}' \in \Omega(\tilde{\mathbf{w}}(\mu^{-1}(j)),i,j,p)} \phi_{\mathbf{w}'(i),\mathbf{f}(j)}\Big) - p > \phi_{\tilde{\mathbf{w}}(\mu^{-1}(j)),\mathbf{f}(j)} - \mathbf{p}_{\mu^{-1}(j),j}, \tag{O.19}$$

where $\Omega(w, i, j, p)$ is the set of worker type assignments $\mathbf{w}'$ satisfying

$$\big(\mu, \mathbf{p}, \mathbf{w}', \mathbf{f}\big) \in E, \tag{O.20}$$

$$\mathbf{w}'\big(\mu^{-1}(j)\big) = w, \tag{O.21}$$

and

$$\nu_{\mathbf{w}'(i),\mathbf{f}(j)} + p > \nu_{\mathbf{w}'(i),\mathbf{f}(\mu(i))} + \mathbf{p}_{i,\mu(i)}. \tag{O.22}$$

For each firm j, let $\hat{\Omega}(w, j)$ denote the set of worker type assignments satisfying (O.20) and (O.21). Note that $\Omega(w, i, j, p) \subset \hat{\Omega}(w, j)$ for any $p \in \mathbb{R}$, since $\Omega(w, i, j, p)$ is further restricted by the requirement that worker i is purportedly willing to form a blocking pair at price p (allowing firm j to draw inferences about i's type).

CLAIM O.6: *For any worker–firm pair* (i, j),

$$\min_{\mathbf{w}' \in \hat{\Omega}(\tilde{\mathbf{w}}(\mu^{-1}(j)), j)} \phi_{\mathbf{w}'(i), \mathbf{f}(j)} + \max_{\mathbf{w}' \in \hat{\Omega}(\tilde{\mathbf{w}}(\mu^{-1}(j)), j)} (\nu_{\mathbf{w}'(i), \mathbf{f}(j)} - \nu_{\mathbf{w}'(i), \mathbf{f}(\mu(i))}) \leq \mathbf{p}_{i, \mu(i)} + \phi_{\tilde{\mathbf{w}}(\mu^{-1}(j)), \mathbf{f}(j)} - \mathbf{p}_{\mu^{-1}(j), j}. \tag{O.23}$$

PROOF: Suppose, to the contrary, that the inequality does not hold. Then there exists $\mathbf{w}^* \in \hat{\Omega}(\tilde{\mathbf{w}}(\mu^{-1}(j)), j)$ such that

$$\Big(\min_{\mathbf{w}' \in \hat{\Omega}(\tilde{\mathbf{w}}(\mu^{-1}(j)), j)} \phi_{\mathbf{w}'(i), \mathbf{f}(j)}\Big) + \nu_{\mathbf{w}^*(i), \mathbf{f}(j)} - \nu_{\mathbf{w}^*(i), \mathbf{f}(\mu(i))} > \mathbf{p}_{i, \mu(i)} + \phi_{\tilde{\mathbf{w}}(\mu^{-1}(j)), \mathbf{f}(j)} - \mathbf{p}_{\mu^{-1}(j), j}.$$

Since $\mathbf{w}^*(\mu^{-1}(j)) = \tilde{\mathbf{w}}(\mu^{-1}(j))$, this inequality is the same as

$$\Big(\min_{\mathbf{w}' \in \hat{\Omega}(\mathbf{w}^*(\mu^{-1}(j)), j)} \phi_{\mathbf{w}'(i), \mathbf{f}(j)}\Big) + \nu_{\mathbf{w}^*(i), \mathbf{f}(j)} - \nu_{\mathbf{w}^*(i), \mathbf{f}(\mu(i))} > \mathbf{p}_{i, \mu(i)} + \phi_{\mathbf{w}^*(\mu^{-1}(j)), \mathbf{f}(j)} - \mathbf{p}_{\mu^{-1}(j), j}. \tag{O.24}$$

But (O.24) implies that there exists $p^* \in \mathbb{R}$ such that

$$\nu_{\mathbf{w}^*(i), \mathbf{f}(j)} + p^* > \nu_{\mathbf{w}^*(i), \mathbf{f}(\mu(i))} + \mathbf{p}_{i, \mu(i)}$$

and

$$\Big(\min_{\mathbf{w}' \in \hat{\Omega}(\mathbf{w}^*(\mu^{-1}(j)), j)} \phi_{\mathbf{w}'(i), \mathbf{f}(j)}\Big) - p^* > \phi_{\mathbf{w}^*(\mu^{-1}(j)), \mathbf{f}(j)} - \mathbf{p}_{\mu^{-1}(j), j}. \tag{O.25}$$

Since $\Omega(\mathbf{w}^*(\mu^{-1}(j)), i, j, p^*) \subset \hat{\Omega}(\mathbf{w}^*(\mu^{-1}(j)), j)$, (O.25) implies that

$$\Big(\min_{\mathbf{w}' \in \Omega(\mathbf{w}^*(\mu^{-1}(j)), i, j, p^*)} \phi_{\mathbf{w}'(i), \mathbf{f}(j)}\Big) - p^* > \phi_{\mathbf{w}^*(\mu^{-1}(j)), \mathbf{f}(j)} - \mathbf{p}_{\mu^{-1}(j), j},$$

contradicting the nonexistence of a price p satisfying (O.18)–(O.19). *Q.E.D.*

The following is now an immediate implication of (O.23): there exists $\tilde{\mathbf{P}}_{ij}^{\tilde{\mathbf{w}}(\mu^{-1}(j))} \in \mathbb{R}$ such that

$$\Big(\min_{\mathbf{w}' \in \hat{\Omega}(\tilde{\mathbf{w}}(\mu^{-1}(j)), j)} \phi_{\mathbf{w}'(i), \mathbf{f}(j)}\Big) - \tilde{\mathbf{P}}_{ij}^{\tilde{\mathbf{w}}(\mu^{-1}(j))} \leq \phi_{\tilde{\mathbf{w}}(\mu^{-1}(j)), \mathbf{f}(j)} - \mathbf{p}_{\mu^{-1}(j), j} \tag{O.26}$$

and

$$\max_{\mathbf{w}' \in \hat{\Omega}(\tilde{\mathbf{w}}(\mu^{-1}(j)),j)} (\nu_{\mathbf{w}'(i),\mathbf{f}(j)} - \nu_{\mathbf{w}'(i),\mathbf{f}(\mu(i))}) + \tilde{\mathbf{P}}_{ij}^{\tilde{\mathbf{w}}(\mu^{-1}(j))} \le \mathbf{p}_{i,\mu(i)}.$$

The critical feature of $\tilde{\mathbf{P}}_{ij}^{\tilde{\mathbf{w}}(\mu^{-1}(j))}$ is that it only depends on $\tilde{\mathbf{w}}$ through the value of $\tilde{\mathbf{w}}(\mu^{-1}(j))$.

We are now in a position to extend the set of stable outcomes E to a set of price-taking outcomes C as follows. For each $(\mu, \mathbf{p}, \tilde{\mathbf{w}}, \mathbf{f}) \in E$, define an associated price-taking matching outcome $(\mu, \tilde{\mathbf{P}}, \tilde{\mathbf{w}}, \mathbf{f})$ by

$$\text{(O.27)} \quad \tilde{\mathbf{P}}_{ij} = \begin{cases} \mathbf{p}_{ij}, & j = \mu(i), \\ \tilde{\mathbf{P}}_{ij}^{\tilde{\mathbf{w}}(\mu^{-1}(j))}, & \text{otherwise.} \end{cases}$$

We define C as the set of all price-taking matching outcomes derived from E in this way.

O.2.3.2. *Step 2. The Price Sustainability of C*

Fix a firm j and an outcome $(\mu, \tilde{\mathbf{P}}, \tilde{\mathbf{w}}, \mathbf{f}) \in C$, and consider the set $\Omega'(j)$ of worker type assignments $\mathbf{w}' \in \Omega$ for which there exists $\mathbf{P}'$ such that

$$(\mu, \mathbf{P}', \mathbf{w}', \mathbf{f}) \in C,$$
$$\mathbf{w}'(\mu^{-1}(j)) = \tilde{\mathbf{w}}(\mu^{-1}(j)),$$

and

$$\mathbf{P}'_{i',\mu(i')} = \tilde{\mathbf{P}}_{i',\mu(i')} \quad \text{and} \quad \mathbf{P}'_{i',j} = \tilde{\mathbf{P}}_{i',j}, \quad \forall i' \in I.$$

Observe that, by the definition of C, if $(\mu, \tilde{\mathbf{P}}, \tilde{\mathbf{w}}, \mathbf{f})$, $(\mu, \mathbf{P}', \mathbf{w}', \mathbf{f}) \in C$, then $\mathbf{P}'_{i',\mu(i')} = \tilde{\mathbf{P}}_{i',\mu(i')} = \mathbf{p}_{i',\mu(i')}$ for any $i' \in I$; if, in addition, $\mathbf{w}'(\mu^{-1}(j)) = \tilde{\mathbf{w}}(\mu^{-1}(j))$, then $\tilde{\mathbf{P}}_{i',j} = \mathbf{P}'_{i',j} = \mathbf{P}_{i',j}^{\tilde{\mathbf{w}}(\mu^{-1}(j))}$ for any $i' \in I$. Therefore,

$$\Omega'(j) = \hat{\Omega}(\tilde{\mathbf{w}}(\mu^{-1}(j)), j),$$

where $\hat{\Omega}(w, j)$ is defined just after (O.22). Hence, it follows from (O.26)–(O.27) that, for any $i \in I$ and $j \in J$,

$$\text{(O.28)} \quad \left(\min_{\mathbf{w}' \in \Omega'(j)} \phi_{\mathbf{w}'(i),\mathbf{f}(j)}\right) - \tilde{\mathbf{P}}_{ij} \le \phi_{\tilde{\mathbf{w}}(\mu^{-1}(j)),\mathbf{f}(j)} - \tilde{\mathbf{P}}_{\mu^{-1}(j),j}$$

and

$$\text{(O.29)} \quad \max_{\mathbf{w}' \in \Omega'(j)} (\nu_{\mathbf{w}'(i),\mathbf{f}(j)} - \nu_{\mathbf{w}'(i),\mathbf{f}(\mu(i))}) + \tilde{\mathbf{P}}_{ij} \le \tilde{\mathbf{P}}_{i,\mu(i)}.$$

Inequality (O.28) implies

$$\phi_{\mathbf{w}'(i),\mathbf{f}(j)} - \tilde{\mathbf{P}}_{ij} \leq \phi_{\mathbf{w}(\mu^{-1}(j)),\mathbf{f}(j)} - \tilde{\mathbf{P}}_{\mu^{-1}(j),j}$$

for some $\mathbf{w}' \in \Omega'(j)$. Since $(\mu, \tilde{\mathbf{P}}, \tilde{\mathbf{w}}, \mathbf{f}) \in C$, by definition, $\tilde{\mathbf{w}} \in \hat{\Omega}(\tilde{\mathbf{w}}(\mu^{-1}(j)), j) = \Omega'(j)$, and so (O.29) implies

$$\nu_{\tilde{\mathbf{w}}(i),\mathbf{f}(j)} + \tilde{\mathbf{P}}_{ij} \leq \nu_{\tilde{\mathbf{w}}(i),\mathbf{f}(\mu(i))} + \tilde{\mathbf{P}}_{i,\mu(i)}.$$

Hence, by Definition 11, C is self-sustainable. This completes the proof of Proposition 7.

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