A Folk Theorem with Communication for Repeated Games with Imperfect Private Monitoring*

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Abstract

We prove a new folk theorem for repeated games with private monitoring and communication by exploiting the connection between public monitoring games and private monitoring games via public coordination devices.

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JEL Classifications: C72, C73, D82

1 Introduction

The notion that cooperative outcomes can be sustained as equilibrium outcomes in repeated games has been intensively studied for several decades. For the case in which each player can observe all other players' actions directly (perfect monitoring), Aumann and Shapley [5] and Rubinstein [32] proved a folk theorem without discounting, and Fudenberg and Maskin [13] proved a folk theorem with discounting. For the case in which each player observes a noisy public signal (imperfect public monitoring), Abreu, Pearce and Stacchetti [1] characterized the set of pure strategy sequential equilibrium payoffs and Fudenberg, Levine and Maskin [14] proved a folk theorem. The theory of repeated games has improved our understanding by showing how coordinated threats to punish can prevent deviations from cooperative behavior, but much of the work in repeated games rests on very restrictive assumption that all players share the same public information either perfectly or imperfectly. In a more realistic, albeit more complicated model, players possess only partial information about the environment (imperfect private monitoring) and may not know the information possessed by other players.

In McLean, Obara and Postlewaite (2014) (henceforth MOP), we examined the robustness of equilibria in a game with imperfect public monitoring when that game is replaced with a new game in which monitoring monitoring is private, but which closely approximates the original game with public monitoring. In the leading example presented in that paper, the information contained in the underlying public signal is dispersed among the players in the form of noisy private signals. If the amount of information contained in each player's private signal is negligible, then one can view the "perturbed" game with private signals and the underlying game with public signals as being "close." In that paper, we examined whether an equilibrium in a game with public monitoring remains an equilibrium with respect to private monitoring perturbation when players can communicate.

In this paper, we begin with a repeated game with imperfect private monitoring and prove a new folk theorem for a communication extension. For the robustness result in MOP, we began with a public monitoring game and then considered nearby private monitoring games to check the robustness of equilibria for repeated games with public monitoring. For the folk theorem, we take the opposite path: we begin with a private monitoring game, then generate public monitoring games via public coordinating devices. We then ask the following question: what feasible, individually rational payoffs can be sustained as equilibrium payoffs for sufficiently large discount factors when players truthfully reveal their signals along the equilibrium path and choose their actions as a function of the history of public coordinating signals.

To make these ideas precise, consider a private monitoring game (G, p), where G is the normal form game defined by strategy sets and the associated payoffs and p is mapping from strategy profiles to probabilty measures on signal profiles. In particular, each action profile a generates a private signal profile $s = (s_1, .., s_n)$ with probability $p(s|a)$. In our analysis of the private monitoring game (G, p), we will augment the model with a "public coordination device" ϕ that chooses a public coordinating signal (possibly randomly) from a finite set Y based on the reported profile of private signals. In this expanded game with communication, players choose an action profile a, observe their private signals $(s_1, .., s_n)$, and publicly announce the (not necessarily honest) profile $(s'_1, .., s'_n)$. A public coordinating signal $y \in Y$ is then selected with probability $\phi(y|s'_1, .., s'_n)$. If the players report their private signals truthfully, then the probability that the realized public coordinating signal is y given a and ϕ is equal to $p^\phi(y|a) = \sum_{s \in S} \phi(y|s)p(s|a)$. We call the private monitoring repeated game augmented by such public coordinating devices (which may change over time) a communication extension of the repeated game associated with (G, p).

If informational incentive compatibility constraints can be ignored and players are assumed to announce their private signals truthfully, then the game is essentially one of imperfect public monitoring and a folk theorem is readily obtained as an application of Fudenberg, Maskin and Levine (1994). Hence, our analysis must necessarily be concerned with with revelation constraints.

As in MOP, two concepts are essential in order to deal with informational incentive compatibility constraints: *informational size* and *distributional variability.* Roughly speaking, player i is informationally small if for each action profile a, her private information is unlikely to have a large effect on the distribution of the public coordinating signal $p^\phi(\cdot|a)$. Distributional variability is an index that measures the correlation between a player's private signal and the public coordinating signal which she would expect when she reports her signal truthfully. If this index is large, that means that a player's conditional belief about the public coordinating signal varies widely with respect to her private information. The larger this index is, the easier it is to detect and punish a dishonest report.

The way to induce honest reporting is roughly as follows. If the same coordinating devise ϕ is employed in every period, then $p^\phi(\cdot|a)$ is the same in every period and players will typically have an incentive to send false reports. To address this, we employ different public coordinating devices ϕ_{h^t} at different public histories h^t, where each ϕ_{h^t} is a perturbation of ϕ. When every player's informational size is small relative to her distributional variability, we can construct ϕ_{h^t} so that every revelation constraint is satisfied on the equilibrium path (i.e. after a player has played the equilibrium action in the same period), while keeping each perturbation small so that the players have no incentive to deviate with respect to actions.

There are also several technical contributions in this paper. First, we prove a *uniformly strict* folk theorem. That is, we prove a folk theorem by using uniformly strict equilibria where every player would lose at least a certain amount of payoffs by deviating from the equilibrium action at any history. As a special case, this result implies a uniformly strict folk theorem for some class of repeated games with imperfect public monitoring. Another technical contribution of the paper, which might be of independent interest, is to prove the theorem corresponding to Theorem 4.1 in [14] without relying on their smoothness condition, which is commonly used to prove a folk theorem in the literature.

The model is described in Section 2 and the concepts of informational size and distributional variability are introduced in Section 3. In Section 4, we prove a new folk theorem for repeated games with private monitoring and communication. Section 5 discusses the related literature. Some proofs are provided in the appendix (Section 6).

2   Preliminaries

2.1   Repeated Games with Private Monitoring

In this section, we recall some of the notation, definitions and concepts presented in MOP (2014). The set of players is $N = \{1, ..., n\}$. The game proceeds in stages and in each stage t, player i chooses an action from a finite set A_i. An action profile is denoted by $a = (a_1, ..., a_n) \in \Pi_i A_i := A$. Player i's stage game payoff function is

$g_i : A \to \mathbb{R}$ and we denote the resulting stage game by $G = (N, A, g)$. Actions are not publicly observable. Instead, each player i observes a private signal s_i from a finite set S_i. A private signal profile is denoted $s = (s_1, .., s_n) \in \Pi_i S_i := S$. For each $a \in A$, $p(\cdot|a) \in \triangle(S)$ is the distribution on S given action profile a. We assume that the marginal distributions have full support, that is, $p(s_i|a) := \sum_{s_{-i}} p(s_i, s_{-i}|a) > 0$ for all $s_i \in S_i$, $a \in A$ and $i \in N$. Let $p(s_{-i}|a, s_i) := \frac{p(s_i, s_{-i}|a)}{p(s_i|a)}$ denote the conditional probability of $s_{-i} \in S_{-i}$ given (a, s_i).

We normalize payoffs so that each player's *pure strategy* minmax payoff is 0 in the stage game. Note that the mixed minmax payoff may be smaller than the pure strategy minmax payoff. The set of feasible payoff profiles is $V(G) = co\{G(a) | a \in A\}$ and $V^*(G) = \{v \in V(G) | v \gg 0\}$ is the set of feasible, strictly individually rational payoff profiles.

We next introduce the repeated game augmented with communication. Let Y denote a finite set of public signals. A *public coordinating device* is a function $\phi : S \to \triangle(Y)$ that generates a public signal $y \in Y$ with probability $\phi(y|s)$. A convex combination of two public coordination devices ϕ and ϕ' is denoted by $(1 - \lambda)\phi + \lambda\phi'$ and is defined by

$$\left((1 - \lambda)\phi + \lambda\phi'\right)(y|s) := (1 - \lambda)\phi(y|s) + \lambda\phi'(y|s).$$

Players communicate directly each period. At the end of each period, each player i publicly announces some $s_i \in S_i$ but player i may or may not report her signal truthfully. If s is the actual realized signal profile and if all payers report their private signals honestly, then the distribution of the signal generated by ϕ given a with honest reporting is denoted

$$p^\phi(y|a) := \sum_{s \in S} \phi(y|s) p(s|a).$$

We denote expectations with respect to $p^\phi(\cdot|a)$ by $E^\phi[\cdot|a]$. Player i may not report her signal truthfully. Define a reporting rule for player i as a function $\rho_i : S_i \to S_i$. Let R_i be the set of all reporting rules for player i and $\tau_i \in R_i$ be the honest reporting rule defined by $\tau_i(s_i) = s_i$ for all $s_i \in S_i$. When player i uses a reporting rule $\rho_i \in R_i$, we will abuse notation and define

$$p^\phi(y|a, \rho_i) := \sum_{s \in S} \phi(y|\rho_i(s_i), s_{-i}) p(s|a)$$

as the distribution of the generated public signal given action profile a when i uses the reporting rule ρ_i and the other players report their private signals truthfully. We denote expectation with respect to $p^\phi(\cdot|a, \rho_i)$ by $E^\phi[\cdot|a, \rho_i]$. Assuming honest reporting by players $j \neq i$, player i's conditional belief regarding the realization of

the public coordinating signal given (a, s_i) and report s_i' is given by

$$p^\phi\left(y|a, s_i, s_i'\right) := \sum_{s_{-i}} \phi(y|s_i', s_{-i}) p\left(s_{-i}|a, s_i\right).$$

We often use $p^\phi\left(y|a, s_i\right)$ for $p^\phi\left(y|a, s_i, s_i\right)$ to economize on notation.

This formulation of communication employing a public communication device does not require a trusted mediator who receives and sends confidential private information from and to the players[1].

In the repeated game (G, p) augmented with communication as described above, a public history in period t consists of a sequence of realized public coordinating signals $h^t \in Y^t$ and a sequence of public announcements $h_R^t \in S^t$. We allow different coordinating devices to be employed at different $h^t \in Y^t$. Given a private monitoring game (G, p), a *public communication device* for (G, p) is a collection $\Phi = \{\phi_{h^t} : h^t \in Y^t, t \geq 0,\}$ where each $\phi_{h^t} : S \to \Delta(Y)$ is a public coordination device. Given a private monitoring game (G, p), a discount factor δ, and a public communication device Φ, let $G_p'^\infty\left(\delta, \Phi\right)$ denote the *public communication extension* of the repeated game with private monitoring $G_p'^\infty\left(\delta\right)$.

In $G_p'^\infty\left(\delta, \Phi\right)$, play proceeds in the following way. At the beginning of period t, player i chooses an action contingent on $\left(h_i^t, h^t, h_R^t\right)$, where $h_i^t \in A_i^t \times S_i^t$ is a sequence of her own private actions and private signals. If the resulting action profile is a, then players receive private signals according to the distribution $p(\cdot|a)$. Let s denote the realized signal profile. Then player i makes a public announcement s_i' contingent on $(h_i^t, h^t, h_R^t, a_i, s_i)$. Of course, s_i' may differ from s_i. Let $s' \in S$ denote the profile of announcements. Then a public coordinating signal is chosen according to the probability measure $\phi_{h^t}(\cdot|s')$. If s' is announced and y is realized in period t, then h_R^{t+1} and h^{t+1} in period $t+1$ are defined as follows: $h_R^{t+1} = (h_R^t, s')$ and $h^{t+1} = (h^t, y)$.

To describe a strategy in $G_p'^\infty\left(\delta, \Phi\right)$, let $H^t = Y^t$ denote the set of histories of realized public coordinating signals in period t, $H_R^t = S^t$ denote the set of public reporting histories, and $H_i^t = A_i^t \times S_i^t$ denote the set of private histories for player i in period t. Player i's (pure) strategy consists of two components, an "action strategy" $\alpha_i^t : H_i^t \times H^t \times H_R^t \longrightarrow A_i$ and a "reporting strategy" $\rho_i^t : H_i^t \times H^t \times H_R^t \times A_i \longrightarrow R_i$ where R_i is the set of all mappings from S_i to S_i. Let $\alpha_i = (\alpha_i^0, \alpha_i^1, ...), \rho_i = (\rho_i^0, \rho_i^1, ...), \alpha = \{\alpha_i\}_{i \in N}, \rho = \{\rho_i\}_{i \in N}$ and let $\sigma = (\alpha, \rho)$. A pure strategy profile σ induces a probability measure on A^∞. Player i's discounted expected payoff in $G_p'^\infty\left(\delta, \Phi\right)$ is

$$w_i^{\sigma, \delta}\left(\Phi\right) = (1 - \delta) \sum_{t=0}^\infty \delta^t E\left[g\left(\tilde{a}^t\right)|\sigma, \Phi\right].$$

We usually drop Φ when it is clear from the context which public communication device is used.

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[1] See Forges [11] and Myerson [27] for mediated communication in dynamic games.

A strategy $\sigma_i = (\alpha_i, \rho_i)$ for player i is *truthful* if player i reports her private signal truthfully whenever she played according to α_i in the same period, i.e., $\rho_i^t \left(s_i | h_i^t, h^t, h_R^t, \alpha_i^t \left(h_i^t, h^t, h_R^t \right) \right) = s_i$ for every $\left(h_i^t, h^t, h_R^t \right)$ and s_i. Note that we allow players to lie immediately after a deviation in action. That is, we do not require that $\rho_i^t \left(s_i | h_i^t, h^t, h_R^t, a_i \right) = s_i$ if $a_i \neq \alpha_i^t \left(h_i^t, h^t, h_R^t \right)$. A strategy $\sigma_i = (\alpha_i, \rho_i)$ is public if α_i^t only depends on $h^t = \left(y^0, ..., y^{t-1} \right) \in H^t$ and ρ_i^t depends only on $\left(h^t, a_i^t \right)$. Since we focus on this class of strategies in the public communication extension, we will write $\alpha_i^t \left(h^t \right)$ instead of $\alpha_i^t \left(h_i^t, h^t, h_R^t \right)$ and $\rho_i^t \left(h^t, \alpha_i^t \left(h^t \right) \right)$ instead of $\rho_i^t \left(h_i^t, h^t, h_R^t, \alpha_i^t \left(h_i^t, h^t, h_R^t \right) \right)$. Notice that there is a natural one-to-one relationship between public strategies in $G_\pi^\infty (\delta)$ and the action strategy components of public strategies in $G_p'^\infty (\delta, \Phi)$. Note also that we can ignore incentive constraints across different $\left(h_i^t, h_R^t \right)$ in $G_p'^\infty (\delta, \Phi)$ when every player uses a public strategy, as we can ignore incentive constraints across different h_i^t with public strategies for $G_\pi^\infty (\delta)$.

We extend the standard definition of perfect public equilibrium of Fudenberg, Maskin and Levine (1994) to the public communication extension as follows: a strategy profile σ for the public communication extension is a *perfect public equilibrium with communication* (which we will refer to as PPE from now on) if σ is a profile of *truthful* public strategies and the continuation (public) strategy profile constitutes a Nash equilibrium at the beginning of every period. A strategy profile σ is an $\eta-$*uniformly strict perfect public equilibrium with communication* if σ is a perfect public equilibrium and any player would lose at least η in term of discounted average payoff at any moment when she deviates from the *equilibrium action*. Formally, we have the following.

Definition 1 *A strategy profile* $\sigma = (\alpha_i, \rho_i)_{i \in N}$ *is an* $\eta-$*uniformly strict perfect public equilibrium with communication in* $G_p'^\infty (\delta, \Phi)$ *if the following conditions are satisfied:*
(i)

$$(1 - \delta) g_i \left(\alpha^t \left(h^t \right) \right) + \delta \sum_{s \in S} \left[\sum_{y \in Y} w_i^\sigma \left(h^t, y \right) \phi_{h^t} \left(y | s_i, s_{-i} \right) \right] p(s | \alpha^t \left(h^t \right)) - \eta \geq$$

$$(1 - \delta) g_i \left(a_i, \alpha_{-i}^t \left(h^t \right) \right) + \delta \sum_s \left[\sum_y w_i^\sigma \left(h^t, y \right) \phi_{h^t} \left(y | f_i(s_i), s_{-i} \right) \right] p(s | a_i, \alpha_{-i}^t \left(h^t \right))$$

for all $h^t \in H^t, t \geq 0$, *and* $i \in N$.
(ii)

$$(1 - \delta) g_i \left(\alpha^t \left(h^t \right) \right) + \delta \sum_{s \in S} \left[\sum_{y \in Y} w_i^\sigma \left(h^t, y \right) \phi_{h^t} \left(y | s_i, s_{-i} \right) \right] p(s_{-i} | \alpha^t \left(h^t \right), s_i) \geq$$

$$(1 - \delta) g_i \left(\alpha^t \left(h^t \right) \right) + \delta \sum_s \left[\sum_y w_i^\sigma \left(h^t, y \right) \phi_{h^t} \left(y | s_i', s_{-i} \right) \right] p(s_{-i} | \alpha^t \left(h^t \right), s_i)$$

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for all $h^t \in H^t, t \geq 0,\ s_i, s_i' \in S_i$ *and* $i \in N$.

3 Informational Size and Incentive Compatibility

3.1 Informational Size, Distributional Variability, and One-Shot Revelation Game

We turn to the issue of truthful revelation of private information in this subsection. Although our main interest is in repeated games, it is useful to consider the following simple *one-shot information revelation game* first. Fix any private monitoring game (G, p). For any public coordination device ϕ, any profile of payoff functions $w : Y \to \mathbb{R}^n$, and any $a \in A$, the one-shot information revelation game (G, p, ϕ, w, a) is defined as follows. Player i observes a private signal s_i, which is distributed according to $p(s|a)$. Players report s', then a public coordinating signal y is generated with probability $\phi(y|s')$. Finally, player i receives payoff $w_i(y)$ if the realized value of the public signal is y. In the context of repeated games, this payoff will be interpreted as player i's continuation payoff. Consequently, (G, p, ϕ, w, a) defines a game of incomplete information in which a strategy for player i is a function $\rho_i : S_i \to S_i$ and truthful reporting is an equilibrium if for each i,

$$\sum_{s_{-i} \in S_{-i}} \sum_{y \in Y} w_i(y)\phi(y|s_i, s_{-i})\, p(s_{-i}|a, s_i) \geq \sum_{s_{-i} \in S_{-i}} \sum_{y \in Y} w_i(y)\phi\left(y|s_i', s_{-i}\right) p(s_{-i}|a, s_i)$$

for each $s_i, s_i' \in S_i$.

When do players have incentive to report their private signals truthfully in this game? To fix ideas, consider the extreme case in which (p, ϕ) is nonexclusive. Then no player has an incentive to lie because what she reports is irrelevant and does not affect the generated public signal at all. Hence truth-telling can be implemented in a one-shot revelation game for any specification of the payoff function and any action when (p, ϕ) is nonexclusive.

We wish to generalize this simple observation. In general, it should become "easier" to induce truth-telling as each player's influence on the public coordinating signal becomes "smaller." The following index measures the size of this influence for each player.

Definition 2 *(Informational Size)*

Player $i's$ informational size $v_i^\phi(s_i, s_i', a)$ given ϕ and $(s_i, s_i', a) \in S_i \times S_i \times A$ is the smallest ϵ satisfying

$$\Pr(\left\| \phi\left(\cdot|s_i, \widetilde{s}_{-i}\right) - \phi\left(\cdot|s_i', \widetilde{s}_{-i}\right)\right\| > \epsilon|s_i, a) \leq \epsilon.$$

This means that, conditional on (s_i, a), player i believes that the probability of her being able to manipulate the public signal distribution by more than $v_i^\phi(s_i, s_i', a)$

by announcing s_i' is at most $v_i^\phi(s_i, s_i', a)$. Of course, a player's informational size alone is not enough to induce honest reporting. Since players may still have incentive to misreport their signals, however small it is, we need to introduce some scheme to punish dishonest reporting. So we consider the following mechanism design problem: given that $a \in A$ is played, find a public coordination device ϕ' that generates approximately the same distribution as $p^\phi(\cdot|a)$ and makes truthful reporting a Bayesian Nash equilibrium for the one-shot revelation game (G, p, ϕ', w, a). For this purpose, we construct a certain scoring rule that relies on a player's distributional variability.

Definition 3 *(Distributional Variability)*

$$\Lambda_i^\phi\left(s_i, s_i', a\right) = \left\| \frac{p^\phi\left(\cdot|a, s_i\right)}{\|p^\phi\left(\cdot|a, s_i\right)\|} - \frac{p^\phi\left(\cdot|a, s_i'\right)}{\|p^\phi\left(\cdot|a, s_i'\right)\|} \right\|^2$$

This measures the extent to which player i's conditional (normalized) beliefs regarding the public coordinating signal are different given s_i and s_i' (assuming honest reporting by others). Let $\Lambda_i^\phi(s_i, a) := \min_{s_i' \neq s_i} \Lambda_i^\phi(s_i, s_i', a)$ be player i's minimum distributional variability given (s_i, a).

Intuitively, it must be easier to induce players to report their private signals truthfully when the first indices are smaller and the second indices are larger. It turns out that what is important for truthful revelation is the ratio of these two indexes at each (s_i, s_i', a).

Definition 4 *The measure p^ϕ is $\gamma-$regular for ϕ if $v_i^\phi(s_i, s_i', a) \leq \gamma\Lambda_i^\phi(s_i, s_i', a)$ for all u, $s_i \in S_i$, $s_i' \in S_i$, $a \in A$ and $i \in N$.*

For example, p^ϕ is $0-$regular if (p, ϕ) is nonexclusive. When we say a player is *informationally small*, we mean that the ratio of her informational size to her distributional variability given every (s_i, s_i', a) in this sense. We can now prove the following theorem, stated without proof in MOP (2014).

Theorem 1 *If (G, p) is a private monitoring game (G, p) and if $\lambda \in (0, 1)$, then there exists a $\gamma > 0$ such that the following holds: if p^ϕ is $\gamma-$regular for some ϕ, then for any $a \in A$ and any payoff function $w : Y \to \mathbb{R}^n$, there exists a public coordination device $\phi'_{a,w} : S \to \triangle(Y)$ such that truthful reporting is a Bayesian Nash equilibrium for the one-shot information revelation game $(G, p, (1 - \lambda)\phi + \lambda\phi'_{a,w}, w, a)$.*

Proof. See Appendix B. ∎

This theorem means that honest reporting can be induced for any one-shot revelation game by perturbing ϕ slightly. The smaller γ is, the smaller the size of the required perturbation. Note that γ depends on λ but is independent of the

payoff function and the underlying action. These properties will be important when this result is applied to repeated games. It is natural that γ is independent of a, since $\gamma-$regularity requires a certain property across all actions. The reason why γ is also independent of w is as follows. We construct a punishment by perturbing ϕ slightly so that the distribution of the generated signal remains similar, but truth-telling is incentive compatible. When w is large, the temptation to deviate may be high, but the size of punishments is large in the same proportion. So the size of w does not matter.

When will p^ϕ satisfy $\gamma-$regularity? The following example is taken from MOP.

Definition 5 *Example 1 A private monitoring game (G, p) is called $\beta-$conditionally independent (G, π) if $S_i = Y$ for all i and there exists for each $a \in A$ a $\pi(\cdot|a) \in \Delta(Y)$ for all y and i $q_i(\cdot|y) \in \triangle(S_i)$ such that*

$$p(s|a) = \sum_{y \in Y} \prod_i q_i(s_i|y)\pi(y|a)$$

and $q_i(y|y) \geq \beta$ for any y and i.

Suppose that (G, p) is a $\beta-$conditionally independent. Let $\phi_M : S \to Y$ denote the public coordination device that chooses that signal y reported by the largest number of players (with some tie-breaking rule). Then

$$
\begin{aligned}
p^{\phi_M}(y|a) &= \sum_{s \in S} \phi_M(y|s)p(s|a) \\
&= \sum_{s \in S} \phi_M(y|s) \sum_{y \in Y} \left[\prod_{i=1}^{n} q_i(s_i|y)\right] \pi(y|a)
\end{aligned}
$$

and p^{ϕ_M} can generate almost the same signal distribution as π as $\beta \to 1$. Furthermore, player i's maximum informational size $v_i^\phi(s_i, a)$ converges to 0 given any (s_i, a) as long as $n \geq 3$ and player i's minimum distributional variability $\Lambda_i^\phi(s_i, a)$ converges to a positive constant given any (s_i, a) as $\beta \to 1$ for every $i \in N$. Consequently, it follows that if $n \geq 3$. then for each $\gamma > 0$, there exists $\beta \in (0, 1)$ p^{ϕ_M} is $\gamma-$regular.

4   Folk Theorem

In this section, we prove a new folk theorem for repeated games with private monitoring and communication when players are informationally small. We exploit a connection between public monitoring games and private monitoring games and adapt some standard techniques for a public-monitoring folk theorem to the domain of private monitoring games.

Our folk theorem asserts the following. Suppose that for some public coordination device ϕ for (G, p) the associated p^ϕ satisfies a certain condition that guarantees a folk theorem in the repeated game with public monitoring game (G, p^ϕ). Then there exists a $\gamma > 0$ such that a folk theorem is also obtained for a communication extension of the repeated game with private monitoring game $G_p'^\infty(\delta, \Phi)$ for some public communication device Φ when p^ϕ is γ-regular. Furthermore, our folk theorem is a uniformly strict folk theorem, i.e., a folk theorem with $\eta-$uniformly strict PPE for some $\eta > 0$.

To state the theorem more precisely, we need to clarify the "certain condition" to which we have alluded in the previous paragraph. Given any public signal distribution π, let $T_i^\pi(a) \subset R^{|Y|}$ be defined as

$$T_i^\pi(a) = co\left\{\pi\left(\cdot|a_i', a_{-i}\right) - \pi\left(\cdot|a\right) : a_i' \neq a_i\right\}$$

and let $\widehat{T}_i^\pi(a) = co\left\{T_i^\pi(a) \cup \{0\}\right\}.$[2] The set $T_i^\pi(a)$ consists of those distributional changes that player i can induce by choosing a strategy different from a_i when the remaining players choose a_{-i}. The set $\widehat{T}_i^\pi(a)$ consists of all feasible distributional changes that player i can induce. We say that a public signal distribution π satisfies *distinguishability* at $a \in A$ if for each pair of distinct players i and j, the following conditions are satisfied:

$$0 \notin T_i^\pi(a) \cup T_j^\pi(a) \tag{1}$$

$$\widehat{T}_i^\pi(a) \cap \widehat{T}_j^\pi(a) = \{0\} \tag{2}$$

$$\left(-\widehat{T}_i^\pi(a)\right) \cap \widehat{T}_j^\pi(a) = \{0\}. \tag{3}$$

We say that π satisfies distinguishability if it satisfies distinguishability at every $a \in A$. (1) means that a unilateral deviation by player i or player j must be statistically detectable. (2) and (3) are conditions regarding the distinguishability of player i's deviation and player j's deviation. It is known that these conditions are sufficient for a folk theorem for repeated games with public monitoring.[3]

Now we can state our folk theorem. Let $E(\delta, \Phi, \eta) \subset \mathbb{R}^n$ be the set of $\eta-$uniformly strict PPE payoff profiles of $G_p'^\infty(\delta, \Phi)$ given δ and Φ.

Theorem 2 *Fix any private monitoring game (G, p). Suppose that $intV^*(G) \neq \varnothing$ and there exists ϕ such that p^ϕ is distinguishable. Then there exists a $\gamma > 0$ such that, if p^ϕ is $\gamma-$regular, then the following holds: for each $v \in intV^*(G)$, there exists an $\eta > 0$ and a $\underline{\delta} \in (0, 1)$ such that, for each $\delta \in (\underline{\delta}, 1)$, there exists a public*

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[2] coX denote the convex hull of X in $\mathbb{R}^n$.

[3] These conditions guarantee that the incentive constraints of player i and j are satisfied simultaneously by using appropriate transfers (=continuation payoffs) even when their transfers are required to lie on any hyperplane. They are parallel to (A2) and (A3) in Kandori and Matsushima [18].

communication device Φ and a $(1-\delta)\,\eta-$uniformly strict truthful PPE of $G_p'^{\infty}(\delta, \Phi)$ with payoff v.

Proof. See Appendix D. ∎

Note that γ depends only on the underlying stage game (G, p) but not on v. On the other hand, η depends on v, while Φ depends on both v and δ.[4]

Remark.

- The assumption $intV^*(G) \neq \varnothing$ requires that $V^*(G)$ is full dimensional. When $V^*(G)$ is not full-dimensional, we may strengthen the distinguishability condition to prove the same result. To prove this theorem, for each $a \in A$ and $q \in \mathbb{R}^n$ such that $\|q\| = 1$ and $|q_i| < 1$ for all i, we construct $x : Y \to \mathbb{R}^n$ that satisfies $E^{\phi}[x_i|a] > E^{\phi}[x_i|a_i', a_{-i}]$ for all $a_i' \neq a_i$ and all i. If $V^*(G)$ is not full dimensional, the range of x needs to be the affine space that contains $V^*(G)$, instead of $\mathbb{R}^n$. This additional restriction can be addressed by strengthening the distinguishability condition. The bottom line is that every proof goes through if we restrict our attention to the affine space that contains $V^*(G)$.[5] See Fudenberg, Levine and Takahashi [15] for the characterization of the limit equilibrium payoff when $V^*(G)$ is not full dimensional.

- Is a folk theorem obvious given our robustness result? Take any private monitoring game (G, p) for which there exists ϕ such that p^{ϕ} satisfies distinguishability. Why not prove a folk theorem with $\eta-$uniformly strict PPE for some $\eta > 0$ for the public monitoring repeated game with (G, p^{ϕ}) (which is not difficult to do) and apply our robustness result? However, this approach is not satisfactory because we need to tailor the informational size to each target equilibrium payoff profile and given discount factor to do so, i.e. γ depends on both v and δ. The strength of the above folk theorem is that we can find a *fixed* size of informational smallness for which the folk theorem is obtained, rather than including γ as a parameter that depends on each payoff profile in the statement of the folk theorem.

4.1 Overview of Proof

We prove our folk theorem in several steps. Some proofs are provided in the appendix.

Self Decomposability with Private Monitoring and Public Coordinating Device

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[4]However, η can be chosen independent of v for generic stage games, namely when the solution for $\max_{a \in A} g_i(a)$ and $\min_{a \in A} g_i(a)$ is unique for every i.

[5]In particular, we need to state Lemma 9 with respect affine spaces and in terms of relative interior.

In the following, a private monitoring game (G, p) is fixed. Rather than analyzing the repeated game directly, we begin by decomposing discounted average payoffs of a repeated game into current payoffs and continuation payoffs, and then analyze a collection of one-shot revelation games.

For a public monitoring game (G, π), an action profile $a \in A$ is said to be *enforceable* with respect to $W \subset \mathbb{R}^n$ and $\delta \in (0, 1)$ if there exists a function $w : Y \to W$ such that

$$(1 - \delta) g_i(a) + \delta E[w_i(\cdot) | a] \geq (1 - \delta) g_i(a_i', a_{-i}) + \delta E[w_i(\cdot) | a_i', a_{-i}]$$
$$\text{for all } a_i' \neq a_i \text{ and } i \in N.$$

If $v = (1 - \delta) g(a) + \delta E[w(\cdot) | a]$ for some enforceable action a and $w : Y \to W$, then we say that v is decomposable with respect to W and δ. Let $B(\delta, W)$ be the set of payoff profiles that are decomposable with respect to W and δ. It is a known result that, if any bounded set W' is self decomposable i.e. $W' \subset B(\delta, W')$, then every payoff in $B(\delta, W')$ (hence in W') can be supported by a PPE (Abreu, Pearce and Stacchetti [1]).

We now extend these ideas to the private monitoring game (G, p) with public coordination devices. Recall that, given action profile $a \in A$, $E^\phi[\cdot | a]$ denotes expectation with respect to $p^\phi(\cdot | a)$, $E^\phi[\cdot | a, \rho_i]$ denotes expectation with respect to $p^\phi(\cdot | a, \rho_i)$ and $\tau_i : S_i \to S_i$ denotes the honest reporting rule for player i defined by $\tau_i(s_i) = s_i$ for all $s_i \in S_i$.

Definition 6 *An action profile $a \in A$ is $\eta-$enforceable with respect to $W \subset \mathbb{R}^n$ and $\delta \in (0, 1)$ if there exists a public coordinating device $\phi : S \to \Delta(Y)$ and $w : Y \to W$ such that for all $i \in N$,*
(i) $(1 - \delta) g(a) + \delta E^\phi[w_i(\cdot) | a] - \eta \geq (1 - \delta) g(a_i', a_{-i}) + \delta E^\phi[w_i(\cdot) | (a_i', a_{-i}), \rho_i]$ for all $a_i' \neq a_i$, $\rho_i : S_i \to S_i$
(ii) $(1 - \delta) g(a) + \delta E^\phi[w_i(\cdot) | a] \geq (1 - \delta) g(a) + \delta E^\phi[w_i(\cdot) | a, \rho_i]$ for all $\rho_i \neq \tau_i$.

The inequality (i) means that a player would lose more than η when deviating from a. Inequality (ii) means that dishonest reporting is not profitable after a is played. If $a \in A$ is $\eta-$enforceable with respect to W and δ with some v and w and $v = (1 - \delta) g(a) + \delta E^\phi[w_i(\cdot) | a]$, then we say that the triple (a, ϕ, w) $\eta-enforces$ v with respect to W and δ. We say that v is $\eta-decomposable$ with respect to W and δ when there exists a triple (a, ϕ, w) that $\eta-$enforces v with respect to W and δ.

Next define the set of $\eta-$decomposable payoffs with respect to W and δ as follows.

$$B(\delta, W, \eta) := \{v \in \mathbb{R}^n | v \text{ is } \eta - \text{decomposable with respect to } W \text{ and } \delta\}.$$

We say that W is $\eta-self\ decomposable$ with respect to $\delta \in (0, 1)$ if $W \subset B(\delta, W, \eta)$.

It is easy to see that a "uniformly strict" version of Theorem 1 in Abreu, Pearce, and Stacchetti [1] holds here when $\eta > 0$: if W is $\eta-$self decomposable with respect

to δ, then every $v \in W$ can be supported by a $\eta-$uniformly strict PPE of $G_p'^\infty (\delta, \Phi)$ for some public communication device Φ. Note that each payoff profile may need to be supported by using a different public coordinating device. Hence different public coordinating devices need to be used at different public histories. Since the following lemma is a straightforward implication of the result in Abreu, Pearce and Stacchetti [1], its proof is omitted.

Lemma 1 *If $W \subset \mathbb{R}^n$ is bounded and $\eta-$self decomposable with respect to $\delta \in (0,1)$, then for any $v \in W$, there exists Φ such that $v \in E(\delta, \Phi, \eta)$.*

Local Decomposability is Enough

Fudenberg, Levine, and Maskin [14] showed that local self decomposability is sufficient for self decomposability of any convex, compact set for large δ. Here we prove the corresponding lemma in our setting. First, we prove a lemma that establishes a certain monotonicity property of B. The Lemma implies that, if W is $\eta-$self decomposable with respect to $\delta \in (0,1)$, then W is $\frac{1-\delta'}{1-\delta}\eta-$self decomposable for every $\delta' \in (\delta, 1)$.

Lemma 2 *If $W \subseteq \mathbb{R}^n$ is convex and $C \subseteq B(\delta, W, \eta) \cap W$, then $C \subseteq B\left(\delta', W, \frac{1-\delta'}{1-\delta}\eta\right)$ for every $\delta' \in (\delta, 1)$.*

Proof. Suppose that $v \in C$. Since $v \in B(\delta, W, \eta)$, v is $\eta-$decomposable with respect to W and δ, there exists a triple (a, ϕ, w) that $\eta-$enforces v. For any $\delta' > \delta$, define $w^{\delta'} : Y \to W$ as the following convex combination of v and w:

$$w^{\delta'}(y) = \frac{\delta' - \delta}{\delta'(1-\delta)}v + \frac{\delta(1-\delta')}{\delta'(1-\delta)}w(y).$$

Clearly, $w^{\delta'}(y) \in W$ for each $y \in Y$ since W is convex. Furthermore, we can show that, for every $\delta' \in (\delta, 1)$, the triple $\left(a, \phi, w^{\delta'}\right) \frac{1-\delta'}{1-\delta}\eta-enforces$ v with respect to W and δ'. To see this, note that, for every $\delta' \in (\delta, 1)$, $a \in A$, and $i \in N$,

$$\left(1 - \delta'\right) g(a) + \delta' \sum_{y,s} w_i^{\delta'}(y)\phi(y|s) p(s|a)$$

$$= \left(1 - \delta'\right) g(a) + \frac{\delta(1-\delta')}{1-\delta} \sum_{y,s} w_i(y)\phi(y|s) p(s|a) + \frac{\delta' - \delta}{1-\delta}v_i$$

$$= \frac{1-\delta'}{1-\delta}\left\{(1-\delta)g(a) + \delta \sum_{y,s} w_i(y)\phi(y|s) p(s|a)\right\} + \frac{\delta' - \delta}{1-\delta}v_i.$$

Consequently,

$$v = \left(1 - \delta'\right) g(a) + \delta' \sum_{y,s} w^{\delta'}(y)\phi(y|s) p(s|a)$$

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and conditions (i) and (ii) of Definition 6 hold for $w^{\delta'}$ when η is replaced with $\frac{1-\delta'}{1-\delta}\eta$. Therefore, the triple $\left(a, \phi, w^{\delta'}\right)$ $\frac{1-\delta'}{1-\delta}\eta-$enforces v with respect to W and δ implying that v is $\frac{1-\delta'}{1-\delta}\eta-$self decomposable with respect to W and δ'. Therefore $C \subseteq B\left(\delta', W, \frac{1-\delta'}{1-\delta}\eta\right)$ for every $\delta' \in (\delta, 1)$. ∎

Now we show that local self decomposability implies self decomposability. A set $W \subseteq \mathbb{R}^n$ is *locally strictly self-decomposable* if, for any $v \in W$, there exists $\eta > 0$, $\delta \in (0, 1)$ and an open set U containing v such that $U \cap W \subset B(\delta, W, \eta)$.

Lemma 3 *If $W \subset \mathbb{R}^n$ is compact, convex, and locally strictly self decomposable, then there exists $\eta > 0$ and $\underline{\delta} \in (0, 1)$ such that W is $(1 - \delta)\eta-$self decomposable with respect to δ for any $\delta \in (\underline{\delta}, 1)$.*

Proof. Choose $v \in W$. Since W is locally strictly self decomposable, there exists $\delta_v \in (0, 1)$, $\eta_v > 0$, and an open ball U_v around v such that

$$U_v \cap W \subseteq B(\delta_v, W, \eta_v).$$

Since W is compact, there exists a finite subcollection $\{U_{v_k}\}_{k=1}^K$ that covers W. Define $\underline{\delta} = \max_{k=1,\dots,K}\{\delta_{v_k}\}$ and $\eta = \min_{k=1,\dots,K}\{\eta_{v_k}\}$. Then

$$U_{v_k} \cap W \subseteq B\left(\delta_{v_k}, W, \eta_{v_k}\right) \subseteq B\left(\delta_{v_k}, W, \eta\right).$$

Lemma 2 and the convexity of W imply that

$$U_{v_k} \cap W \subseteq B\left(\delta, W, \frac{(1-\delta)}{1-\delta_{v_k}}\eta\right) \subseteq B(\delta, W, (1-\delta)\eta)$$

for any $\delta \in (\underline{\delta}, 1)$ and for $k = 1, \dots, K$. Consequently,

$$W = \cup_{k=1}^K (U_{v_k} \cap W) \subseteq B(\delta, W, (1-\delta)\eta).$$

∎

Proving Local Decomposability

Given Lemma 1 and Lemma 3, the proof of Theorem 3 will be complete if, for every individually rational and feasible payoff profile $v \in intV^*(G)$, we can find a compact, convex, locally self decomposable set that contains it. We call a set in $\mathbb{R}^n$ *smooth* if it is closed and convex with an interior point in $\mathbb{R}^n$ and there exists the unique tangent hyperplane at every boundary point.[6] Since any such v can be contained in some smooth set in $intV^*(G)$, we are done if we can show that every smooth set in $intV^*(G)$ is locally self decomposable. Hence the following lemma completes the proof of Theorem 2.

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[6] This notion of smoothness is slightly more general than the one in [14] in the sense that the surface does not need to be twice continuously differentiable.

Lemma 4 *Fix a private monitoring game* (G, p). *Suppose that there exists* ϕ *such that* p^ϕ *is distinguishable. Then there exists a* $\gamma > 0$ *such that, if* p^ϕ *is* $\gamma-$*regular, then every smooth set* $W \subseteq intV^*(G)$ *is locally strictly self decomposable.*

Proof. See appendix D. ∎

To prove this, we follow the argument of Fudenberg, Levine and Maskin [14] in Theorem 4.1. Suppose that we can induce players to report their private signals truthfully. Then the stage game is essentially a public monitoring game (G, p^ϕ). In this case, we can show that almost every boundary point v on a smooth set $W \subseteq intV^*(G)$ is decomposable with respect to the hyperplane that is parallel to the tangent hyperplane at v if p^ϕ satisfies distinguishability. Since we need to induce truthful reporting at the same time, we need to strengthen this requirement and show that every such boundary point v is *strictly decomposable.* We then perturb ϕ and continuation payoffs slightly as in the previous section so that these boundary points remain strictly decomposable and every player has an incentive to report honestly. This can be done when every player is informationally small.

A few comments are in order. First, it may be of some technical interest that we prove this step using some smoothness condition that is weaker than the one in [14], which is commonly invoked to prove a folk theorem in the literature. Second, we choose γ independent of the target payoff profiles as we emphasized. Third, it may not possible to obtain strict decomposability when the continuation payoffs lie on the tangent hyperplane that is not "regular" (i.e. it is "vertical" or "horizontal"; all the coefficients except one are 0) because the continuation payoffs are constant for some player. In this case, we obtain strict decomposability by choosing continuation payoffs from a half space bounded away from the target payoff profile. Finally, our result clearly implies that a uniformly strict folk theorem is obtained for repeated games with imperfect public monitoring when distinguishability is satisfied, because there is no incentive constraint regarding the revelation of private information in this case.

5   Related Literature and Discussion

There is a large literature on repeated games with private monitoring and communication. Most papers in the literature focus on a folk theorem rather than robustness. Our approach to folk theorem is similar to Ben-Porath and Kahneman [6]. They prove a folk theorem when a player's action is perfectly observed by at least two other players. For each individually rational and feasible payoff profile, they fix a strategy to support it with perfect monitoring, then augment it with a reporting strategy to support the same payoff profile with direct communication. Their strategies employ draconian punishments when a player's announcement is inconsistent with others' announcements ("shoot the deviator"). Our paper differs from their paper in many

respects. Firstly, our paper uses not only perfect monitoring but also imperfect public monitoring as a benchmark. Secondly, private signals can be noisy in our paper. Aoyagi [4] proves a Nash-threat folk theorem in a setting similar to Ben-Porath and Kahneman [6], but with noisy private monitoring. In his paper, each player is monitored by a subset of players. Private signals are noisy, but reflect the action of the monitored player very accurately when they are jointly evaluated. That is, private monitoring is jointly almost perfect. In his paper, players have access to a more general communication device than ours, namely, *mediated communication*. Tomala [35] introduces a concept called perfect communication equilibrium and proves a folk theorem with private monitoring and mediated communication. Compte [8] and Kandori and Matsushima [18] provide general sufficient conditions for a folk theorem with noisy private monitoring and with direct communication like us. Our sufficient conditions are different from their conditions. Compte [8] assumes that players' private signals are independent conditional on action profiles. This condition is not satisfied by any β-perturbation of public monitoring game, to which our Proposition ?? and Theorem 2 apply. Obara [28] finds a sufficient condition to extend Compte [8]'s result to the case where private signals are correlated. Kandori and Matsushima [18] assume that, among others, a deviation by one player and a deviation by another player can be statistically distinguished based on the private signals of the remaining players. This condition is similar to, but different from our condition (2) and (3). Their condition and our condition impose the same restriction on the set of probability measures, but they impose it on the marginal distributions of private signals for each subset of $n - 2$ players, whereas we impose it on the public signal distribution that is approximated by the private signal distribution via some public coordination device. Fudenberg and Levine [12] prove a folk theorem for repeated games with private monitoring and communication when private monitoring is *almost perfect messaging*. Our folk theorem allows for more general perturbations, but their result (as well as Obara [28]'s) applies to the two player games unlike our results. Anderlini and Lagunoff [3] consider dynastic repeated games with communication where short-lived players care about their offsprings. As in our paper, players may have an incentive to conceal bad information so that future generations do not suffer from mutual punishments. Their model is based on perfect monitoring and their focus is on characterizing the equilibrium payoff set rather than establishing the robustness of equilibria or proving a folk theorem. Mailath and Morris [21] also prove a folk theorem for general repeated games with almost-perfect and almost-public private monitoring without any communication.[7]

There is an extensive literature on repeated games with private monitoring without communication, starting with Sekiguchi [33]. Bhaskar and Obara [7], Ely and Välimäki [9], Piccione [29] prove a folk theorem for a repeated prisoners' dilemma game with private almost-perfect monitoring. The most successful approach to pri-

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[7]However, the proof of the folk theorem in [21] is flawed. A correct proof can be found in Mailath and Olszewski [23].

vate monitoring games, which is taken by [9] and [29], is to rely on a class of equilibria called *Belief-free equilibria.*[8,9] Belief-free equilibrium is formalized and generalized by Ely, Hörner and Olszewski [16]. Its limit payoff characterization is first given by [16], then improved by Yamamoto [37]. In general, belief-free equilibrium does not deliver a folk theorem except for the special game such as prisoners' dilemma. Various extensions of belief-free equilibrium have been proposed and successfully applied to prove a folk theorem for more general games. Matsushima [24] employs *Belief-free review strategies* to prove a folk theorem for a repeated prisoner's dilemma game with conditionally independent noisy private monitoring, which is neither almost perfect nor almost public.[10] Hörner and Olszewski [16] also uses a type of belief-free review strategies to prove a folk theorem for general stage games with private almost-perfect monitoring. Sugaya [33] pushes this idea of belief-free review strategies further and proves a folk theorem for general stage games with noisy private monitoring when the signal spaces are large enough. Kandori [17] introduces a notion of weakly belief-free equilibrium that includes belief-free equilibrium as a special case. Finally, Miyagawa, Miyahara, and Sekiguchi [26] consider private monitoring games where each player can observe the other players' actions perfectly with some cost, and proves a folk theorem without any assumption on imperfect monitoring structure.

---

[8]However, the *Belief-based approach* by [7] also has been studied and refined in recent papers such as Phelan and Skrzypacz [31] and Kandori and Obara [20].

[9]A type of belief-free equilibrium has also appeared in the context of repeated games with *imperfect public monitoring* in Kandori and Obara [19].

[10]For an extension of Matsushima's construction to N player games, see Yamamoto [36].

6 Appendix

A. Preliminary Lemma

Here we prove several useful lemmas. First we derive a few upper bounds on player i's ability to manipulate the distribution of a public coordinating signal.

Lemma 5 $\left\| p^\phi\left(\cdot|a,s_i\right) - p^\phi\left(\cdot|a,s_i,s_i'\right) \right\| \leq \left(1+\sqrt{2}\right) v_i^\phi\left(s_i,s_i',a\right)$ *for all* $s_i', s_i \in S_i$, $a \in A$ *and* $i \in N$.

Proof.

$$
\begin{aligned}
\left\| p^\phi\left(\cdot|a,s_i\right) - p^\phi\left(\cdot|a,s_i,s_i'\right) \right\| &\leq E\left[\left\| \phi(\cdot|s) - \phi(\cdot|s_i',s_{-i}) \right\| \, | a, s_i \right] \quad \text{(Jensen's inequality)} \\
&\leq \left(1 - v_i^\phi\left(s_i,s_i',a\right)\right) v_i^\phi\left(s_i,s_i',a\right) + v_i^\phi\left(s_i,s_i',a\right) \cdot \max_{c,d \in \triangle(Y)} \|c-d\| \\
&\leq \left(1+\sqrt{2}\right) v_i^\phi\left(s_i,s_i',a\right)
\end{aligned}
$$

∎

The next lemma provides an upper bound on player i's distributional variability.

Lemma 6

$$
\begin{aligned}
\Lambda_i^\phi\left(s_i,s_i',a\right) &\leq 2\left(1 - \frac{p^\phi\left(\cdot|a,s_i\right) \cdot p^\phi\left(\cdot|a,s_i'\right)}{\left\| p^\phi\left(\cdot|a,s_i\right) \right\| \left\| p^\phi\left(\cdot|a,s_i'\right) \right\|}\right) \\
&\text{for all } s_i, \ s_i' \ \neq \ s_i, \ a \in A \text{ and } i \in N.
\end{aligned}
$$

Proof.

$$
\begin{aligned}
\Lambda_i^\phi\left(s_i,s_i',a\right) &\leq \left\| \frac{p^\phi\left(\cdot|a,s_i\right)}{\left\| p^\phi\left(\cdot|a,s_i\right) \right\|} - \frac{p^\phi\left(\cdot|a,s_i'\right)}{\left\| p^\phi\left(\cdot|a,s_i'\right) \right\|} \right\|^2 \\
&= 2\left(1 - \frac{p^\phi\left(\cdot|a,s_i\right) \cdot p^\phi\left(\cdot|a,s_i'\right)}{\left\| p^\phi\left(\cdot|a,s_i\right) \right\| \left\| p^\phi\left(\cdot|a,s_i'\right) \right\|}\right)
\end{aligned}
$$

∎

B. Proof of Theorem 1

Proof. Fix a private monitoring game (G,p) and $\lambda \in (0,1)$. Pick any payoff function $w : Y \to \mathbb{R}^n$ and $a \in A$. Without loss of generality, we will assume that $\min_{y \in Y} w_i(y) = 0$ for all $i \in N$. First we define a public coordination device $\phi_{a,w}'$. To begin, define the following function $\psi_i : A \times S \to [0,1]$ for each $i \in N$

$$
\psi_i\left(a,s\right) := \sum_{y \in Y} \frac{p^\phi\left(y|a,s_i\right)}{\left\| p^\phi\left(\cdot|a,s_i\right) \right\|} \cdot \phi\left(y|s\right)
$$

Next, for any pair of probability distributions $\overline{\mu}_i, \underline{\mu}_i \in \triangle(Y)$, let

$$\phi'_{a,\overline{\mu}_i,\underline{\mu}_i}(y|s) := \overline{\mu}_i(y)\psi_i(a,s) + \underline{\mu}_i(y)(1 - \psi_i(a,s))$$

and define $\phi'_{a,\overline{\mu},\underline{\mu}} := \frac{\sum_{i=1}^n \phi'_{a,\overline{\mu}_i,\underline{\mu}_i}}{n}$ as the average of $\phi'_{i,a,\overline{\mu}_i,\underline{\mu}_i}, i = 1, ..., n$, where $(\overline{\mu}, \underline{\mu}) = (\overline{\mu}_1, \underline{\mu}_1, ..., \overline{\mu}_n, \underline{\mu}_n)$.

Next let $\overline{\mu}_{i,w}$ and $\underline{\mu}_{i,w}$ be any pair of probability distributions on Y that satisfy

$$\overline{\mu}_{i,w} \in \arg\max_{q \in \triangle(Y)} \sum_{y \in Y} q(y)w_i(y)$$

and

$$\underline{\mu}_{i,w} \in \arg\min_{q \in \triangle(Y)} \sum_{y \in Y} q(y)w_i(y).$$

That is $\overline{\mu}_{i,w}$ is a distribution on Y that maximizes player i's expected value of w_i and $\underline{\mu}_{i,w}$ is a distribution that minimizes player i's expected value of w_i. Finally, define $\phi'_{a,w} := \phi'_{a,\overline{\mu}_w,\underline{\mu}_w}$ and let

$$\phi^\lambda_{a,w} := (1 - \lambda)\phi + \lambda\phi'_{a,w}.$$

We prove the following claim. Note that this completes the proof of Theorem 1 because γ is chosen independent of w and a.

Claim: Suppose that

$$0 < \gamma < \frac{1}{\left((1-\lambda)\sqrt{|Y|} + \lambda\right)(1 + \sqrt{2})} \frac{\lambda}{2\sqrt{|Y|}n}.$$

If p^ϕ is γ-regular, then truthful reporting is a Bayesian Nash equilibrium in the one-shot information revelation game $\left(G, p, \phi^\lambda_{a,w}, w, a\right)$.

Proof of Claim: We will prove that, if γ satisfies the condition of the claim and if p^ϕ is γ-regular, then

$$\sum_{s_{-i} \in S_{-i}} \sum_{y \in Y} w_i(y) \left\{\phi^\lambda_{a,w}(y|s_i, s_{-i}) - \phi^\lambda_{a,w}(y|s'_i, s_{-i})\right\} p(s_{-i}|a, s_i) \geq 0$$

for each $i \in N$ and each $s_i, s'_i \in S_i$. Let $\overline{w}_i = \max_{y \in Y} w_i(y)$ and note $\overline{w}_i \geq 0$ since $\min_{y \in Y} w_i(y) = 0$.

We prove this claim in four steps. In Step 1, we derive a lower bound for player i's expected loss from misreporting that comes from $\phi'_{a,\overline{\mu}_i,\underline{\mu}_i}$. In Step 2 and 3, we derive an upper bound for player i's maximum expected gain from misreporting that

comes from two other terms in $\phi^\lambda_{a,w}$: ϕ and $\phi'_{a,\overline{\mu}_j,\underline{\mu}_j}$ respectively. Then we show that the expected loss always outweigh the expected gains in Step 4.

Fix player i and suppose that player i's true signal is s_i, but she dishonestly reports s'_i.

Step 1: In this step, we show that

$$\sum_{s_{-i} \in S_{-i}} \sum_{y \in Y} w_i(y) \left\{ \phi'_{a,\overline{\mu}_{i,w},\underline{\mu}_{i,w}}(y|s_i, s_{-i}) - \phi'_{a,\overline{\mu}_{i,w},\underline{\mu}_{i,w}}(y|s'_i, s_{-i}) \right\} p(s_{-i}|a, s_i)$$

$$\geq \ \overline{w}_i \left(\frac{\Lambda^\phi_i(s_i, s'_i, a)}{2\sqrt{|Y|}} - \left(1 + \sqrt{2}\right) v^\phi_i(s_i, s'_i, a) \right)$$

To see this, note that

$$\sum_{s_{-i} \in S_{-i}} \sum_{y \in Y} w_i(y) \left\{ \phi'_{a,\overline{\mu}_{i,w},\underline{\mu}_{i,w}}(y|s_i, s_{-i}) - \phi'_{a,\overline{\mu}_{i,w},\underline{\mu}_{i,w}}(y|s'_i, s_{-i}) \right\} p(s_{-i}|a, s_i)$$

$$= \ \overline{w}_i \sum_{s_{-i} \in S_{-i}} \left(\psi_i(a, s) - \psi_i\left(a, (s'_i, s_{-i})\right) \right) p(s_{-i}|a, s_i)$$

$$= \ \overline{w}_i \left[\sum_{y \in Y} \frac{p^\phi(y|a, s_i)}{\|p^\phi(\cdot|a, s_i)\|} p^\phi(y|a, s_i) - \sum_{y \in Y} \frac{p^\phi(y|a, s'_i)}{\|p^\phi(\cdot|a, s'_i)\|} p^\phi(y|a, s_i, s'_i) \right]$$

$$= \ \overline{w}_i \sum_{y \in Y} \left(\frac{p^\phi(y|a, s_i)}{\|p^\phi(\cdot|a, s_i)\|} - \frac{p^\phi(y|a, s'_i)}{\|p^\phi(\cdot|a, s'_i)\|} \right) p^\phi(y|a, s_i)$$

$$- \overline{w}_i \sum_{y \in Y} \frac{p^\phi(y|a, s'_i)}{\|p^\phi(\cdot|a, s'_i)\|} \left(p^\phi(y|a, s_i) - p^\phi(y|a, s_i, s'_i) \right)$$

$$= \ \overline{w}_i \left\| p^\phi(\cdot|a, s_i) \right\| \left(1 - \frac{p^\phi(\cdot|a, s_i) \cdot p^\phi(\cdot|a, s'_i)}{\|p^\phi(\cdot|a, s_i)\| \, \|p^\phi(\cdot|a, s'_i)\|} \right)$$

$$- \overline{w}_i \sum_{y \in Y} \frac{p^\phi(y|a, s'_i)}{\|p^\phi(\cdot|a, s'_i)\|} \left(p^\phi(y|a, s_i) - p^\phi(y|a, s_i, s'_i) \right)$$

$$\geq \ \overline{w}_i \frac{\Lambda^\phi_i(s_i, s'_i, a)}{2\sqrt{|Y|}} - \overline{w}_i \left(1 + \sqrt{2}\right) v^\phi_i(s_i, s'_i, a)$$

where the final inequality follows from Lemma 5 and Lemma 6 and the Cauchy-Schwartz inequality.

Step 2: We claim that

$$\sum_{s_{-i} \in S_{-i}} \sum_{y \in Y} w_i(y) \left\{ \phi\left(y|s'_i, s_{-i}\right) - \phi(y|s) \right\} p(s_{-i}|a, s_i) \leq \overline{w}_i \sqrt{|Y|} \left(1 + \sqrt{2}\right) v^\phi_i(s_i, s'_i, a).$$

To see this, note that

$$\sum_{s_{-i} \in S_{-i}} \sum_{y \in Y} w_i(y) \left\{ \phi\left(y|s_i', s_{-i}\right) - \phi\left(y|s\right) \right\} p\left(s_{-i}|a, s_i\right)$$

$$= \sum_{y \in Y} w_i(y) \left(p^\phi\left(y|a, s_i, s_i'\right) - p^\phi\left(y|a, s_i\right) \right)$$

$$\leq \|w_i(\cdot)\| \left\| p^\phi\left(\cdot|a, s_i, s_i'\right) - p^\phi\left(\cdot|a, s_i\right) \right\|$$

$$\leq \overline{w}_i \sqrt{|Y|} \left(1 + \sqrt{2} \right) v_i^\phi\left(s_i, s_i', a\right)$$

where the final inequality follows from Lemma 5.

Step 3: We claim that, if $j \neq i$, then

$$\sum_{s_{-i} \in S_{-i}} \sum_{y \in Y} w_i(y) \left\{ \phi'_{a, \overline{\mu}_{j,w}, \underline{\mu}_{j,w}}\left(y|s_i', s_{-i}\right) - \phi'_{a, \overline{\mu}_{j,w}, \underline{\mu}_{j,w}}\left(y|s\right) \right\} p\left(s_{-i}|a, s_i\right) \leq \overline{w}_i \left(1 + \sqrt{2} \right) v_i^\phi\left(s_i, s_i', a\right).$$

To see this, note that

$$\sum_{s_{-i} \in S_{-i}} \sum_{y \in Y} w_i(y) \left\{ \phi'_{a, \overline{\mu}_{j,w}, \underline{\mu}_{j,w}}\left(y|s_i', s_{-i}\right) - \phi'_{a, \overline{\mu}_{j,w}, \underline{\mu}_{j,w}}\left(y|s\right) \right\} p\left(s_{-i}|a, s_i\right)$$

$$= \sum_{s_{-i} \in S_{-i}} \sum_{y \in Y} w_i(y) \left(\overline{\mu}_j(y) - \underline{\mu}_j(y) \right) \left(\psi_j\left(a, (s_i', s_{-i})\right) - \psi_j(a, s) \right) p\left(s_{-i}|a, s_i\right)$$

$$\leq \left| \sum_{y \in Y} w_i(y) \left(\overline{\mu}_j(y) - \underline{\mu}_j(y) \right) \right| \left| \sum_{s_{-i} \in S_{-i}} \left(\psi_j\left(a, (s_i', s_{-i})\right) - \psi_j(a, s) \right) p\left(s_{-i}|a, s_i\right) \right|$$

$$\leq \overline{w}_i \left| \sum_{s_{-i} \in S_{-i}} \sum_{y \in Y} \frac{p^\phi(y|s_j)}{\|p^\phi(\cdot|s_j)\|} \left(\phi\left(y|s_i', s_{-i}\right) - \phi(y|s) \right) p\left(s_{-i}|a, s_i\right) \right|$$

$$= \overline{w}_i \left| \sum_{y \in Y} \frac{p^\phi(y|s_j)}{\|p^\phi(\cdot|s_j)\|} \left(p^\phi\left(y|a, s_i, s_i'\right) - p^\phi\left(y|a, s_i\right) \right) \right|$$

$$\leq \overline{w}_i \left\| p^\phi\left(\cdot|a, s_i, s_i'\right) - p^\phi\left(\cdot|a, s_i\right) \right\|$$

$$\leq \overline{w}_i \left(1 + \sqrt{2} \right) v_i^\phi\left(s_i, s_i', a\right)$$

where the final inequality follows from Lemma 5.

Step 4: Combining Steps 1-3, it follows that

$$\sum_{s_{-i} \in S_{-i}} \sum_{y \in Y} w_i(y) \left\{ \phi_{a,w}^{\lambda}(y|s_i, s_{-i}) - \phi_{a,w}^{\lambda}(y|s_i', s_{-i}) \right\} p(s_{-i}|a, s_i)$$

$$\geq \frac{\lambda}{n} \overline{w}_i \left(\frac{\Lambda_i^{\phi}(s_i, s_i', a)}{2\sqrt{|Y|}} - \left(1 + \sqrt{2}\right) v_i^{\phi}(s_i, s_i', a) \right) - (1 - \lambda)\overline{w}_i \sqrt{|Y|} \left(1 + \sqrt{2}\right) v_i^{\phi}(s_i, s_i', a)$$

$$- \frac{\lambda(n-1)}{n} \overline{w}_i \left(1 + \sqrt{2}\right) v_i^{\phi}(s_i, s_i', a)$$

$$= \frac{\lambda}{n} \overline{w}_i \left(\frac{\Lambda_i^{\phi}(s_i, s_i', a)}{2\sqrt{|Y|}} \right) - \left(1 + \sqrt{2}\right) \overline{w}_i v_i^{\phi}(s_i, s_i', a) \left[(1 - \lambda)\sqrt{|Y|} + \lambda \right].$$

Finally, note that

$$\frac{\lambda}{n} \overline{w}_i \left(\frac{\Lambda_i^{\phi}(s_i, s_i', a)}{2\sqrt{|Y|}} \right) - \left(1 + \sqrt{2}\right) \overline{w}_i v_i^{\phi}(s_i, s_i', a) \left[(1 - \lambda)\sqrt{|Y|} + \lambda \right] \geq 0$$

if

$$v_i^{\phi}(s_i, s_i', a) \leq \frac{1}{\left((1 - \lambda)\sqrt{|Y|} + \lambda\right)\left(1 + \sqrt{2}\right)} \frac{\lambda}{2\sqrt{|Y|}n} \Lambda_i^{\phi}(s_i, s_i', a).$$

It follows immediately that

$$\sum_{s_{-i} \in S_{-i}} \sum_{y \in Y} w_i(y) \left\{ \phi_{a,w}^{\lambda}(y|s_i, s_{-i}) - \phi_{a,w}^{\lambda}(y|s_i', s_{-i}) \right\} p(s_{-i}|a, s_i) \geq 0$$

for any s_i, s_i' if p^{ϕ} is γ−regular for any γ satisfying

$$0 < \gamma < \frac{1}{\left((1 - \lambda)\sqrt{|Y|} + \lambda\right)\left(1 + \sqrt{2}\right)} \frac{\lambda}{2\sqrt{|Y|}n}. \tag{4}$$

∎

D. Proof of Theorem 2

Let $Q = \{q \in \mathbb{R}^n | \|q\| = 1\}$ and $e^i = (0, 0, .., 1, ..., 0)^{\top} \in Q$ with the ith coordinate equal to 1. First we prove two lemmata to prove Lemma 4.

Lemma 7 *Suppose that p^{ϕ} is distinguishable for some public coordinating device ϕ. Then there exists $\gamma > 0$ such that, if p^{ϕ} is γ−regular, then for any $q \in Q$ and $a \in A$, there exists $\xi : Y \to \mathbb{R}^n$ and another public coordinating device ϕ' that satisfy the following conditions:*
(i)

$$E^{\phi'}\left[\xi_j|a\right] = 0 \text{ for } j = 1, ..., n \tag{5}$$

(ii) if $0 \leqq |q_i| < 1$ for each $i \in N$, then

$$E^{\phi'}[\xi_j|a] > E^{\phi'}[\xi_j|a_j', a_{-j}, \rho_j] \text{ for all } (a_j', \rho_j) \text{ with } a_j' \neq a_j \text{ and for all } j \in N \quad (6)$$

$$E^{\phi'}[\xi_j|a] \geq E^{\phi}[\xi_j|a, \rho_j] \text{ for all } \rho_j \text{ and for all } j \in N \quad (7)$$

$$q \cdot \xi(y) = 0 \text{ for all } y \in Y \quad (8)$$

(iii) if $|q_i| = 1$ for some $i \in N$ (hence $q_j = 0$ for every $j \neq i$), then

$$E^{\phi'}[\xi_j|a] > E^{\phi'}[\xi_j|a_j', a_{-j}, \rho_j] \text{ for all } (a_j', \rho_j) \text{ with } a_j' \neq a_j \text{ and for all } j \in N \quad (9)$$

$$E^{\phi'}[\xi_j|a] \geq E^{\phi}[\xi_j|a, \rho_j] \text{ for all } \rho_j \text{ and for all } j \in N \quad (10)$$

Proof. Step 1: For each $a \in A$ and each pair (i, j) with $i \neq j$, there exist functions $x_a^{i,j,+}, x_a^{i,j,-} : Y \to \mathbb{R}$ satisfying the following conditions

$$
\begin{aligned}
E^{\phi}[zx_a^{i,j,z}|a] &> E^{\phi}[zx_a^{i,j,z}|a_i', a_{-i}] \text{ for all } a_i' \neq a_i \text{ for } z = +, - \\
E^{\phi}[x_a^{i,j,z}|a] &> E^{\phi}[x_a^{i,j,z}|a_j', a_{-j}] \text{ for all } a_j' \neq a_j \text{ for } z = +, -
\end{aligned}
$$

and

$$||x_a^{i,j,+}|| = 1 = ||x_a^{i,j,-}||.$$

Such functions $x_a^{i,j,+}$ and $x_a^{i,j,-}$ exist as a consequence of (1)-(3) and an application of the separating hyperplane theorem.

Step 2: We first consider the case of (ii). Take any $q \in Q$ such that $|q_j| < 1$ for any j. This q is fixed throughout steps 2-4. Let i be a player such that $|q_i| \geq |q_j|$ for all j. If $q_i < 0$, then define $x^{(a,q)} : Y \to \mathbb{R}^n$ as follows: for each $y \in Y$,

$$
\begin{aligned}
x_j^{(a,q)}(y) \quad &:\quad = x_a^{i,j,+}(y) \text{ if } q_j \geq 0 \text{ and } j \neq i \\
x_j^{(a,q)}(y) \quad &:\quad = x_a^{i,j,-}(y) \text{ if } q_j < 0 \text{ and } j \neq i \\
x_i^{(a,q)}(y) \quad &:\quad = -\sum_{j \neq i} \frac{q_j}{q_i} x_j^{(a,q)}(y).
\end{aligned}
$$

If $q_i > 0$, then define $x^{(a,q)} : Y \to \mathbb{R}^n$ as follows: for each $y \in Y$,

$$
\begin{aligned}
x_j^{(a,q)}(y) \quad &:\quad = x_a^{i,j,-}(y) \text{ if } q_j \geq 0 \text{ and } j \neq i \\
x_j^{(a,q)}(y) \quad &:\quad = x_a^{i,j,+}(y) \text{ if } q_j < 0 \text{ and } j \neq i \\
x_i^{(a,q)}(y) \quad &:\quad = -\sum_{j \neq i} \frac{q_j}{q_i} x_j^{(a,q)}(y).
\end{aligned}
$$

From these definitions, it follows that $q \cdot x^{(a,q)}(y) = 0$ for all $y \in Y$ so that condition (8) is satisfied.

Step 3: For each $s \in S$ and $a \in A$, let

$$\psi_i(a, s) := \sum_{y \in Y} \frac{p^\phi(y|a, s_i)}{\|p^\phi(\cdot|a, s_i)\|} \cdot \phi(y|s)$$

as in the proof of Theorem 1. Define $\phi'_{a, x^{(a,q)}} : S \to \Delta(Y)$ as

$$\phi'_{a, x^{(a,q)}} = \frac{\sum_{j=1}^n \phi'_{j, a, x^{(a,q)}}}{n}$$

where

$$\phi'_{j, a, x^{(a,q)}}(s) = \psi_j(a, s) \cdot \overline{\mu}_j + \left(1 - \psi_j(a, s)\right)\underline{\mu}_j$$

and $\overline{\mu}_j$ $(\underline{\mu}_j)$ is a probability measure on Y that assigns probability zero to any y not a member of $\arg\max_{y' \in Y} x_j^{(a,q)}(y')$ $(\arg\min_{y' \in Y} x_j^{(a,q)}(y'))$. Finally, let

$$\phi^\lambda_{a, x^{(a,q)}} := (1 - \lambda)\phi + \lambda\phi'_{a, x^{(a,q)}}$$

for some $\lambda \in (0, 1)$.

Let

$$
\begin{aligned}
\eta_1 &= \min\{E^\phi[zx_a^{i,j,z}|a] - E^\phi[zx_a^{i,j,z}|a'_i, a_{-i}] : \text{for all } i, j, a, a'_i \neq a_i \text{ and } z = +, -\}, \\
\eta_2 &= \min\{E^\phi[x_a^{i,j,z}|a] - E^\phi[x_a^{i,j,z}|a'_j, a_{-j}] \text{ for all } i, j, a, \; a'_j \neq a_j \text{ and } z = +, -\}
\end{aligned}
$$

and define

$$\eta := \min\{\eta_1, \eta_2\}.$$

Note that $\eta > 0$ and it is defined independent of a or q.

Step 4: In this step, we prove that condition (6) hold for $x^{(a,q)} : Y \to \mathbb{R}^n$ if p^ϕ is $\gamma-$regular and the following condition is satisfied for γ and λ.

$$\eta - 2\left(1 + \sqrt{2}\right)\gamma - 4\lambda > 0 \; (*)$$

We need to show the following for all j:

$$E^{\phi^\lambda_{a,x}}[x_j^{(a,q)}|a] > E^{\phi^\lambda_{a,x}}[x_j^{(a,q)}|a'_j, a_{-j}, \rho'_j] \text{ for all } \left(a'_j, \rho'_j\right) \text{ with } a'_j \neq a_j$$

To accomplish this, suppose that $q_i < 0$ and let $x = x^{(a,q)}$ for notational ease. The case with $q_i > 0$ is similar, thus omitted. First consider $j \neq i$ for which $q_j \geq 0$ so

that $x_j := x_a^{i,j,+}$. For any $a'_j \neq a_j$ and ρ_j,

$$
\begin{aligned}
& E^{\phi_{a,x}^\lambda}[x_j|a] - E^{\phi_{a,x}^\lambda}[x_j|a'_j, a_{-j}, \rho_j] \\
=\; & (1-\lambda)\left[E^\phi[x_a^{i,j,+}|a] - E^\phi[x_a^{i,j,+}|a'_j, a_{-j}, \rho_j]\right] \\
& + \lambda\left[E^{\phi'_{a,q}}[x_a^{i,j,+}|a] - E^{\phi'_{a,q}}[x_a^{i,j,+}|a'_j, a_{-j}, \rho_j]\right] \\
\geq\; & E^\phi[x_a^{i,j,+}|a] - E^\phi[x_a^{i,j,+}|a'_j, a_{-j}, \rho_j] - 4\lambda \text{ (because } \|x_a^{i,j,+}\| = 1) \\
=\; & E^\phi[x_a^{i,j,+}|a] - E^\phi[x_a^{i,j,+}|a'_j, a_{-j}] \\
& + E^\phi[x_a^{i,j,+}|a'_j, a_{-j}] - E^\phi[x_a^{i,j,+}|a'_j, a_{-j}, \rho_j] - 4\lambda \\
\geq\; & \eta - \|x_a^{i,j,+}\| \left\|p^\phi(\cdot|a'_j, a_{-j}) - p^\phi(\cdot|a'_j, a_{-j}, \rho_j)\right\| - 4\lambda \\
\geq\; & \eta - \left(1 + \sqrt{2}\right) v_i^\phi(s_i, a) - 4\lambda \\
\geq\; & \eta - 2\left(1 + \sqrt{2}\right)\gamma - 4\lambda \\
>\; & 0
\end{aligned}
$$

We can use the exactly same proof for player $j \neq i$ with $q_j < 0$ (so that $x_j := x_a^{i,j,-}$) to show that

$$
\begin{aligned}
E^{\phi_{a,x}^\lambda}[x_j|a] - E^{\phi_{a,x}^\lambda}[x_j|a'_j, a_{-j}, \rho_j] \;\geq\;& \eta - 2\left(1 + \sqrt{2}\right)\gamma - 4\lambda \\
>\;& 0
\end{aligned}
$$

for any $a'_j \neq a_j$ and ρ_j.

Finally for player i, the same proof implies that

$$
\begin{aligned}
E^{\phi_{a,x}^\lambda}[zx_a^{i,j,z}|a] - E^{\phi_{a,x}^\lambda}[zx_a^{i,j,z}|a'_i, a_{-i}, \rho_i] \;\geq\;& \eta - 2\left(1 + \sqrt{2}\right)\gamma - 4\lambda \\
>\;& 0
\end{aligned}
$$

for any $j, z = +, -$, and a'_i. Hence, whenever $q_j \neq 0$, we obtain

$$
E^{\phi_{a,x}^\lambda}[q_j x_j^{(a,q)}(y)|a] - E^{\phi_{a,x}^\lambda}[q_j x_j^{(a,q)}(y)|a'_i, a_{-i}, \rho_i] > 0.
$$

Observe that $q_j \neq 0$ for some $j \neq i$ and $q_i < 0$ by assumption. Therefore it follows that

$$
\begin{aligned}
& E^{\phi_{a,x}^\lambda}[x_i^{(a,q)}(y)|a] - E^{\phi_{a,x}^\lambda}[x_i^{(a,q)}(y)|a'_i, a_{-i}, \rho_i] \\
=\; & E^{\phi_{a,x}^\lambda}[-\sum_{j \neq i} \frac{q_j}{q_i} x_j^{(a,q)}(y)|a] - E^{\phi_{a,x}^\lambda}[-\sum_{j \neq i} \frac{q_j}{q_i} x_j^{(a,q)}(y)|a'_i, a_{-i}, \rho_i] \\
>\; & 0.
\end{aligned}
$$

Step 5: In this step, we prove that condition (7) holds for $x^{(a,q)} : Y \to \mathbb{R}^n$ if p^ϕ is $\gamma-$regular and the following condition is satisfied for γ and λ :

$$
0 < \gamma < \frac{1}{\left((1-\lambda)\sqrt{|Y|} + \lambda\right)\left(1 + \sqrt{2}\right)} \frac{\lambda}{2\sqrt{|Y|}n} \quad (**).
$$

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If $(**)$ is satisfied, it follows directly from Theorem 1 that truthful reporting is a Bayesian Nash equilibrium in the one-shot information revelation game $\left(G, p, \phi_{a,x}^\lambda, x, a\right)$ for any x and a. Hence we obtain

$$E^{\phi_{a,x}^\lambda}[x_j|a] \geq E^{\phi_{a,x}^\lambda}[x_j|a, \rho_j'] \text{ for all } \rho_j'$$

for any $a \in A$.

Step 6: Next consider the case of (iii). Take any $q \in Q$ such that $|q_i| = 1$ for any i. Then it immediately follows from Step 1 and Step 3-5 that we can construct $x^{(a,q)} : Y \to \mathbb{R}^n$ that satisfy (9) and (10) in this case. This is because condition (8), which requires payoff profiles to be on a certain hyperplane, is not imposed this time.

Step 7: Finally, choose λ and γ small enough so that $(*)$ and $(**)$ are satisfied in each case. Observe that we can choose λ and γ independent of a and q. For each $a \in A$ and $q \in Q$, define $\phi' := \phi_{a,x}^\lambda$ and $\xi := x^{(a,q)} - E^{\phi'}[x^{(a,q)}|a]$. Then ξ and ϕ' satisfy (5) in addition to (6)-(8) in the case of (ii) and (9)-(10) in the case of (iii). Therefore the lemma is proved. ∎

We need one more lemma to prove Lemma 4.

Lemma 8 *Let $M \subset \mathbb{R}^n$ be a closed and convex set with an interior point in $\mathbb{R}^n$. Suppose that each boundary point $v \in M$ is associated with the unique supporting hyperplane and the unique normal vector $\lambda_v (\neq 0) \in \mathbb{R}^n$ such that $\lambda_v \cdot v \geq \lambda_v \cdot x$ for all $x \in M$. Then for any point $y \in \mathbb{R}^n$ such that $\lambda_v \cdot v > \lambda_v \cdot y$, there exists $\alpha^* \in (0,1)$ such that $(1-\alpha)v + \alpha y$ is in the interior of M for any $\alpha \in (0, \alpha^*)$.*

Proof. Suppose that this is not the case, i.e. there does not exist such $\alpha^* > 0$. Let $W = \{x \in \mathbb{R}^n | \exists \alpha \in [0,1], x = (1-\alpha)v + \alpha y\}$.

We first show $W \cap int M = \varnothing$. First v is not an interior point of M by definition. If $(1-\alpha')v + \alpha'y$ is an interior point for any $\alpha' \in (0,1]$. Then $(1-\alpha)v + \alpha y$ is an interior point of M for every $\alpha \in (0, \alpha')$ as it is a strictly positive combination of $v \in M$ and $(1-\alpha')v + \alpha'y \in int M$. This is a contradiction. Hence $W \cap int M = \varnothing$.

Since $W \cap int M = \varnothing$, we can apply the separating hyperplane theorem for each $x_\alpha = (1-\alpha)v + \alpha y \in W$, obtaining $\lambda_\alpha (\neq 0) \in \mathbb{R}^n$ such that (a) $\lambda_\alpha \cdot x_\alpha \geq \lambda_\alpha \cdot x$ for all $x \in M$. Normalize them so that $\|\lambda_\alpha\| = 1$. Since $\lambda_\alpha \cdot x_\alpha \geq \lambda_\alpha \cdot v$, it also follows that (b) $\lambda_\alpha \cdot y \geq \lambda_\alpha \cdot v$ for every $\alpha > 0$ by the definition of $x_\alpha = (1-\alpha)v + \alpha y$.

Take a sequence of λ_{α_n} such that $\alpha_n > 0$ converges to 0 and λ_{α_n} converges to some $\lambda^* (\neq 0) \in \mathbb{R}^n$. Then $\lambda^* \cdot v \geq \lambda^* \cdot x$ for all $x \in M$ (from (a)) and $\lambda^* \cdot y \geq \lambda^* \cdot v$ (from (b)) by continuity.

Finally $\lambda^* \neq \lambda_v$ follows from $\lambda_v \cdot v > \lambda_v \cdot y$. Hence λ^* and λ_v are different normal vectors that separate v from M. This is a contradiction. ∎

We now prove Lemma 4, thus completing the proof of Theorem 2.

.

Proof of Lemma 4

Proof. Choose $\gamma > 0$ satisfying the conditions of Lemma 7 and let $W \subset intV^*(G)$ be a smooth set. We will show that, for any $v \in W$, there exists $\eta > 0$, $\delta \in (0,1)$ and an open set U containing v such that $U \cap W \subset B(\delta, W, \eta)$.

Step 1: Suppose that v is a boundary point of W. Let $q^* \in Q$ be the vector of utility weights such that $v = \arg\max_{v' \in W} q^* \cdot v'$ and $a^* = \arg\max_{a \in A} q^* \cdot g(a)$. We first show that v is strictly enforceable for some $w'_\delta : Y \to \mathbb{R}^n$ such that $q^* \cdot v > q^* \cdot w'_\delta(y)$ for any y.

Let $\xi : Y \to \mathbb{R}^n$ and ϕ' be the payoff function and public coordinating device as defined in the conditions of Lemma 7 given q^* and a^*. Note that, by (ii), we can find $c > 0$ and $\eta' > 0$ such that

$$g_j(a^*) + cE^{\phi'}[\xi_i | a^*] - \eta' > g_j(a_j, a^*_{-j}) + cE^{\phi'}[\xi_j | a'_j, a^*_{-j}, \rho_j] \tag{11}$$
$$\text{for all } (a'_j, \rho_j) \text{ with } a'_j \neq a^*_j \text{ and for all } j \in N.$$

Let $u(\delta) \in \mathbb{R}^n$ be the payoff vector satisfying $v = (1-\delta)g(a^*) + \delta u(\delta)$ for each $\delta \in (0,1)$. Define $w'_\delta : Y \to \mathbb{R}^n$ as follows.

$$w'_\delta(y) := u(\delta) + \frac{1-\delta}{\delta} c\xi(y).$$

Then (a^*, ϕ', w'_δ) clearly $(1-\delta)\eta'$-enforces the payoff profile v for every $\delta \in (0,1)$ (by (11) and (iii)).

Since W is in the interior of the feasible set,

$$q^* \cdot g(a^*) > q^* \cdot v = q^* \cdot [(1-\delta)g(a^*) + \delta u(\delta)].$$

Hence $q^* \cdot g(a^*) > q^* \cdot u(\delta)$ for any $\delta \in (0,1)$. Since $q^* \cdot \xi(y) = 0$ by construction, this implies the desired inequality:

$$q^* \cdot v > q^* \cdot u(\delta) = q^* \cdot w'_\delta(y) \text{ for all } y \in Y.$$

Step 2: We show that $w'_\delta : Y \to intW$ for large enough δ. Fix any δ. Since $v = (1-\delta)g(a^*) + \delta u(\delta)$, $w'_\delta(y)$ can be represented by

$$w'_\delta(y) = \frac{v - (1-\delta)g(a^*)}{\delta} + \frac{1-\delta}{\delta} c\xi(y).$$

Then for any $\delta' \in (\delta, 1)$, we can represent $w'_{\delta'}(y)$ as a positive convex combination of v and $w'_\delta(y)$ as follows.

$$w'_{\delta'}(y) = \frac{(1-\delta')\delta}{\delta'(1-\delta)} w'_\delta(y) + \frac{\delta'-\delta}{\delta'(1-\delta)} v \text{ for any } y \in Y.$$

Since Y is a finite set and $q^* \cdot v > q^* \cdot w'_\delta(y)$ by step 1, it directly follows from Lemma 8 that w'_δ takes a value in the interior of W for large enough discount factor.

Step 3: Next suppose that v is an interior point of W. In this case, it is clear that v is strictly enforceable with some $w'_\delta : Y \to \mathbb{R}^n$ and furthermore $w'_\delta(y)$ is in the interior of W for any $y \in Y$ if δ is close enough to 1.

Step 4: We have shown that, for any $v \in W$, there exists $\delta' \in (0, 1)$, $a^* \in A$, $\eta' > 0$, ϕ' and $w'_{\delta'} : Y \to intW$ such that $\left(a^*, \phi', w'_{\delta'}\right) \left(1 - \delta'\right) \eta'$-enforces v. We can now choose $\varepsilon > 0$ so that $v' \in U = \{z \in \mathbb{R}^N \| \|z - v\| < \varepsilon\}$ implies that $w'_{\delta'}(y) + \frac{v'-v}{\delta'} \in intW$ for each y. Then it follows that each $v' \in W \cap U$ is $(1 - \delta')\eta'$-enforced by $\left(a^*, \phi', w'_{\delta'} + \frac{v'-v}{\delta'}\right)$ with respect to W and δ'. Hence each $v' \in W \cap U$ is $(1 - \delta')\eta'$-decomposable with respect to W and δ'. Define η by $\eta := (1 - \delta')\eta'$. Then we have $W \cap U \subset B(\delta', W, \eta)$, i.e. the local strict self decomposability is established.
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