GAMES AND ECONOMIC BEHAVIOR 1, 131–143 (1989)

Excess Functions and Nucleolus Allocations of Pure Exchange Economies*

RICHARD P. MCLEAN

Department of Economics, Rutgers University, New Brunswick, New Jersey 08903

AND

ANDREW POSTLEWAITE

Department of Economics, University of Pennsylvania, Philadelphia, Pennsylvania 19104

Excess functions and nucleolus allocations are defined for pure exchange economies. These allocations are shown to be a nonempty, compact subset of the efficient allocations and are contained in the core when the core is nonempty. A special class of excess functions based on the coefficient of resource utilization is investigated.

1. INTRODUCTION

Concepts from cooperative game theory have been applied to a wide range of economic problems with great success. The core equivalence and value equivalence theorems are outstanding examples of the fruitful synthesis of game theory and equilibrium analysis. These (and other results from cooperative game theory) have provided insight into market outcomes by investigating the problems in the absence of economic institutions or mechanisms. The nucleolus of a game with side payments, introduced by Schmeidler (1969), heretofore has not been applied to economic problems (although it was extended to games without side payments by Kalai (1975)). In this paper we will define a class of nucleolus allocations for a pure exchange economy.

0899-8256/89 $3.00

The interest in our extension derives from its properties and its relationship to other economic concepts which have been previously analyzed. Before introducing our notion of nucleolus, we recall some facts regarding the nucleolus of a side payment game. The latter may be viewed as the unique payoff vector in a game with side payments that "lexicographically" minimizes the dissatisfaction of coalitions. It has several desirable properties: it always exists (in fact, for TU games, it is singleton) and it lies in the core when the core is nonempty. We will show that the nucleolus that we define for pure exchange economies will inherit these properties. In particular, the set of nucleolus allocations for an economy will always be nonempty even when the core of the economy is empty (we do not assume convexity of preferences). When the core is empty, any nucleolus allocation can obviously be improved upon by some coalition, but since the largest dissatisfaction of coalitions has been minimized (in a sense made precise below) the extent to which any coalition can improve upon the allocation has also been minimized. This property—minimization of dissatisfaction—makes the resulting allocation interesting from both a positive and a normative perspective. If the core of an economy is nonempty, we often restrict our attention to precisely those allocations in the core on the assumption that a coalition that can improve upon an allocation will do so. But from a positive point of view, some allocation must arise even if the core is empty. One plausible prediction in this case would be an allocation for which the incentive of coalitions to reject it has been minimized. Such an allocation which minimizes the dissatisfaction of coalitions may have appeal from a normative point of view as well.

A possible objection to the arguments in the previous paragraph is that the nucleolus treats the dissatisfaction of a coalition in the same way, whether the coalition is large or small. We consider "weighted" nucleolus allocations which result by, first, multiplying the dissatisfaction of each coalition by a positive number and, second, lexicographically minimizing the altered measures of coalitions' dissatisfaction. The resulting weighted nucleolus is of particular interest in the case in which weights of 1 are assigned to any coalition containing a single person and weights of 0 are assigned to all other coalitions. We show that this weighted nucleolus allocation is essentially an egalitarian equivalent allocation (Pazner and Schmeidler, 1978). We also discuss the relationship between core allocations and weighted nucleolus allocations.

The last economic concept to which our nucleolus allocations will be related is Walrasian equilibria. In a large economy, it is known that any core allocation is approximately Walrasian, so if the core of a large economy is nonempty, a nucleolus allocation is approximately Walrasian. We show that in a large economy a nucleolus allocation is approximately Walrasian even in the case where the core is empty.

2. Some Notation and Definitions

We shall be concerned with finite, pure exchange economies with agents indexed by the elements of $N = \{1, \ldots, n\}$. The commodity space is the nonnegative orthant of $\mathbb{R}^l$, denoted Ω. Each consumer is characterized by a preference relation and an initial endowment, so we must specify the set of agents' characteristics. Since we will also be interested in several continuity questions, we will topologize this set in a standard way as well. Let R be a nonempty subset of $\Omega \times \Omega$ satisfying:

(1) R is closed

(2) $(x, y) \in R \Rightarrow (x, x) \in R$ and $(y, y) \in R$

(3) $[(x, y) \in \Omega \times \Omega] \Rightarrow (x, y) \in R$ or $(y, x) \in R$

(4) $[(x, y) \in R$ and $(y, z) \in R] \Rightarrow (x, z) \in R$

(5) $[(x, y) \in R, x \geqq y$ and $x \neq y] \Rightarrow (y, x) \notin R$.

These are conditions of continuity, reflexivity, completeness, transitivity, and (strict) monotonicity. Let $\mathcal{R}$ denote the set of all such sets R in $\Omega \times \Omega$. The set $\mathcal{R}$ may be endowed with a metrizable topological structure, namely the topology of closed convergence. For a detailed explication of this topology, its properties, and its applications see Hildenbrand (1974) and the numerous references cited there. A succinct treatment which we have followed may be found in Hildenbrand (1972). In this paper the symbols "$(x, y) \in R$" and "$x\, Ry$" will be used interchangeably.

Let $\{A_k\}_{k=1}^{\infty}$ be a sequence of subsets of $\mathbb{R}^l$. As in Hildenbrand (1974), define $\text{Ls}\, A_k = \{x \in \mathbb{R}^l | \forall \varepsilon > 0, B_\varepsilon(x) \cap A_k \neq \emptyset$ for infinitely many $k\}$ and $\text{Li}\, A_k = \{x \in \mathbb{R}^l | \forall \varepsilon > 0, \exists k_\varepsilon \colon B_\varepsilon(x) \cap A_k \neq \emptyset, \forall k > k_\varepsilon\}$. ($B_\varepsilon(x)$ denotes the Euclidean open ball of radius ε centered at x.) An important property of the topology of closed convergence on $\mathcal{R}$ is the following result: A sequence $\{R^k\}$ in $\mathcal{R}$ converges to R if and only if $\text{Li}\, R^k = R = \text{Ls}\, R^k$. An economy E is an n-tuple $\{(R_i, a_i)\}_{i \in N}$ where $a_i \in \Omega$ and $R_i \in \mathcal{R}$ for each i. Note that the assumptions on preferences guarantee that if $(R_1, \ldots, R_n) \in \mathcal{R}^n$, then there exists a vector of continuous functions $u = (u_1, \ldots, u_n)$ such that each $u_i \colon \Omega \rightarrow \mathbb{R}$ is a representation of R_i.

An economy E is an element of $(\mathcal{R} \times \Omega)^n = \mathscr{E}$ and if $E \in \mathscr{E}$, then R_i^E and a_i^E will denote the projections of E onto the ith components of $(\mathcal{R} \times \Omega)^n$. $E|_S$ is the subeconomy of E consisting of the members of S. The indifference relation associated with R_i^E will be denoted I_i^E, that is, $x\, I_i^E\, y$ if and only if $x\, R_i^E\, y$ and $y\, R_i^E\, x$. The "strict" preference relation associated with R_i^E will be denoted P_i^E, that is, $x\, P_i^E\, y$ if and only if $x\, R_i^E\, y$ and not $x\, I_i^E\, y$. For each nonempty $S \subseteqq N$ and each $E \in \mathscr{E}$, let $F_S(E) = \{x^S \in \Omega^S | \Sigma_{i \in S}\, x_i^S = \Sigma_{i \in S}\, a_i^E\}$ and let $\Phi_S(E) = \{x^S \in F_S(E)|$ there is no $y^S \in F_S(E)$ such that $y_i^S\, P_i^E\, x_i^S, \forall i \in S\}$. The sets $F_S(E)$ and $\Phi_S(E)$ are, respectively, the $|S|$-

tuples of bundles that are feasible for S and those which are weakly efficient for $E|_S$. Note that efficient and weakly efficient allocations are the same because of our monotonicity assumption. If $(x_1, \ldots, x_n)$ is an allocation, then $x(S)$ will denote $\Sigma_{i \in S} x_i$.

3. Results

3.1. *Excess Functions and Nucleolus Allocations*

In Kalai (1975), a general notion of excess function for a game without side payments is presented. We will utilize such a general framework as well and we adapt Kalai's definition to the pure exchange context.

Definition 1. A function f_S: $\Omega^n \times \mathscr{E} \to \mathbb{R}$ is an excess function for $S \subseteqq N$ if the following conditions are satisfied:

(1) If $x, y \in \Omega^n$, $E \in \mathscr{E}$, and $x_i \, I_i^E \, y_i$ for each $i \in S$, then $f_S(x, E) = f_S(y, E)$.

(2) If $x, y \in \Omega^n$, $E \in \mathscr{E}$, and $x_i \, P_i^E \, y_i$ for each $i \in S$, then $f_S(x, E) < f_S(y, E)$.

(3) If $E \in \mathscr{E}$, $x^S \in \Phi_S(E)$, and x is an allocation with $x_i^S = x_i$, $\forall i \in S$, then $f_S(x, E) = 0$.

(4) f_S: $\Omega^n \times \mathscr{E} \to \mathbb{R}$ is a continuous function.

An excess function is meant to provide a measure of a coalition's "dissatisfaction" with an allocation. Condition 1 of the definition states that if the members of S are indifferent between two allocations, then they are equally dissatisfied with these allocations. Condition 2 says that if each member of S prefers allocation x to allocation y, then S is less dissatisfied with x than with y. Condition 3 is a normalization condition which states that the dissatisfaction of S at an allocation x is zero if x^S is feasible for S and the members of S cannot all be made better off using only their own resources. Finally, condition 4 is a regularity condition.

Remark 1. The efficient allocations of an economy $E \in \mathscr{E}$ are exactly those for which the excess of the coalition N is zero. More precisely, let $E \in \mathscr{E}$ and let f_N be an excess function for N. Then (i) $x \in F_N(E)$ if and only if $f_N(x, E) \geqq 0$ and (ii) $x \in \Phi_N(E)$ if and only if $f_N(x, E) = 0$. If $x \in F_N(E)$, then S *can improve upon* x if there exists $y^S \in \Omega^S$ such that $y^S(S) = a^E(S)$ and $y_i^S \, P_i^E \, x_i$ for each $i \in S$. An allocation $x \in F_N(E)$ is a core allocation for E if no coalition can improve upon x. It follows immediately from the definition that if f_S is any excess function, then S can improve upon $x \in F_N(E)$ if and only if $f_S(x, E) > 0$. Thus, if $\{f_S\}_{S \subseteqq N}$ is a collection of excess functions, then x is a core allocation for E if and only if $f_S(x, E) \leqq 0$ for each S. More generally we define the notion of ε-core as follows.

DEFINITION 2. Let $\mathcal{F} = \{f_S\}_{S \subseteq N}$ be a collection of excess functions. An allocation $x \in F_N(E)$ belongs to the strong ε-core of E with respect to $\mathcal{F}$ (denoted $C^{\mathcal{F}}(\varepsilon, E)$) if $f_S(x, E) \leqq \varepsilon$, for all $S \subsetneqq N$.

Obviously, $x \in F_N(E)$ is a core allocation if and only if x belongs to the 0-core of E with respect to $\mathcal{F}$. In fact, $C^{\mathcal{F}}(0, \mathcal{E}) = C^{\mathcal{F}'}(0, \mathcal{E})$ for any collection $\mathcal{F}$ and $\mathcal{F}'$ though $C^{\mathcal{F}}(\varepsilon, \mathcal{E})$ and $C^{\mathcal{F}'}(\varepsilon, \mathcal{E})$ are generally different if $\varepsilon > 0$.

PROPOSITION 1. *If $\mathcal{F} = \{f_S\}_{S \subseteq N}$ is a collection of excess functions, then the correspondence $C^{\mathcal{F}}$: $\mathbb{R}^l \times \mathcal{E} \to \Omega^n$ has a closed graph.*

3.2. *Nucleolus Allocations*

We now define nucleolus allocations. If $\mathcal{F}$ is a collection of excess functions, $E \in \mathcal{E}$, and $x \in F_N(E)$, let $\theta(x, E)$ be the vector in $\mathbb{R}^{2^n-1}$ consisting of the numbers $\{f_S(x, E)\}_{S \subseteq N}$ arranged in decreasing order. If x and y are members of $F_N(E)$, write $x <_{\text{lex}} y$ if there is an integer m, $1 \leqq m \leqq 2^n - 1$, such that $\theta_k(x, E) = \theta_k(y, E)$ for all $k < m$ and $\theta_m(x, E) < \theta_m(y, E)$.

DEFINITION 3. The nucleolus of E w.r.t. $\mathcal{F}$, denoted $\nu^{\mathcal{F}}(E)$, is the set $\{x \in F_N(E) |$ there is no $y \in F_N(E)$ such that $y <_{\text{lex}} x\}$.

Nucleolus allocations "lexicographically minimize" coalitional dissatisfaction. Note that we have used $F_N(E)$ (the feasible allocations for E) rather than $\Phi_N(E)$ (the efficient allocations for E) in the definition of nucleolus allocations. It is clearly desirable that nucleolus allocations be efficient and Proposition 2(A) below indicates that this is indeed the case.

PROPOSITION 2. *Let $\mathcal{F} = \{f_S\}_{S \subseteq N}$ be a collection of excess functions.*

(A) *For each $E \in \mathcal{E}$, $\nu^{\mathcal{F}}(E)$ is a nonempty, compact subset of $\Phi_N(E)$.*

(B) *If $(u_1, \ldots, u_n)$ is any n-tuple of utility representations for the preferences in E, then $\{(u_1(x_1), \ldots, u_n(x_n)) | (x_1, \ldots, x_n) \in \nu^{\mathcal{F}}(E)\}$ is a finite subset of $\mathbb{R}^n$.*

(C) *$\nu^{\mathcal{F}}(E) \subseteq C^{\mathcal{F}}(\varepsilon, E)$ whenever $C^{\mathcal{F}}(\varepsilon, E) \neq \Phi$.*

The nucleolus defined in terms of excess functions that satisfy the properties of Definition 1 provides a solution concept for markets that is independent of any particular utility representation of preferences. However, we are measuring excess in terms of commodities and not utility, and several individuals have remarked that this approach to the nucleolus does not result in a solution that is independent of the way in which commodities are represented. If the units of commodities are changed, then the nucleolus allocations in the transformed commodity space are not necessarily transformed nucleolus allocations of the original problem. The nucleolus defined in the next section exhibits this phenomenon.

3.3. *A Special Class of Excess Functions*

We now investigate a class of excess functions related to the coefficient of resource utilization of Debreu (see Debreu, 1951). Let $\xi \in \Omega\backslash\{0\}$ and suppose $x \in F_N(E)$. Define $e_S(x, E, \xi)$ as max $[t|z_i \in \Omega$ and $z_i\, R_i^E\, x_i$, $\forall i \in S$, and $z(S) = a^E(S) - t\xi]$. The number $e_S(x, E, \xi)$ is a measure of the amount of the resources of S that can be thrown away and still leave the members of S as satisfied as they are at the allocation x. If $a^E(S) \neq 0$, then $e_S(x, E, a^E(S))$ is the largest fraction of the total resources of S that can be discarded without lowering anyone's utility. In the case $S = N$, $e_N(x, E, a^E(N))$ is the coefficient of resource utilization of Debreu (1951) and may be interpreted as a measure of the inefficiency of the feasible allocation x. In general let $g: \mathscr{E} \to \Omega\backslash\{0\}$ be a function and define $e_S^g(x, E) = e_S(x, E, g(E))$.

PROPOSITION 3. *If* $g: \mathscr{E} \to \Omega\backslash\{0\}$ *is continuous, then* $e_S^g: \Omega^n \times \mathscr{E} \to \mathbb{R}$ *is an excess function.*

Let $\nu^g(E)$ denote the nucleolus of E w.r.t. the excess functions $\{e_S^g\}_{S \subseteq N}$.

PROPOSITION 4. *If* $a_i^E = 0$ *and* $x \in \nu^g(E)$, *then* $x_i = 0$.

Proposition 4 states that an agent who has no endowment gets nothing in any nucleolus allocation. This is not a property of value allocations as Shafer (1980) has shown.

Remark 2. Two agents i and j are called *identical* (in E) if $R_i^E = R_j^E$ and $a_i^E = a_j^E$. Nucleolus allocations possess a weak symmetry property in the following sense: If i and j are identical in E and $x \in \nu^g(E)$, then $y \in \nu^g(E)$ where $y_i = x_j$, $y_j = x_i$ and $y_k = x_k$ otherwise. This says that the nucleolus is a "symmetric set" with respect to identical players (a property shared by certain kinds of stable sets) though identical players need not receive identical bundles in any nucleolus allocation. This is to be expected since we have made no convexity assumptions. In the absence of convexity, identical agents can receive different bundles in core allocations and value allocations as well.

Remark 3. Other related definitions of excess functions are possible. As an example, we will translate a notion of ε-core due to Kannai (1970) into our framework. First, define a class of excess functions $\{\tilde{e}_S\}_{S \subseteq N}$ as

$$\tilde{e}_S(x, E, \xi) = \max[t|\forall i \in S,\ z_i\, R_i^E\, x_i,\ z_i \in \Omega \text{ and } z(S) = a^E(S)\ \theta t\xi],$$

where $a \,\theta\, b$ is a vector whose kth component is $\max\{0, a_k - b_k\}$. An allocation x belongs to the strong ε-core of E as defined in Kannai (1970) if and only if $\tilde{e}_S(x, E, \chi) \leqq \varepsilon$ for all $S \subseteqq N$ where $\chi = (1, \ldots, 1)$ in $\mathbb{R}^m$. It follows from the definition that $e_S(x, E, \xi) \leqq \tilde{e}_S(x, E, \xi)$ so any allocation in the strong ε-core with respect to the collection $\{\tilde{e}_S\}_{S \subseteqq N}$ will belong to the strong ε-core with respect to $\{e_S\}_{S \subseteqq N}$. In addition, any allocation in the nucleolus of E w.r.t. $\{e_S\}_{S \subseteqq N}$ must be an element of any nonempty ε-core using Kannai's definition.

3.4. *Nucleolus and Competitive Allocations*

It is well known that, in a large economy, core allocations are "approximately Walrasian" (see, for example, Anderson, 1978) and a similar statement can be made for the strong ε-core as well. Let $E \in \mathscr{E}$ be an exchange economy, define $M = \sup\{\|a_{i_1} + \cdots + a_{i_l}\| \,|\, i_1, \ldots, i_l \in N\}$, and let $\nu^{\xi}(E)$ denote the nucleolus of E w.r.t. $\{e_S(x, E, \xi)\}_{S \subseteqq N}$. An easy modification of the proof of Theorem 1 in Anderson (1978) yields the following result.

PROPOSITION 5. *If $x \in \nu^{\xi}(E)$ and $\varepsilon = \max_S e_S(x, E, \xi)$, then there exists $p \in \{q \in \mathbb{R}^l_+ \,|\, \|q\|_1 = 1\}$ such that*

(1) $\Sigma_{i \in N} |p \cdot (x_i - a_i^E)| \leqq 2(M + \varepsilon p \cdot \xi)$,

(2) $\Sigma_{i \in N} |\inf\{p \cdot (z_i - a_i^E) | z_i \, R_i^E \, x_i\}| \leqq 2(M + \varepsilon p \cdot \xi)$.

Thus any condition that guarantees that ε/n is small when n is large will also guarantee that allocations in $\nu^{\xi}(E)$ are approximately Walrasian in the sense of Proposition 5. For example, this can be established for very general sequences of economies by combining Remark 3 with Theorem 2 in Hildenbrand *et al.* (1973). Stronger results could be obtained if ε itself were small when n is large. We conjecture that this is true under additional smoothness hypotheses.

3.5. *Weighted Nucleolus Allocations and Egalitarian Equivalence*

A social planner may be interested in allocations that are lexicographically minimal with respect to some subset of coalitions. Let g be a function as in Proposition 3 and let $\mu^g(E)$ denote the nucleolus of $F_N(E)$ w.r.t. the collection $\{e^g_{\{1\}}, \ldots, e^g_{\{n\}}\}$.

PROPOSITION 6. *$\mu^g(E)$ is a nonempty, compact subset of $\Phi_N(E)$. Furthermore if $x \in \mu^g(E)$, then $e^g_{\{i\}}(x, E) = e^g_{\{j\}}(x, E)$ for all $i, j \in N$.*

As a result of Proposition 6, we can see that if $x \in \mu^g(E)$, then x solves the problem $\max_{x \in \Phi_N(E)} \min_{i \in N} \{e^g_{\{i\}}(x, E)\}$. Allocations possessing maximin properties are related to a fairness criterion called egalitarian equivalence due to Pazner and Schmeidler (1978). In particular, $x \in \Phi_N(E)$ is egalitarian equivalent for E if there exists a bundle $b \in \Omega$ such that $x_i \, I_i^E \, b$, for each $i \in N$. We consider a slight extension.

DEFINITION 4. A vector of net trades $z = (z_1, \ldots, z_n)$ is egalitarian equivalent for E if $z + a^E \in \Phi_N(E)$ and, for some $b \in \Omega$, $(z_i + a_i^E) \, I_i^E (b + a_i^E)$, $\forall i \in N$.

If $a_i^E = a_j^E = a$, for all $i, j \in N$, and if z is an egalitarian equivalent net trade for E, then $z_i + a$ is an egalitarian equivalent allocation for E. As a consequence of Proposition 6, we have the result that if $x \in \mu^g(E)$, then $x - a^E$ is an egalitarian equivalent net trade for E.

The egalitarian equivalent net trades described above can be considered weighted nucleoli. It is clear that for any coalition S we could multiply the excess function of S by a positive weight α_S and obtain a new excess function. For any set of positive weights $(\alpha_S)_{S \subseteq N}$, the nucleolus with respect to the new excess functions created in this way will have the properties stated in Propositions 3 and 4. Further, it is clear that for any such weights, the nucleolus will be contained in the core of the economy if the core is nonempty. A partial converse of this statement is also true. Suppose x is a core allocation for which the excess of every coalition (except, of course, the set of all agents N) is negative. Define α_S to be the reciprocal of the negative of the excess for coalition S. Then with the new weighted excess functions, the excess for every coalition except N will be -1. It will not be possible to reduce the excess of any coalition without increasing the excess for some other coalition since any movement from a Pareto efficient allocation to another allocation which does not leave all agents indifferent must make some agent worse off, thus increasing his excess. Thus this prescribed core allocation will be the nucleolus with respect to the new excess functions. Hence, the set of core allocations with the above-described "interiority" property is contained in the set of weighted nucleolus allocations with positive weights.

Weighted nucleolus allocations with nonnegative weights will not correspond to the entire core, however. Without restricting the set of coalitions which receive 0 weight, the resulting allocation may not be in the core at all. There is one interesting example of weighted nucleolus allocations which allow 0 weights. Consider a set of weights which are 0 for all coalitions except the singletons and which puts equal weight on the singleton sets. The nucleolus that results from these weights is precisely the set of egalitarian equivalent allocations of the previous section.

4. PROOFS OF PROPOSITIONS

Proof of Proposition 1. Follows from the continuity of the excess functions.

Proof of Proposition 2. The proof that $\nu^{\mathcal{F}}$ is nonempty and compact proceeds exactly as that of Theorem 1 in Schmeidler (1969) (see also Aumann, 1975/1976, Theorem 7.17). Now suppose $x \in F_N(E)$ but $x \notin \Phi_N(E)$. Then $f_N(x, E) > 0$ and by the monotonicity assumption on preferences, there exists $y \in F_N(E)$ such that $y_i \, P_i^E \, x_i$, $\forall i \in N$. By property (2) in the definition of excess functions, $f_S(y, E) < f_S(x, E)$ for all $S \subseteqq N$. Thus x cannot be in $\nu^{\mathcal{F}}$ and we conclude that $\nu^{\mathcal{F}} \subseteqq \Phi_N(E)$. To show (B), let $(u_1, \ldots, u_n)$ be a vector of representations for $R_1, \ldots, R_n$. If x is any element of $\nu^{\mathcal{F}}(E)$, let $B(x) = \{B_1(x), \ldots, B_k(x)\}$ be a partition of $2^n \backslash \emptyset$ such that $B_j(x)$ contains those coalitions with the jth largest excess at x. Following Kalai (1975, Theorem 1(viii)), we show that $(u_1, \ldots, u_n)(\nu^{\mathcal{F}}(E))$ is a finite set. If ξ and η belong to $\nu^{\mathcal{F}}(E)$, then $B(\xi)$ and $B(\eta)$ have the following properties: (a) $|B(\xi)| = |B(\eta)|$; (b) $|B_j(\xi)| = |B_j(\eta)|, \forall j$; and (c) if $S \in B_j(\xi)$ and $T \in B_j(\eta)$, then $f_S(\xi, E) = f_T(\eta, E)$. Now $f_{\{i\}}(\cdot, E)$ determines the unique indifference classes to which ξ_i and η_i belong, i.e., $\xi_i \sim \eta_i$ if and only if $f_{\{i\}}(\xi, E) = f_{\{i\}}(\eta, E)$. If ξ_i and η_i belong to different indifference classes, then the conditions above imply that $B(\xi) \neq B(\eta)$. Since there are finitely many coalition orderings, it must be true that $(u_1, \ldots, u_n)(\nu^{\mathcal{F}}(E))$ is a finite set. Part (C) follows from the definitions, and the proof of Proposition 2 is complete.

Before proving Proposition 3, we prove a lemma that may be of some independent interest, as it provides conditions that are strong enough to show that the optimal value function of an optimization problem is continuous but not strong enough to show upper hemicontinuity of the optimizer correspondence.

LEMMA 1. *Let X, Y be metric spaces and suppose Φ: $X \to Y$ is a nonempty valued correspondence. Let f: $X \times Y \to \mathbb{R}$ be a continuous function such that $g(x) = \max_{y \in \Phi(x)} f(x, y)$ is well defined, $\forall x \in X$. Suppose* (i) $\lim_n x_n = x_0 \Rightarrow Ls\Phi(x_n) = \Phi(x_0)$ *(i.e., Φ is closed and lower hemicontinuous) and* (ii) *whenever $c \in \mathbb{R}$ and $\{x_n\} \subseteqq X$ is a convergent sequence with $g(x_n) \geqq c$, $\forall n$, there is a subsequence $\{x_{n_k}\}$ and a convergent sequence $\{y_k\} \subseteqq Y$ such that $y_k \in \Phi(x_{n_k})$ and $f(x_{n_k}, y_k) \geqq c$. Then g: $X \to \mathbb{R}$ is continuous.*

Proof. It suffices to show that g is upper semicontinuous and lower semicontinuous. Let $c \in \mathbb{R}$.

Part 1. Let $\{x_n\} \subseteqq \bar{A}_c = \{x \in X | g(x) \geqq c\}$ be convergent with limit x_0. We must show that $x_0 \in \bar{A}_c$. By definition of g, $\bar{A}_c = \{x \in X | \exists y \in \Phi(x)$: $f(x, y) \geqq c\}$. Since $g(x_n) \geqq c$, $\forall n$, it follows from assumption (ii) that there is a convergent subsequence $\{x_{n_k}\}$ and a convergent sequence $\{y_k\} \subseteqq Y$ (with limit y_0) such that $y_k \in \Phi(x_{n_k})$ and $f(x_{n_k}, y_k) \geqq c$, $\forall k$. Since f is continuous, $f(x_0, y_0) \geqq c$. By assumption (i), Ls $\Phi(x_n) \subseteqq \Phi(x_0)$. Thus $y_0 \in \Phi(x_0)$ and we conclude that $g(x_0) \geqq c$.

Part 2. The lower semicontinuity of g follows immediately from a result of Berge (see Aubin, 1979, Theorem 2, p. 69), and we include a proof for the sake of completeness. Let $\{x_n\} \subseteq \underline{A}_c = \{x \in X | g(x) \leqq c\}$ be convergent with limit x_0. Again by the definition of g, $\underline{A}_c = \{x \in X | y \in \Phi(x) \Rightarrow f(x, y) \leqq c\}$. We must show that $x_0 \in \underline{A}_c$, so choose $y_0 \in \Phi(x_0)$ (recall that Φ is nonempty valued). From assumption (i), $\Phi(x_0) \subseteq \text{Ls}\, \Phi(x_n)$, so there exists a subsequence $\{x_{n_k}\}$ and a sequence $\{y_k\}$ such that $y_k \in \Phi(x_{n_k})$, $\forall k$, and $\lim y_k = y_0$. Since $\{x_{n_k}\} \subseteq \underline{A}_c$ and $y_k \in \Phi(x_{n_k})$, it follows that $f(x_{n_k}, y_k) \leqq c$, $\forall k$. By continuity of f, $f(x_0, y_0) \leqq c$. Since y_0 is an arbitrarily chosen element of $\Phi(x_0)$, we conclude that $x_0 \in \underline{A}_c$. This completes the proof.

Remark 4. Berge's theorem (see Hildenbrand, 1974, for example) is a consequence of this result since if Φ is a compact valued, continuous correspondence, then g is well defined and conditions (i) and (ii) hold. However, our less restrictive assumptions do not allow us to conclude that the maximizer correspondence is compact valued or upper hemicontinuous. For example, let $X = \mathbb{R}^1_+$, $Y = -\mathbb{R}^1_+$, $f(x, y) = xy^2 + y$, and

$$\Phi(x) = \begin{cases} \{0, -1/x\}, & x > 0 \\ \{0\}, & x = 0. \end{cases}$$

Then f and Φ satisfy the conditions of the theorem, but Φ is not upper hemicontinuous. Furthermore, $g(x) \equiv 0$ and the maximizer correspondence is exactly Φ. Since we are interested only in the continuity of g, one might ask if assumption (ii) could be dropped. This is not the case. Let $X = Y = \mathbb{R}^1_+$, $f(x, y) = y$, and

$$\Phi(x) = \begin{cases} \{0, 1/x\}, & x > 0 \\ \{0\}, & x = 0. \end{cases}$$

The optimal value function is well defined and

$$g(x) = \begin{cases} 1/x, & x > 0 \\ 0, & x = 0. \end{cases}$$

However, condition (ii) is violated and g is obviously not continuous.

To prove Proposition 3, we need another lemma that describes the continuity properties of the feasible set in the optimization problem that must be solved to compute a coalition's excess. For each $S \subseteq N$, define

$$\Phi_S(x, E, \xi) = \{(t, z) \in \mathbb{R}^1 \times \mathbb{R}^{nl} | (z_i, x_i) \in R_i^E, \forall i \in S, z(S) = a^E(S) - t\xi\}.$$

Hence, $e_S(x, E, \xi) = \max[t | (t, z) \in \Phi_S(x, E, \xi)]$.

LEMMA 2. *For all* $S \subseteqq N$, $\Phi_S: \mathbb{R}^{nl}_+ \times \mathscr{E} \times \Omega\backslash 0 \to \mathbb{R}^1 \times \mathbb{R}^{nl}$ *is closed and lower hemicontinuous.*

Proof. Let $\{(x^n, E^n, \xi^n)\}_{n=1}^{\infty} \subseteqq \mathbb{R}^{nl}_+ \times \mathscr{E} \times \Omega\backslash 0$ be convergent with limit (x^0, E^0, ξ^0). We will write a_i^n rather than $a_i^{E_n}$, R_i^n rather than $R_i^{E_n}$, etc.

Step 1. To show that Φ_S is closed, we must prove that Ls $\Phi_S(x^n, E^n, \xi^n) \subseteqq \Phi_S(x^0, E^0, \xi^0)$. Let $(t^0, z^0) \in$ Ls $\Phi_S(x^n, E^n, \xi^n)$. There is a subsequence $\{(x^{n_k}, E^{n_k}, \xi^{n_k})\}_{k=1}^{\infty}$ and a sequence (t^k, z^k) such that $(t^k, z^k) \in \Phi_S(x^{n_k}, E^{n_k}, \xi^{n_k})$, for all k, and $\lim_k (t^k, z^k) = (t^0, z^0)$. Since $z^{n_k}(S) = a^{n_k}(S) - t^k\xi^{n_k}$, for all k, it follows that $z^0(S) = a^0(S) - t^0\xi^0$. It remains to show that $(x_i^0, z_i^0) \in R_i^0$, $\forall i \in S$. Now $(z_i^k, x_i^{n_k}) \in R_i^{n_k}$, $\forall k$, and $\lim_k (z_i^k, x_i^{n_k}) = (z_i^0, x_i^0)$, so $(z_i^0, x_i^0) \in$ Ls R_i^n. Since $\lim_n E^n = E^0$, it follows that Ls $R_i^n = R_i^0$, $\forall i \in S$, by the definition of the topology of closed convergence, and $(z_i^0, x_i^0) \in R_i^0$.

Step 2. To show that Φ_S is lower hemicontinuous, we must prove that $\Phi_S(x^0, E^0, \xi^0) \subseteqq$ Ls $\Phi_S(x^n, E^n, \xi^n)$. Let $(t^0, z^0) \in \Phi_S(x^0, E^0, \xi^0)$. We must show that there is a subsequence $\{(x^{n_k}, E^{n_k}, \xi^{n_k})\}_{k=1}^{\infty}$ and a sequence $((t^k, z^k)\}_{k=1}^{\infty}$ such that $(t^k, z^k) \in F_S(x^{n_k}, E^{n_k}, \xi^{n_k})$, for all k, and $\lim_k (t^k, z^k) = (t^0, z^0)$. To do this, fix a positive integer k and for each $i \in S$, define $z_i^{n,k}$ as

$$z_i^{n,k} = z_i^0 + \frac{1}{|S|}[a^n(S) - a^0(S)] + \frac{1}{|S|}\left(\frac{1}{k} - t^0\right)[\xi^n - \xi^0] + \frac{1}{k|S|}\xi^0.$$

Note that $\lim_n z_i^{n,k} = z_i^0 + (1/k|S|)\,\xi^0$ and, since $\xi^0 \in \Omega\backslash 0$, $\lim_n z_i^{n,k} P_i^0 z_i^0$, $\forall i \in S$. Thus, for sufficiently large n, $z_i^{n,k} P_i^0 z_i^0$, $\forall i \in S$. Since $\lim_n x_i^n = x_i^0$, $z_i^0 P_i^0 x_i^0$, and $\lim_n z_i^{n,k} P_i^0 z_i^0$, it follows that for sufficiently large n, $z_i^{n,k} P_i^0 x_i^n$. Summarizing the above, for all k there is an $\hat{n}_k$ such that $z_i^{n,k} P_i^0 x_i^n$, for all $n > \hat{n}_k$ and for each $i \in S$. We claim that for some $n > \hat{n}_k$, $(z_i^{n,k}, x_i^n) \in R_i^n$. If this were not true, then $(x_i^n, z_i^{n,k}) \in R_i^n$, for all $n > \hat{n}_k$ by completeness. Since $\lim_n (x_i^n, z_i^{n,k}) = (x_i^0, z_i^0 + (1/|S|k)\,\xi^0)$ and Li $R_i^n = R_i^0$ (by the definition of closed convergence), it would follow that $(x_i^0, z_i^0 + (1/|S|k)\,\xi^0) \in R_i^0$, which is an obvious contradiction since $(z_i^0, x_i^0) \in R_i^0$. So for each k, let $n_k > \hat{n}_k$ be an integer such that $(z_i^{n_k,k}, x_i^k) \in R_i^{n_k}$. Let $z_i^k = z_i^{n_k,k}$ and let $t^k = t^0 - 1/k$. It is easy to verify that $z^k(S) = a^{n_k}(S) - t_k\xi^{n_k}$ so that $(z_i^k, t^k) \in \Phi_S(x^{n_k}, E^{n_k}, \xi^{n_k})$. Since $\lim_k z_i^k = z_i^0$ and $\lim_k t^k = t^0$, the result is proved.

Proof of Proposition 3. It is easy to check that for each $(x, E, \xi) \in \mathbb{R}^{nl}_+ \times \mathscr{E} \times \Omega\backslash 0$, the number $e_S(x, E, \xi) := \max[t|(t, z) \in \Phi(x, E, \xi)]$ is well defined. To show that e_S is a continuous function, we need to verify that Φ satisfies the conditions of Lemma 1. Condition (i) is satisfied by Lemma 2. To show that condition (ii) holds, let $c \in \mathbb{R}^1$ and let $\{(x^n, E^n, \xi^n)\}_{n=1}^{\infty}$ be a convergent sequence with limit (x^0, E^0, ξ^0) such that $e_S(x^n, E^n, \xi^n) \geqq c$, $\forall n$. Choose a sequence $\{(t^n, z^n)\}_{n=1}^{\infty}$ such that $(t^n, z^n) \in \Phi(x^n, E^n, \xi^n)$ and $t^n \geqq c$. (Such a sequence exists since $e_S(x^n, E^n, \xi^n) \geqq c$.) We claim that $\{(t^n, z^n)\}$ contains a convergent subsequence. To see this, note that $\forall n$, $z^n(S) = a^n(S) - t^n\xi^n$. Since $\lim_n a^n(S) = a^0(S)$, $\lim_n \xi^n = \xi^0$, and $z^n(S) \geqq 0$, it follows that there exists a number M such that $t_n \leqq M$ for all n. Since $t_n \geqq c$, $\forall n$, the sequence $\{t_n\}$ has a convergent subsequence $\{t_{n_k}\}$. Thus $\{z_{n_k}\}$ is convergent as well and by applying Lemma 1, we conclude that $e_S(\cdot, \cdot, \cdot)$ is continuous. Since g is continuous, e_S^g is continuous and the proof of the proposition is complete.

Proof of Proposition 4. Suppose $a_i = 0$ and $x \in \nu^g(E)$. (For notational ease, the dependence of endowments and preferences on E is suppressed.) If $x_i\, P_i\, 0$, then $e_{S \cup i}(x, E) < e_S(x, E)$, $\forall S \subseteqq N \backslash i$. To see this, let z_k, $k \in S$, and z_i in $\mathbb{R}^l_+$ satisfy the following conditions: $z_k\, R_k\, x_k$, $\forall k \in S$, $z_i\, R_i\, x_i$, $\Sigma_{k \in S}\, z_k + z_i = a(S) + a_i - e_{S \cup i}(x, E)\, g(E)$. Note that $z_i \neq 0$ by monotonicity. By defining $\hat{z}_k = z_k + (1/|S|)\, z_i$ for each $k \in S$, we obtain $\hat{z}_k\, P_k\, x_k$, $\forall k \in S$, and $\Sigma_{k \in S}\, \hat{z}_k = \Sigma_{k \in S}\, z_k + z_i = a(S) - e_{S \cup i}(x, E)\, g(E)$. Hence, $e_{S \cup i}(x, E) < e_S(x, E)$. Now define an allocation ζ with $\zeta_i = 0$ and $\zeta_j = x_j + (1/(n-1))\, x_i$, $\forall_j \neq i$. Since $x_i \neq 0$, $\zeta_j\, P_j\, x_j$, $\forall_j \neq i$, so for $S \subseteqq N \backslash i$, $e_S(x, E) > e_S(\zeta, E)$. Since $a_i = 0$, $e_{S \cup i}(\zeta, E) = e_S(\zeta, E)$. Thus we have, for each $S \subseteqq N \backslash i$,

(1) $e_S(x, E) > e_S(\zeta, E)$

(2) $e_S(x, E) > e_{S \cup i}(x, E)$

(3) $e_S(\zeta, E) = e_{S \cup i}(\zeta, E)$.

This implies that $\theta(x) >_{\text{lex}} \theta(\zeta)$, contradicting the hypothesis that $x \in \nu^g(E)$. This completes the proof.

Proof of Proposition 6. The proof that $\mu^g(E)$ is a nonempty, compact subset of $\Phi_N(E)$ is the same as that of Proposition 2(A). Let

$$K = \{k \in N \mid e^g_{\{k\}}(x) = \max_{i \in N} \{e^g_{\{i\}}(x)\}\} \quad \text{and}$$

$$L = \{l \in N \mid e^g_{\{l\}}(x) = \min_{i \in N} \{e^g_{\{i\}}(x)\}\}.$$

If $L \neq K$ then define a new allocation y as

$$\begin{aligned} y_k &= x_k + \frac{1}{|K|}\, w, && \text{if } k \in K \\ y_l &= x_l - \frac{1}{|L|}\, w, && \text{if } l \in L \\ y_i &= x_i, && \text{otherwise,} \end{aligned}$$

where $w \in \mathbb{R}^l$. By continuity, there exists a $w \in \mathbb{R}^l$ such that $e^g_{\{l\}}(x) < e^g_{\{l\}}(y) < e^g_{\{i\}}(y) < e^g_{\{k\}}(y) < e^g_{\{k\}}(x)$ whenever $l \in L$, $k \in K$, and $i \in N \backslash (K \cup L)$. Thus x cannot be a nucleolus allocation if $K \neq L$. This finishes the proof of Proposition 6.

Acknowledgments

We thank E. Kalai, H. Moulin, R. Myerson, D. Samet, D. Schmeidler, and a referee for their helpful remarks. We also thank the National Science Foundation for financial support.

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