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Informational size and incentive compatibility with aggregate uncertainty ☆

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Abstract

In McLean and Postlewaite (Econometrica 56, 1992, p. 2421), we analyzed pure exchange economies with asymmetrically informed agents. We defined a notion of informational size and showed that, when the aggregate information of all agents resolves nearly all the uncertainty regarding the state of nature, the conflict between incentive compatibility and (ex post) efficiency can be made small if agents have sufficiently small informational size. This paper investigates the relationship between informational size and efficiency for the case in which nontrivial aggregate uncertainty is present, i.e., when significant uncertainty about the world persists even when the information of all agents is known.

JEL classification: D82

1. Introduction

It is well understood that, in the presence of asymmetric information, incentive compatibility and Pareto efficiency often conflict: agents may benefit from misrepresenting their private information when that information is to be used in making decisions that affect them. In McLean and Postlewaite (2002), we addressed certain continuity issues when

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agents have small amounts of information that is not common knowledge. Much research in economics ignores the issue of asymmetry of information, implicitly or explicitly assuming that the characteristics of the economic environment are common knowledge among agents even though the assumption that all agents are *identically* informed is implausible. This working assumption greatly simplifies the analysis and is based on a belief that behavior should be continuous with respect to the information structure. When the asymmetry of information is 'small enough,' predicted behavior when asymmetries are ignored should be close to predicted behavior when these asymmetries are properly modeled.

McLean and Postlewaite (2002) formally address this question by introducing a measure of an agent's informational size and show that, in some cases, the unavoidable inefficiencies caused by incentive problems can be made small when agents have sufficiently small informational size. In that paper, two aspects of the informational structure are important. The first aspect is *aggregate uncertainty*: to what extent does the agents' information *in toto* resolve nearly all the uncertainty in the economy? The second is *informational variability* which, very roughly, measures the degree to which an agent's information is correlated with the information of others.[1] McLean and Postlewaite (2002) restrict attention to the case of *negligible aggregate uncertainty* in which the agents' information resolves nearly all uncertainty regarding the state of the world and show that many individually rational Pareto efficient allocations can be approximated by incentive compatible allocations when each agent is sufficiently informationally small relative to his informational variability.

In this paper we investigate the relationship between informational size and efficiency in the presence of nontrivial aggregate uncertainty, i.e., when significant uncertainty regarding the world is present even when the information of all agents is known. In the next section, we present the formal model; following this, we discuss in general the question of aggregate uncertainty and the difficulty that aggregate uncertainty presents when trying to identify nearly efficient incentive compatible allocations, even when agents are informationally small.

We then prove the existence of incentive compatible, individually rational and nearly ex post efficient allocations without assuming negligible aggregate uncertainty when each agent has small informational size relative to informational variability. We also show that, for general exchange economies with asymmetric information, the conflict between incentive compatibility and efficiency asymptotically vanishes when an economy is replicated.

Both this paper and McLean and Postlewaite (2002) are related to Cremer and McLean (1985, 1988) and to subsequent work by McAfee and Reny (1992). The possibility of full extraction in mechanism design problems also depends on the correlation of agents' information in a way that is related to, but not precisely the same, as our concept of informational variability. This is discussed in detail when that concept is formally introduced below.

[1] This is defined formally below.

2. Private information economies

The model is similar to that in McLean and Postlewaite (2002). The interested reader will find a detailed discussion of the model and many of the assumptions there.

Throughout the paper, $||\cdot||$ will denote the 1-norm unless specified otherwise. Let $N = \{1, 2, \ldots, n\}$ denote the set of *economic agents*. Let $\Theta = \{\theta_1, \ldots, \theta_m\}$ denote the (finite) *state space* and let $T_1, T_2, \ldots, T_n$ be finite sets where T_i represents the set of possible *signals* that agent i might receive. Let $T \equiv T_1 \times \cdots \times T_n$ and $T_{-i} \equiv \times_{j \neq i} T_i$. If $t \in T$, we will often write $t = (t_{-i}, t_i)$. If X is a finite set, define

$$\Delta_X := \left\{\rho \in \Re^{|X|} \mid \rho(x) \geqslant 0, \ \sum_{x \in X} \rho(x) = 1\right\}.$$

In our model, nature chooses an element $\theta \in \Theta$. The state of nature is unobservable but each agent i receives a 'signal' t_i that may be correlated with nature's choice of θ. More formally, let $(\tilde{\theta}, \tilde{t}_1, \tilde{t}_2, \ldots, \tilde{t}_n)$ be an $(n+1)$-dimensional random vector taking values in $\Theta \times T$ with associated distribution $P \in \Delta_{\Theta \times T}$ where

$$P(\theta, t_1, \ldots, t_n) = \text{Prob}\left\{\tilde{\theta} = \theta, \tilde{t}_1 = t_1, \ldots, \tilde{t}_n = t_n\right\}.$$

Without loss of generality, we will make the following 'full support' assumptions regarding the marginal distributions: for each $\theta \in \Theta$,

$$P(\theta) = \text{Prob}\left\{\tilde{\theta} = \theta\right\} > 0$$

and for each $i \in N$ and for each $t_i \in T_i$,

$$P(t_i) = \text{Prob}\left\{\tilde{t}_i = t_i\right\} > 0.$$

Let $T^* = \{t \in T \mid P(t) > 0\}$. We note that T^* need *not* be equal to T. If $t \in T^*$, let $P_\Theta(\cdot \mid t) \in \Delta_\Theta$ denote the induced conditional probability measure on Θ. Let $\chi_\theta \in \Delta_\Theta$ denote the degenerate measure concentrated on state θ.

The *consumption set* of each agent is $\Re^\ell_+$ and $w_i \in \Re^\ell_+$ denotes the *initial endowment* of agent i (an agent's initial endowment is independent of the state θ). For each $\theta \in \Theta$, let $u_i(\cdot, \theta) : \Re^\ell_+ \to \Re$ be the utility function of agent i in state θ. We note that this formulation differs from the more standard formulation that specifies utility functions of the form $\tilde{u}_i(\cdot, t)$, where t is the profile of agents' types. Clearly, our formulation is without loss of generality, since one can always define the space $\Theta \equiv T$. The formulation $u_i(\cdot, \theta)$ that we have chosen is advantageous for our purposes in that it focuses attention on the manner in which the types of agents different from i will affect agent i's utility. The utility of i for any bundle of goods x depends only on the state of nature θ, and the information of other agents affects i only through the way that information changes the likelihood of different states of nature. This focus simplifies our formulation of informational size below.

Throughout the paper, we make the following assumptions regarding utilities and endowments for each agent $i \in N$:

- *Continuity*: for each $\theta \in \Theta$, $u_i(\cdot, \theta)$ is continuous.
- *Monotonicity*: if $x, y \in \Re^\ell_+$, $x \geqslant y$ and $x \neq y$, then $u_i(x, \theta) > u_i(y, \theta)$.
- *Normalization*: $u_i(0, \theta) = 0$.

- *Non-zero endowment*: $w_i \neq 0$.

Each $\pi \in \Delta_\Theta$ can be associated with a pure exchange economy in which each agent's utility for any bundle x is the expected utility of that bundle given the distribution π on Θ. More formally, the *expected economy corresponding to* π (expected economy for short) is the pure exchange economy in which agent i has endowment w_i and utility

$$v_i(x,\pi) := \sum_{\theta\in\Theta} u_i(x,\theta)\pi(\theta).$$

The expected economy corresponding to π will be denoted $e(\pi) = \{w_i, v_i(\cdot,\pi)\}_{i\in N}$ and we will define $e(\chi_\theta) : = e(\theta)$. For each $\pi \in \Delta_\Theta$, an allocation for $e(\pi)$ is a collection $\{x_i(\pi)\}_{i\in N}$ satisfying $x_i(\pi) \in \Re^\ell_+$ for each i and $\sum_{i\in N}(x_i(\pi) - w_i) \leqslant 0$. For each $\pi \in \Delta_\Theta$, an allocation $\{x_i(\pi)\}_{i\in N}$ for the expected economy $e(\pi)$ is *efficient* if there is no other allocation $\{y_i(\pi)\}_{i\in N}$ for $e(\pi)$ such that

$$v_i\big(y_i(\pi),\pi\big) > v_i\big(x_i(\pi),\pi\big)$$

for each $i \in N$. For each $\varepsilon \geqslant 0$, an allocation $\{x_i(\pi)\}_{i\in N}$ for the expected economy $e(\pi)$ is ε*-efficient* if there is no other allocation $\{y_i(\pi)\}_{i\in N}$ for $e(\pi)$ such that

$$v_i\big(y_i(\pi),\pi\big) > v_i\big(x_i(\pi),\pi\big) + \varepsilon$$

for each $i \in N$.

The collection $(\{e(\theta)\}_{\theta\in\Theta}, \tilde\theta, \tilde t, P)$ will be called a *private information economy* (PIE for short). A PIE allocation $z = (z_1, z_2, \dots, z_n)$ for the PIE is a collection of functions $z_i : T \to \Re^\ell_+$ satisfying $\sum_{i\in N}(z_i(t) - w_i) \leqslant 0$ for all $t \in T$. We will not distinguish between $w_i \in \Re^\ell_+ \setminus \{0\}$ and the constant allocation that assigns the bundle w_i to agent i for all $t \in T$.

Recall that $P_\Theta(\cdot \mid t) \in \Delta_\Theta$ denotes the conditional distribution on Θ given $t \in T^*$. Given a PIE $(\{e(\theta)\}_{\theta\in\Theta}, \tilde\theta, \tilde t, P)$, we can define a natural expected economy $e(t) := e(P_\Theta(\cdot \mid t))$ for each $t \in T^*$. In this notation, every PIE allocation z for the PIE $(\{e(\theta)\}_{\theta\in\Theta}, \tilde\theta, \tilde t, P)$ induces an allocation $z(t)$ in the expected economy $e(t)$ for each $t \in T^*$. Note that $e(t)$ depends on P and we are suppressing this dependence.

A PIE allocation z for the PIE $(\{e(\theta)\}_{\theta\in\Theta}, \tilde\theta, \tilde t, P)$ is:

- *incentive compatible* (IC) if

$$\sum_{\theta\in\Theta}\sum_{t_{-i}\in T_{-i}} u_i\big(z_i(t_{-i},t_i),\theta\big)P(\theta,t_{-i} \mid t_i) \geqslant \sum_{\theta\in\Theta}\sum_{t_{-i}\in T_{-i}} u_i\big(z_i(t_{-i},t_i'),\theta\big)P(\theta,t_{-i} \mid t_i)$$

  for all $i \in N$, and all $t_i, t_i' \in T_i$.
- *ex post individually rational* (XIR) if $z(t)$ is individually rational in $e(t)$ for all $t \in T^*$.
- *ex post* ε*-efficient* ($X_\varepsilon E$) if $z(t)$ is ε-efficient in $e(t)$ for all $t \in T^*$.

Note that allocations can depend on agents' types (their information) but not on θ, which is assumed to be unobservable. Hence, our use of the term 'ex post' refers to events that occur *after* the realization of the signal vector t but *before* the realization of the state θ.

2.1. Nonexclusive information, incentive compatibility, and ex post efficiency

In this section, we address the following question: When can we find a PIE allocation $z(\cdot)$ for the PIE $(\{e(\theta)\}_{\theta\in\Theta}, \tilde{\theta}, \tilde{t}, P)$ satisfying IC, XIR, and X_0E? To illustrate the ideas, we begin with two examples.

Example 1 (*Independence*). Suppose that $\tilde{\theta}$ and $\tilde{t}$ are independent. Let P_Θ denote the marginal of P on Θ. Choose an allocation $\{\bar{x}_i\}_{i\in N}$ that is individually rational and Pareto efficient for the expected economy $e(P_\Theta)$ with utilities

$$v_i(x_i, P_\Theta) = \sum_{\theta\in\Theta} u_i(x_i, \theta) P_\Theta(\theta)$$

and endowments w_i. If we define an allocation $z(\cdot)$ for the PIE as

$$z_i(t) = \begin{cases} \bar{x}_i & \text{if } t \in T^*, \\ w_i & \text{if } t \notin T^*, \end{cases}$$

then $z(\cdot)$ is XIR, X_0E, and IC. Note that IC is trivial in this case; when all agents are announcing their types truthfully, the only possible effect that an agent can have on the outcome by misreporting his type is to change the resulting allocation from $\bar{x}$ to w. This cannot increase any agent's utility since $\bar{x}$ is individually rational.

Example 2 (*Perfect correlation*). Suppose that

$$T_i = \Theta = \{\theta_1, \dots, \theta_m\}$$

for each i and that, for each k,

$$P(\theta_k, t) = \begin{cases} P_\Theta(\theta_k) & \text{if } t = (\theta_k, \dots, \theta_k), \\ 0 & \text{if } t \neq (\theta_k, \dots, \theta_k). \end{cases}$$

In words, each agent learns the state of nature θ precisely. Hence,

$$T^* = \{(\theta_k, \dots, \theta_k)\}_{k=1}^m \quad \text{and} \quad P_\Theta(\cdot \mid t) = \chi_{\theta_k} \quad \text{if } t = (\theta_k, \dots, \theta_k).$$

For each k, choose an efficient, individually rational allocation $\{x_i(\theta_k)\}_{i\in N}$ for the (degenerate) expected economy $e(\theta_k)$. If we define an allocation $z(\cdot)$ for the PIE as

$$z_i(t) = \begin{cases} x_i(\theta_k) & \text{if } t = (\theta_k, \dots, \theta_k), \\ w_i & \text{if } t \notin T^*, \end{cases}$$

then $z(\cdot)$ is XIR, X_0E, and IC. As in the first example, incentive compatibility follows from the fact that, whenever all other agents are announcing truthfully, the only possible effect that an agent can have on the outcome by misreporting his type is to change the resulting allocation to w.

The two examples presented above are special cases of the more general concept of *nonexclusive information* (Postlewaite and Schmeidler, 1986).

Definition. A measure $P \in \Delta_{\Theta\times T}$ satisfies *nonexclusive information* (NEI) if

$$t \in T^* \quad \Rightarrow \quad P_\Theta(\cdot \mid t) = P_\Theta(\cdot \mid t_{-i}) \quad \text{for all } i \in N.$$

The following (easily proved) lemma provides a simple but useful characterization of NEI.

Lemma 1. *$P \in \Delta_{\Theta\times T}$ satisfies NEI if and only if, for each $i \in N$ and for all $t_i, t_i' \in T_i$,*

$$\big[(t_{-i}, t_i) \in T^* \text{ and } (t_{-i}, t_i') \in T^*\big] \quad \Rightarrow \quad P_\Theta(\cdot \mid t_{-i}, t_i) = P_\Theta(\cdot \mid t_{-i}, t_i').$$

Examples 1 and 2 above demonstrate that for two particular instances in which NEI holds, there exist incentive compatible, ex post individually rational and ex post efficient mechanisms. The logic of those examples can be generalized, and we show next that if P satisfies NEI, then we can always find incentive compatible, ex post individually rational, ex post efficient mechanisms.

Proposition 1. *Let $\{e(\theta)\}_{\theta\in\Theta}$ be a collection of degenerate expected economies and suppose that $P \in \Delta_{\Theta\times T}$ satisfies NEI. Then there exists a PIE allocation $z(\cdot)$ for the PIE $(\{e(\theta)\}_{\theta\in\Theta}, \tilde{\theta}, \tilde{t}, P)$ satisfying XIR, IC, and X_0E.*

Proof. Let $z(\cdot)$ be a PIE allocation for $(\{e(\theta)\}_{\theta\in\Theta}, \tilde{\theta}, \tilde{t}, P)$ satisfying

(i) $z(t)$ is an efficient, individually rational allocation for the expected economy $e(t)$ for each $t \in T^*$,
(ii) $z(t) = z(\hat{t})$ if $t, \hat{t} \in T^*$ and $P_\Theta(\cdot \mid t) = P_\Theta(\cdot \mid \hat{t})$, and
(iii) $z(t) = (w_1, \ldots, w_n)$ if $t \notin T^*$.

Clearly, the PIE allocation $z(\cdot)$ is XIR and $\mathrm{X_0E}$. To prove incentive compatibility, note that

$$\begin{aligned}
&\sum_{\theta\in\Theta}\sum_{t_{-i}\in T_{-i}} \big[u_i\big(z_i(t_{-i}, t_i), \theta\big) - u_i\big(z_i(t_{-i}, t_i'), \theta\big)\big] P(\theta, t_{-i} \mid t_i) \\
&\quad= \sum_{\substack{t_{-i}\in T_{-i}:\\ (t_{-i},t_i)\in T^*\\ (t_{-i},t_i')\in T^*}} \big[v_i\big(z_i(t_{-i}, t_i), P_\Theta(\cdot \mid t)\big) - v_i\big(z_i(t_{-i}, t_i'), P_\Theta(\cdot \mid t)\big)\big] P(t_{-i} \mid t_i) \\
&\qquad+ \sum_{\substack{t_{-i}\in T_{-i}:\\ (t_{-i},t_i)\in T^*\\ (t_{-i},t_i')\notin T^*}} \big[v_i\big(z_i(t_{-i}, t_i), P_\Theta(\cdot \mid t)\big) - v_i\big(z_i(t_{-i}, t_i'), P_\Theta(\cdot \mid t)\big)\big] P(t_{-i} \mid t_i) \\
&\quad= \sum_{\substack{t_{-i}\in T_{-i}:\\ (t_{-i},t_i)\in T^*\\ (t_{-i},t_i')\notin T^*}} \big[v_i\big(z_i(t_{-i}, t_i), P_\Theta(\cdot \mid t)\big) - v_i\big(w_i, P_\Theta(\cdot \mid t)\big)\big] P(t_{-i} \mid t_i) \\
&\quad\geqslant 0. \qquad \square
\end{aligned}$$

2.2. *Informational size and variability of beliefs*

2.2.1. *Informational size*

In the mechanism of Proposition 1, agents reveal their types and the announced type profile t is used to construct an updated probability $P_\Theta(\cdot \mid t)$ distribution on Θ. The mechanism then specifies an efficient, individually rational allocation for the economy $e(P_\Theta(\cdot \mid t))$. The mechanism is incentive compatible because each agent i is 'informationally small' in the following sense: when agents other than i announce truthfully, there is no residual uncertainty about the state that can be resolved using i's announcement. In other words, i's information is irrelevant if all other agents are announcing truthfully.

An investigation of these issues in a more general framework requires a formal notion of *informational size*. McLean and Postlewaite (2002) introduced such a notion, which we review.

If $t \in T^*$, recall that $P_\Theta(\cdot \mid t) \in \Delta_\Theta$ denotes the induced conditional probability measure on Θ and $\chi_\theta \in \Delta_\Theta$ denotes the degenerate measure concentrated on θ. A natural notion of an agent's informational size is the degree to which he can alter the best estimate of the state θ when other agents are announcing truthfully. In our setup, that estimate is the conditional probability distribution on Θ given a profile of types t. Any profile of agents' types $t = (t_{-i}, t_i) \in T^*$ induces a conditional distribution on Θ and, if agent i unilaterally changes his announced type from t_i to t_i', this conditional distribution will (in general) change. We consider agent i to be informationally small if, for each t_i, there is a 'small' probability that he can effect a non-negligible change in the induced conditional distribution on Θ by changing his announced type from t_i to some other t_i'. This is formalized in the following definition.

Definition. Let

$$I_\varepsilon^i(t_i', t_i) = \big\{t_{-i} \in T_{-i} \mid (t_{-i}, t_i) \in T^*, (t_{-i}, t_i') \in T^*, \text{ and } \\ \big\| P_\Theta(\cdot \mid t_{-i}, t_i) - P_\Theta(\cdot \mid t_{-i}, t_i') \big\| > \varepsilon\big\}.$$

The *informational size* of agent i is defined as

$$\nu_i^P = \max_{t_i \in T_i} \max_{t_i' \in T_i} \min\big\{\varepsilon \geqslant 0 \,\big|\, \mathrm{Prob}\big\{\tilde{t}_{-i} \in I_\varepsilon^i(t_i', t_i) \,\big|\, \tilde{t}_i = t_i\big\} \leqslant \varepsilon\big\}.$$

Loosely speaking, we will say that agent i is *informationally small* with respect to P if his informational size ν_i^P is 'small.' If agent i receives signal t_i but reports $t_i' \neq t_i$, the effect of this misreport is a change in the conditional distribution on Θ from $P_\Theta(\cdot \mid t_{-i}, t_i)$ to $P_\Theta(\cdot \mid t_{-i}, t_i')$. If $t_{-i} \in I_\varepsilon(t_i', t_i)$, then this change is non-negligible in the sense that $\|P_\Theta(\cdot \mid \hat{t}_{-i}, t_i) - P_\Theta(\cdot \mid \hat{t}_{-i}, t_i')\| > \varepsilon$. Therefore, $\mathrm{Prob}\{\tilde{t}_{-i} \in I_\varepsilon(t_i', t_i) \mid \tilde{t}_i = t_i\}$ is the probability that i can have a non-negligible influence on the conditional distribution on Θ by reporting t_i' instead of t_i when his observed signal is t_i. An agent is informationally small if for each of his possible types t_i, he assigns small probability to the event that he can have a non-negligible influence on the distribution $P_\Theta(\cdot \mid t_{-i}, t_i)$, given his observed type.

If all agents have zero informational size, then P must satisfy NEI. In fact, we have the following result which follows easily from Lemma 1.

Proposition 2. *$P \in \Delta_{\Theta\times T}$ satisfies NEI if and only if $\nu_i^P = 0$ for each $i \in N$.*

We conclude this section with the observation that informational size is not related to the 'quality' of an agent's information regarding the state of nature. In Example 1, an agent's private signal provides no information regarding the realization of $\tilde{\theta}$ (since $P_\Theta(\cdot \mid t_i) = P_\Theta(\cdot)$ for each $t_i \in T_i$) while in Example 2, an agent's private signal provides perfect information regarding the realization of $\tilde{\theta}$ (since $P_\Theta(\cdot \mid t_i) = \chi_{t_i}$ for each $t_i \in T_i$). Hence, agents may have very good estimates of the true state conditional on their own types, yet each agent is informationally small.

2.2.2. Variability of agents' beliefs

Whether an agent i can be given incentives to reveal his information will depend on the magnitude of the difference between $P_{T_{-i}}(\cdot \mid t_i)$ and $P_{T_{-i}}(\cdot \mid t_i')$, the conditional distributions on T_{-i} given different types t_i and t_i' for agent i. To formally define the measure of variability that is convenient for our purposes, we first define a metric d on Δ_Θ as follows: for each $\alpha, \beta \in \Delta_\Theta$, let

$$d(\alpha,\beta) = \left\| \frac{\alpha}{\|\alpha\|_2} - \frac{\beta}{\|\beta\|_2} \right\|_2,$$

where $\|\cdot\|_2$ denotes the 2-norm. Hence, $d(\alpha,\beta)$ measures the Euclidean distance between the Euclidean normalizations of α and β.

If $P \in \Delta_{\Theta\times T}$, recall that $P_\Theta(\cdot \mid t_i) \in \Delta_\Theta$ is the conditional distribution on Θ given that i receives signal t_i and define

$$\Omega_i^P = \min_{t_i\in T_i} \min_{t_i'\in T_i\setminus t_i} d\big(P_{T_{-i}}(\cdot \mid t_i), P_{T_{-i}}(\cdot \mid t_i')\big)^2.$$

This is the measure of the 'variability' of the conditional distribution $P_{T_{-i}}(\cdot \mid t_i)$ as a function of t_i and we will refer to this informally as the *variability of agents' beliefs*.[2]

As mentioned in the introduction, our work is related to that of Cremer and McLean (1985, 1988). Those papers and subsequent work by McAfee and Reny (1992) demonstrated how one can use correlation to fully extract the surplus in certain mechanism design problems. The key ingredient there is the assumption that the collection of conditional distributions $\{P_{T_{-i}}(\cdot \mid t_i)\}_{t_i\in T_i}$ is a linearly independent set for each i. This of course, implies that $P_{T_{-i}}(\cdot \mid t_i) \neq P_{T_{-i}}(\cdot \mid t_i')$ if $t_i \neq t_i'$ and, therefore, that $\Omega_i^P > 0$. While linear independence implies that $\Omega_i^P > 0$, the actual (positive) size of Ω_i^P is not relevant in the Cremer–McLean constructions, and full extraction will be possible. In the present work, we do not require that the collection $\{P_{T_{-i}}(\cdot \mid t_i)\}_{t_i\in T_i}$ be linearly independent (or

[2] Other essentially equivalent definitions of variability are possible (e.g., $\min_{t_i\in T_i} \min_{t_i'\in T_i\setminus t_i} \|P_{T_{-i}}(\cdot \mid t_i) - P_{T_{-i}}(\cdot \mid t_i')\|^2$) and we have chosen one that is computationally convenient. This notion of variability of agents' beliefs differs from that in McLean and Postlewaite (2002) and the difference is discussed below.

satisfy the weaker cone condition in Cremer and McLean, 1988). However, the 'closeness' of the members of $\{P_{T_{-i}}(\cdot \mid t_i)\}_{t_i \in T_i}$ is an important issue. It can be shown that for each i, there exists a collection of numbers $z_i(t)$ satisfying $0 \leqslant z_i(t) \leqslant 1$ and

$$\sum_{t_{-i}} \big[z_i(t_{-i}, t_i) - z_i(t_{-i}, t_i')\big] P_{T_{-i}}(t_{-i} \mid t_i) > 0$$

for each $t_i, t_i' \in T_i$ if and only if $\Omega_i^P > 0$. This means that, if the posteriors $\{P_{T_{-i}}(\cdot \mid t_i)\}_{t_i \in T_i}$ are all distinct, then the 'incentive compatibility' inequalities above are strict. However, the expression on the left-hand side decreases as $\Omega_i^P \to 0$. Hence, the difference in the expected reward from a truthful report and from a false report will be very small if the conditional posteriors are very close to each other. Our results require that informational size be small relative to the variation in these posteriors.

Small incentives for truthful reporting (i.e., small values of Ω_i^P) are not a serious problem in the surplus extraction problem studied by Cremer and McLean since the rewards and punishments can be rescaled so that a false report results in a large negative expected payment. Of course, the punishments themselves may then become very large. However, such rescaling is not possible in our framework for two reasons. First, we deal with pure exchange economies where the feasibility requirement limits the size of punishments. Second, we do not restrict attention to quasilinear preferences. Since agents may be risk averse, punishments and rewards that have small (or zero) expected value can have large negative welfare effects.

2.3. Aggregate uncertainty

As mentioned in the introduction, the main concern of this paper is to provide conditions under which nearly efficient incentive compatible allocations exist in the presence of aggregate uncertainty. Before proceeding, we will discuss aggregate uncertainty and its role in McLean and Postlewaite (2002).

In McLean and Postlewaite (2002), three concepts play an important role in constructing individually rational, approximately efficient allocations for private information economies: informational size, variability of beliefs and negligible aggregate uncertainty. The definition of informational size in McLean and Postlewaite (2002) is essentially the same as that presented in this paper. However, McLean and Postlewaite (2002) use a different notion of variability: there we define

$$\Lambda_i^P = \min_{t_i \in T_i} \min_{t_i' \in T_i \setminus t_i} d\big(P_\Theta(\cdot \mid t_i), P_\Theta(\cdot \mid t_i')\big)^2$$

as our measure of the variability of agents' beliefs where d is the metric of the previous section. In this definition, $P_\Theta(\cdot \mid t_i)$ is the conditional distribution on the state space Θ given $\tilde{t}_i = t_i$. Hence, Ω_i^P measures the variation in agent i's beliefs regarding the signals of other agents while Λ_i^P measures the variation in agent i's beliefs regarding the state of nature. Both definitions of variability open the possibility of providing that agent with incentives for truthful revelation.

McLean and Postlewaite (2002) require that the information structure exhibit *negligible aggregate uncertainty*. Informally, we say that a probability measure P exhibits *negligible* *aggregate uncertainty* if, for a set of t's with high probability, $P_\Theta(\cdot \mid t) \approx \chi_\theta$ for some $\theta \in \Theta$. More formally, define

$$\mu_i^P = \max_{t_i \in T_i} \min\bigl\{\varepsilon \geqslant 0 \bigm| \operatorname{Prob}\bigl\{\tilde{t} \in T \text{ and } \bigl\| P_\Theta(\cdot \mid \tilde{t}) - \chi_\theta \bigr\| > \varepsilon \text{ for all } \theta \in \Theta \bigm| t_i\bigr\} \leqslant \varepsilon\bigr\}.$$

We define the *aggregate uncertainty* as $\mu^P \equiv \max_i \mu_i^P$ and we will say that P exhibits negligible aggregate uncertainty if μ^P is small. In this case, each agent knows that, conditional on his own signal, the aggregate information of all agents will, with high probability, provide a good prediction of the true state. The main result of McLean and Postlewaite (2002) may be stated as follows.

Proposition 3. *Let $\{e(\theta)\}_{\theta\in\Theta}$ be a collection of (degenerate) expected economies and suppose that there exists a strictly individually rational, efficient allocation for each $e(\theta)$. Then for every $\varepsilon > 0$, there exists a $\delta > 0$ such that, whenever $P \in \Delta_{\Theta\times T}$ and satisfies*

$$\max_i \mu_i^P \leqslant \delta \min_i \Lambda_i^P \quad \text{and} \quad \max_i \nu_i^P \leqslant \delta \min_i \Lambda_i^P,$$

there exists an allocation $z(\cdot)$ for the PIE $(\{e(\theta)\}_{\theta\in\Theta}, \tilde{\theta}, \tilde{t}, P)$ satisfying XIR and IC. Furthermore, there exists a set $E \subseteq T^*$ such that $\operatorname{Prob}\{\tilde{t} \in E\} \geqslant 1 - \varepsilon$ *and $z(t)$ is ε-efficient in $e(t)$ for all $t \in E$.*

In the statement of Proposition 3, aggregate uncertainty is small relative to informational variability—$\max_i \mu_i^P \leqslant \delta \min_i \Lambda_i^P$—and it is instructive to outline the method of proof for this proposition in order to illustrate the role aggregate uncertainty plays.

Let $\mathcal{A} = \{x(\theta)\}_{\theta\in\Theta}$ be a collection where $x(\theta)$ is a strictly individually rational, Pareto efficient allocation for $e(\theta)$. In the presence of negligible aggregate uncertainty, we can partition T into $m+1$ disjoint sets with $A_k = \{t \in T \mid P(\cdot \mid t) \approx \chi_{\theta_k}\}$ for $k = 1, \ldots, m$, and $A_0 = T \setminus [\bigcup_{k\geqslant 1} A_k]$. In words, A_k with $k \geqslant 1$ is the set of $t \in T$ for which the posterior distribution on Θ is close to the degenerate distribution χ_{θ_k} that puts probability 1 on θ_k. Therefore, A_0 is the set of $t \in T$ for which the posterior is not close to χ_θ for any θ. We next choose a PIE allocation $y(\cdot)$ with $y(t) = x(\theta_k)$ for $t \in A_k$, $k = 1, \ldots, m$, and $y(t) = w$ (the initial endowment) for $t \in A_0$. When aggregate uncertainty is small, the information profile $t \in T$ will, with high probability, resolve most of the uncertainty regarding the state of nature θ. There are two consequences of small aggregate uncertainty: the event A_0 has small probability and for each $t \in A_k$ with $k \geqslant 1$, $P_\Theta(\theta_k \mid t)$ is close to 1. Therefore, $\sum_\ell u_i(y_i(t), \theta_\ell) P_\Theta(\theta_\ell \mid t)$ is close to $u_i(x_i(\theta_k), \theta_k)$ whenever $t \in A_k$. Since $y(t)$ is efficient for the economy $e(\theta_k)$, the PIE allocation $y(\cdot)$ is approximately ex post efficient for most realizations of the signal profile t.

The PIE $y(\cdot)$ as constructed is not incentive compatible in general. Suppose that i receives signal t_i, i reports t_i' and the other agents truthfully report t_{-i}. If $(t_{-i}, t_i) \in A_0$, then no trade takes place and i simply consumes his initial endowment. Since each $x_i(\theta_k)$ is individually rational, agent i has an incentive to misreport if $(t_{-i}, t_i') \in A_k$ for some $k \geqslant 1$. However, the (conditional) probability that $(\tilde{t}_{-i}, t_i) \in A_0$ is small as a consequence of negligible aggregate uncertainty so the utility gain will be small if utility is bounded.

Now suppose that $(t_{-i}, t_i) \in A_k$ for some $k \geqslant 1$. Hence, i receives $x_i(\theta_k)$ if he reports t_i. If $(t_{-i}, t'_i) \in A_0$, then i receives his initial endowment and he does not gain by lying. If $(t_{-i}, t'_i) \in A_j$ for some $j \geqslant 1$, $j \neq k$, he receives $x_i(\theta_j)$. If $x_i(\theta_j)$ results in higher utility than $x_i(\theta_k)$, agent i may have an incentive to misreport. If agent i is informationally small, however, then the (conditional) probability that agent i can 'move' the profile from $(\tilde{t}_{-i}, t_i) \in A_k$ to $(\tilde{t}_{-i}, t'_i) \in A_j$ (and hence 'move' his allocation from $x_i(\theta_k)$ to $x_i(\theta_j)$) is small. Therefore, $(\tilde{t}_{-i}, t'_i) \in A_k \cup A_0$ with high probability, i.e., with high probability, no utility gain is possible.

In summary, the negligible aggregate uncertainty implies that the contribution to the total expected gain from lying is small whenever if $(t_{-i}, t_i) \in A_0$ while small informational size implies that the contribution to the total expected gain from lying is small whenever if $(t_{-i}, t_i) \notin A_0$. Hence, the total expected gain from a lie is small if aggregate uncertainty and agents' informational size are both small. In order to offset this (small) potential gain that i might receive from misreporting, we modify the bundle $x_i(\theta_k)$ that i receives when $t \in A_k$. If the difference between $P(\theta_k \mid t_i)$ and $P(\theta_k \mid t'_i)$ is sufficiently large for different types t_i and t'_i, relative to informational size and aggregate uncertainty, then we can construct a PIE allocation $z(t)$ by slightly modifying each $y(t)$ to ensure incentive compatibility. The final allocation $z(\cdot)$ will be incentive compatible and approximately efficient for nearly all t.

As we have outlined above, incentive compatibility essentially follows from the fact that, with high probability, a lie results in no utility gain. This is a consequence of the partition construction made possible by negligible aggregate uncertainty. If aggregate uncertainty is large, we may still be able to construct an incentive compatible mechanism in which the gains from lying, while not zero with high probability, will be small with high probability. This is accomplished in the next section. The technique involves the construction of an individually rational, efficient allocation $z(t)$ for the expected economy $e(t)$ for each $t \in T^*$ with associated utilities satisfying a *Lipschitzian* property: there exists a constant K such that for any two probability distributions $P(\cdot \mid t_{-i}, t_i)$ and $P(\cdot \mid t_{-i}, t'_i)$,

$$\begin{aligned}
&\big|v_i\big(z_i(t_{-i}, t_i), P(\cdot \mid t_{-i}, t_i)\big) - v_i\big(z_i(t_{-i}, t'_i), P(\cdot \mid t_{-i}, t'_i)\big)\big| \\
&\quad \leqslant K\big\|P(\cdot \mid t_{-i}, t_i) - P(\cdot \mid t_{-i}, t'_i)\big\|.
\end{aligned}$$

While the allocation $z(t)$ is efficient for any vector of announced types, incentive compatibility will typically fail to hold. However, an agent's potential gain from misreporting his type is essentially determined by his informational size. His informational size places a bound on the change in the posterior on Θ he can induce by misreporting, and the Lipschitz property puts a bound on the consequent change in utility. Hence, as an agent's informational size goes to zero, his potential gain from misreporting goes to zero. When informational variability is large enough relative to informational size, the allocation can be slightly modified to insure incentive compatibility, as in the case of negligible aggregate uncertainty. We demonstrate this next.

3. Informational size, incentive compatibility, and approximate ex post efficiency

3.1. Preliminaries and the main approximation lemma

Let $\{e(\theta)\}_{\theta\in\Theta}$ be a collection of (degenerate) expected economies. For each $\pi \in \Delta_\Theta$, let

$$\Phi(\pi) = \big\{\big(v_1(x_1,\pi),\ldots,v_n(x_n,\pi)\big) \bigm| (x_1,\ldots,x_n) \text{ is feasible for } e(\pi)\big\}.$$

That is, $\Phi(\pi)$ is the set of feasible utility profiles for the agents in the expected economy $e(\pi)$. Before proceeding, we will introduce several definitions. A map $f:\Delta_\Theta \to \Re^n$ is a *Φ-selection* if $f(\pi)\in\Phi(\pi)$ for all $\pi\in\Delta_\Theta$. A Φ-selection f is Lipschitz if each f_i is Lipschitz on Δ_Θ. That is, for each i there exists a $K_i > 0$ such that $|f_i(\pi) - f_i(\pi')| \leqslant K_i\|\pi - \pi'\|$ for each $\pi,\pi'\in\Delta_\Theta$. A Φ-selection is *positive* if $f_i(\pi) > 0$ for all $\pi\in\Delta_\Theta$ and all $i\in N$. A Φ-selection is *individually rational* if $f_i(\pi)\geqslant v_i(w_i,\pi)$ for all $\pi\in\Delta_\Theta$ and all $i\in N$. The monotonicity, normalization, and non-zero endowment assumptions imply that every individually rational Φ-selection is positive.

We will show that certain PIE allocations can be approximated by incentive compatible PIE allocations when agents are informationally small relative to variability. The idea is as follows. Let $f:\Delta_\Theta\to\Re^n$ be a Lipschitz Φ-selection. Since $f(\pi) = (f_1(\pi),\ldots,f_n(\pi))$ is a feasible utility profile for $e(\pi)$, there exists an allocation $x(\pi) = (x_1(\pi),\ldots,x_n(\pi))$ such that $f(\pi) = (v_1(x_1(\pi),\pi),\ldots,v_n(x_n(\pi),\pi))$. Next, define a PIE allocation $y(\cdot)$ where $y(t) = x(P_\Theta(\cdot\mid t))$ for each $t\in T^*$ and $P_\Theta(\cdot\mid t)$ is the posterior distribution on Θ given t. That is, $y(t)$ is an allocation that generates the desired utility for the distribution $P_\Theta(\cdot\mid t)$. Of course, $y(\cdot)$ need not be incentive compatible. When agents are informationally small, however, any agent who unilaterally misreports his type can change the posterior by only a small amount. If the selection f is Lipschitz, then the utility change resulting from that agent's misreport will also be small. When the variability condition is satisfied, agents' types are correlated in a way that allows us to construct small rewards and punishments for the agents. These rewards have the property that, by truthfully announcing his type, an agent maximizes his utility—including his reward—if other agents are announcing truthfully.Given these rewards and punishments, we can modify the allocation $y(\cdot)$ in a way that the modified allocation will be incentive compatible when informational size is small relative to variability for all agents. The next proposition formalizes this.

Proposition 4. *Let $\{e(\theta)\}_{\theta\in\Theta}$ be a collection of degenerate expected economies and suppose that f is a positive Lipschitz selection for Φ. Then for every $\varepsilon > 0$, there exists a $\delta > 0$ such that, whenever $P\in\Delta_{\Theta\times T}$ and satisfies*

$$\max_i \nu_i^P \leqslant \delta \min_i \Omega_i^P,$$

there exists an incentive compatible PIE allocation $z(\cdot)$ for the PIE $(\{e(\theta)\}_{\theta\in\Theta},\tilde\theta,\tilde t,P)$ satisfying

$$f_i\big(P_\Theta(\cdot\mid t)\big) \geqslant v_i\big(z_i(t),P_\Theta(\cdot\mid t)\big) \geqslant f_i\big(P_\Theta(\cdot\mid t)\big) - \varepsilon$$

for each $t\in T^*$ and for all $i\in N$. Moreover,

$$v_i\big(\zeta_i(t),P_\Theta(\cdot\mid t)\big) \geqslant v_i\big(z_i(t),P_\Theta(\cdot\mid t)\big) \geqslant v_i\big(\zeta_i(t),P_\Theta(\cdot\mid t)\big) - \varepsilon$$

for any PIE allocation $\zeta(\cdot)$ *for* $(\{e(\theta)\}_{\theta\in\Theta}, \tilde{\theta}, \tilde{t}, P)$ *satisfying*

$$v_i\big(\zeta_i(t), P(\cdot \mid t)\big) = f_i\big(P_\Theta(\cdot \mid t)\big)$$

for all $t \in T^*$.

For proof see Appendix A.

3.2. The main result for economies of fixed size

Proposition 4 of the previous section provides conditions on the information structure under which the utilities associated with a Lipschitz selection can be approximated by an incentive compatible PIE allocation. We next prove the existence of a Lipschitz selection that gives rise to an approximately efficient, strictly individually rational, incentive compatible PIE allocation.

In the previous subsection, we defined Φ, the correspondence that associates with every $\pi \in \Delta_\Theta$ the set of feasible utility vectors for the agents in the expected economy $e(\pi)$, and considered Lipschitz selections from that correspondence. We will now be interested in selections f that have the property that $f(\pi)$ is on the frontier of the utility possibility set $\Phi(\pi)$ for the expected economy $e(\pi)$. Toward this end, define $\Phi^0(\pi) = \{(v_1(x_1,\pi),\dots,v_n(x_n,\pi)) \mid (x_1,\dots,x_n)$ is efficient and IR for $e(\pi)\}$. The following definition will be useful in proving our main result.

Definition. Let $\lambda > 0$ and $\pi \in \Delta_\Theta$. An economy $e(\pi)$ satisfies λ-Individual Rationality (λ-IR) if there exists an allocation $x(\pi)$ for $e(\pi)$ such that $v_i(x_i(\pi),\pi) - v_i(w_i,\pi) \geqslant \lambda$ for all $i \in N$.

We next show that there exist Lipschitz selections that associate with each expected economy $e(\pi)$ a utility vector on the frontier of the feasible set for $e(\pi)$.

Lemma 2. *Let* $\{e(\theta)\}_{\theta\in\Theta}$ *be a collection of degenerate expected economies. The correspondence* $\Phi^0 : \Delta_\Theta \to \Re^n$ *admits an individually rational* (*hence positive*) *Lipschitz selection* f. *Furthermore, the selection* f *has the following property*: *if the economy* $e(\pi)$ *satisfies* λ*-IR for some* $\lambda > 0$, *then* $f_i(\pi) - v_i(w_i,\pi) \geqslant \lambda$ *for each* i.

For proof see Appendix A.

To prove Lemma 2, we choose $\pi \in \Delta_\Theta$ and solve the problem

$$\mu(\pi) = \arg\max\big[\mu \mid v(w,\pi) + \mu e \in \Phi(\pi)\big],$$

where $v(w,\pi) := (v_1(w_1,\pi),\dots,v_n(w_n,\pi))$ and e is the vector of ones in $\Re^n_+$. We then choose a feasible allocation $x(\pi)$ for $e(\pi)$ satisfying $v_i(x_i(\pi),\pi) = v_i(w_i,\pi) + \mu(\pi)e$ for each $i \in N$ and show that the selection $f : \Delta_\Theta \to \Re^n$ defined by $f_i(\pi) = v_i(x_i(\pi)\pi)$ for each i has the desired properties. In this construction, we measure utility along the ray determined by the vector e. Had we measured utility along a different ray, say p where $p_i > 0$ for each i, then the associated maximization problem would have generated a different selection, but one that was still efficient and Lipschitz. Alternatively, we could have maximized the minimal 'weighted' utility gain, where different agents were assigned different weights. These examples illustrate the existence of a large set of Pareto efficient Lipschitz selections that can be approximated by incentive compatible allocations when agents are informationally small. It would be interesting to characterize the set of all such 'weighted' selections, but that is beyond the scope of the present paper.

Using Lemma 2 and Proposition 4 of the previous section, we can prove the following theorem.

Theorem 1. *Let $\{e(\theta)\}_{\theta\in\Theta}$ be a collection of degenerate expected economies. For every $\varepsilon > 0$ and every $\lambda > 0$, there exists a $\delta > 0$ such that the following holds*:

if $P \in \Delta_{\Theta\times T}$ satisfies

$$\max_i \nu_i^P \leqslant \delta \min_i \Lambda_i^P$$

and if $e(P(\cdot \mid t))$ satisfies λ-IR for all $t \in T^*$, then there exists a PIE allocation $z(\cdot)$ for the PIE $(\{e(\theta)\}_{\theta\in\Theta}, \tilde{\theta}, \tilde{t}, P)$ satisfying XIR, IC, and $X_\varepsilon E$.

Proof. Applying Lemma 2, there exists an individually rational Lipschitz selection f for Φ^0 with the property that $f_i(P(\cdot \mid t)) - v_i(w_i, P(\cdot \mid t)) \geqslant \lambda$ for each i and for each $t \in T^*$. Choose $\varepsilon > 0$ and let $0 < \eta < \min\{\varepsilon, \lambda\}$. Applying Proposition 4, there exists a $\delta > 0$ such that, whenever $P \in \Delta_{\Theta\times T}$ and satisfies

$$\max_i \nu_i^P \leqslant \delta \min_i \Omega_i^P,$$

there exists an incentive compatible PIE allocation $z(\cdot)$ for the PIE $(\{e(\theta)\}_{\theta\in\Theta}, \tilde{\theta}, \tilde{t}, P)$ satisfying

$$f_i\big(P_\Theta(\cdot \mid t)\big) \geqslant v_i\big(z_i(t), P_\Theta(\cdot \mid t)\big) \geqslant f_i\big(P_\Theta(\cdot \mid t)\big) - \eta$$

for each $t \in T^*$ and for all $i \in N$. Let $\zeta(\cdot)$ be a PIE allocation for $(\{e(\theta)\}_{\theta\in\Theta}, \tilde{\theta}, \tilde{t}, P)$ satisfying $v_i(\zeta_i(t), P(\cdot \mid t)) = f_i(P_\Theta(\cdot \mid t))$ for all $t \in T^*$. Obviously, $\zeta(t)$ is efficient and

$$v_i\big(\zeta_i(t), P_\Theta(\cdot \mid t)\big) \geqslant v_i\big(z_i(t), P_\Theta(\cdot \mid t)\big) \geqslant v_i\big(\zeta_i(t), P_\Theta(\cdot \mid t)\big) - \eta$$

for all $t \in T^*$. Since $e(P(\cdot \mid t))$ satisfies λ-IR for all $t \in T^*$, it follows that

$$v_i\big(\zeta_i(t), P_\Theta(\cdot \mid t)\big) - v_i\big(w_i, P_\Theta(\cdot \mid t)\big) = f\big(P_\Theta(\cdot \mid t)\big) - v_i\big(w_i, P_\Theta(\cdot \mid t)\big) \geqslant \lambda$$

for each $t \in T^*$. Therefore,

$$\begin{aligned} v_i\big(z_i(t), P_\Theta(\cdot \mid t)\big) - v_i\big(w_i, P_\Theta(\cdot \mid t)\big) &= v_i\big(z_i(t), P_\Theta(\cdot \mid t)\big) - v_i\big(\zeta_i(t), P_\Theta(\cdot \mid t)\big) \\ &\quad + v_i\big(\zeta_i(t), P_\Theta(\cdot \mid t)\big) - v_i\big(w_i, P_\Theta(\cdot \mid t)\big) \\ &\geqslant \lambda - \eta > 0 \end{aligned}$$

for each i and we conclude that $z(t)$ is individually rational in $e(t)$.

To show that $z(t)$ is ε-efficient in $e(t)$, suppose instead that there exists an allocation y for $e(t)$ such that $v_i(y_i, P_\Theta(\cdot \mid t)) > v_i(z_i(t), P_\Theta(\cdot \mid t)) + \varepsilon$ for each i. Then

$$\begin{aligned} v_i\big(y_i, P_\Theta(\cdot \mid t)\big) &> v_i\big(z_i(t), P_\Theta(\cdot \mid t)\big) + \varepsilon \geqslant v_i\big(\zeta_i(t), P_\Theta(\cdot \mid t)\big) - \eta + \varepsilon \\ &> v_i\big(\zeta_i(t), P_\Theta(\cdot \mid t)\big) \end{aligned}$$

for all i contradicting the efficiency of $\zeta(t)$ in the expected economy $e(t)$. This completes the proof of Theorem 1. □

To conclude this section, we will now describe in somewhat more detail how the results in this paper relate to the main result in McLean and Postlewaite (2002). Throughout this discussion, we will assume that $T = T^*$.

As suggested in the discussion on aggregate uncertainty, we must follow a different approach when the assumption of negligible aggregate uncertainty is dropped. Instead of choosing a collection $\mathcal{A} = \{x(\theta)\}_{\theta\in\Theta}$ where $x(\theta)$ is a strictly individually rational, Pareto efficient allocation for $e(\theta)$, we begin with a strictly individually rational Lipschitz selection f from the correspondence Φ^0. We can 'invert' this Lipschitz selection to generate a mapping ζ from Δ_Θ into allocations such that, for any $\pi \in \Delta_\Theta$, $\zeta(\pi)$ is a strictly individually rational, Pareto efficient allocation for the expected economy $e(\pi)$. In the current paper, the mapping $\zeta(\cdot)$ plays a role analogous to that of the collection $\mathcal{A} = \{x(\theta)\}_{\theta\in\Theta}$ in our previous work, and it is important to point out that $\zeta(\cdot)$ need not even be continuous on Δ_Θ. For any announced profile of types t, we begin with the allocation $\zeta(P_\Theta(\cdot \mid t))$, and then modify it. When agents are informationally small, any agent who unilaterally misreports his type will change the posterior distribution on Θ by only a small amount, and hence, change his resulting utility by only a small amount since his *utility* depends on $P_\Theta(\cdot \mid t)$ in a Lipschitzian manner. As long as variability is sufficiently large relative to informational size, the requisite modifications to $\zeta(P_\Theta(\cdot \mid t))$ can be made that ensure incentive compatibility.

4. The replica problem

In the presence of a large number of agents, we might expect any single agent to be informationally small, and replica economies are a natural framework in which to investigate this conjecture.

4.1. Notation and definitions

Let $\{e(\theta)\}_{\theta\in\Theta}$ be a collection of complete information economies and let $J_r = \{1, 2, \ldots, r\}$. For each positive integer r and each θ, let $e^r(\theta) = \{w_{is}, u_{is}(\cdot, \theta)\}_{(i,s)\in N\times J_r}$ denote the r replication of $e(\theta)$ corresponding to state θ satisfying:

(1) $w_{is} = w_i$ for all $s \in J_r$,
(2) $u_{is}(z, \theta) = u_i(z, \theta)$ for all $z \in \Re^\ell_+$, $i \in N$, and $s \in J_r$.

For any positive integer r, let $T^r = T \times \cdots \times T$ denote the r-fold Cartesian product and let $t^r = (t^r(1), \ldots, t^r(r))$ denote a generic element of T^r where $t^r(s) = (t^r_1(s), \ldots, t^r_n(s)) \in T$. If $P^r \in \Delta_{\Theta\times T^r}$, then $e^r = (\{e^r(\theta)\}_{\theta\in\Theta}, \tilde{\theta}, \tilde{t}^r, P^r)$ is a PIE with nr agents.

4.2. Replica economies and the replica theorem

Definition. A sequence of replica economies $\{(\{e^r(\theta)\}_{\theta\in\Theta}, \tilde{\theta}, \tilde{t}^r, P^r)\}_{r=1}^{\infty}$ is a conditionally independent sequence if there exists a $P \in \Delta_{\Theta\times T}$ with $P(\theta, t) > 0$ for each $(\theta, t) \in \Theta \times T$ such that

(a) For each r, each $s \in J_r$ and each $(\theta, t) \in \Theta \times T$,

$$\mathrm{Prob}\{\tilde{\theta} = \theta, \tilde{t}^r(s) = t\} = P(\theta, t_1, t_2, \ldots, t_n).$$

(b) For each r and each θ, the random vectors

$$\tilde{t}^{\,r}(1), \tilde{t}^{\,r}(2), \ldots, \tilde{t}^{\,r}(r)$$

are independent given $\tilde{\theta} = \theta$.

(c) For every $\theta, \hat{\theta}$ with $\theta \neq \hat{\theta}$, there exists a $t \in T$ such that $P(t \mid \theta) \neq P(t \mid \hat{\theta})$.

(d) For each i and each $t_i, t_i' \in T_i$, $P_{T_{-i}}(\cdot \mid t_i) \neq P_{T_{-i}}(\cdot \mid t_i')$.

Thus a conditionally independent sequence is a sequence of PIE's with nr agents containing r 'copies' of each agent $i \in N$. Each copy of an agent i is identical, i.e., has the same endowment and the same utility function. Furthermore, the realizations of type profiles across cohorts are independent given the true value of $\tilde{\theta}$. As r increases each agent is becoming 'small' in the economy in terms of endowment, and each agent is also becoming informationally small. Note that, for large r, an agent may have a small amount of private information regarding the preferences of everyone through his information about $\tilde{\theta}$.

Theorem 2. *Let $\{e(\theta)\}_{\theta\in\Theta}$ be a collection of (degenerate) expected economies. Suppose that $u_i(\cdot, \theta)$ is concave for each i and θ and that the expected economy $e(\pi)$ has at least one strictly individually rational allocation for each $\pi \in \Delta_\Theta$. Let $\{(\{e^r(\theta)\}_{\theta\in\Theta}, \tilde{\theta}, \tilde{t}^r, P^r)\}_{r=1}^{\infty}$ be a conditionally independent sequence. Then for every $\varepsilon > 0$, there exists an integer $\hat{r} > 0$ such that for all $r > \hat{r}$, there exists an allocation z^r for the PIE $(\{e^r(\theta)\}_{\theta\in\Theta}, \tilde{\theta}, \tilde{t}^r, P^r)$ which satisfies IC, XIR, and $X_\varepsilon E$.*

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Appendix A. Proofs

A.1. Proof of Proposition 4

Choose $\varepsilon > 0$. In the next three steps, we will construct a PIE allocation with the desired properties. Throughout the proof, we will make use of the continuity, normalization, monotonicity and non-zero endowment assumptions.

Step 1. Let f be a positive Lipschitz selection for Φ and let $K_i > 0$ denote the modulus of continuity of f_i. Since each $f_i(\cdot)$ is continuous on Δ_Θ and positive on Δ_Θ, and since Δ_Θ is compact, it follows that there exists a $\lambda > 0$ such that $f_i(\pi) \geqslant \lambda$ for all i and π. For each $\pi \in \Delta_\Theta$, choose an allocation $\{\zeta_i(\pi)\}_{i\in N}$ for $e(\pi)$ satisfying

$$v_i\big(\zeta_i(\pi), \pi\big) = f_i(\pi)$$

for each i and note that each $\zeta_i(\pi) \neq 0$. Define

$$M_i = u_i\left(\sum_{j\in N} w_j, \theta\right).$$

Suppose that

$$0 < \eta < \min\{\lambda, \varepsilon\}.$$

Then for each i and π there exists $\beta_i(\pi)$ such that $0 < \beta_i(\pi) < 1$ and

$$v_i\big(\zeta_i(\pi), \pi\big) - v_i\big(\beta_i(\pi)\zeta_i(\pi), \pi\big) = \eta.$$

To see this, define

$$\psi(\beta) = v_i\big(\zeta_i(\pi), \pi\big) - v_i\big(\beta\zeta_i(\pi), \pi\big).$$

Note that

$$\psi(1) = 0 < \eta \quad \text{and} \quad \psi(0) = v_i\big(\zeta_i(\pi), \pi\big) = f_i(\pi) \geqslant \lambda > \eta.$$

The continuity of $v_i(\cdot, \pi)$ implies that $\psi(\cdot)$ is continuous and the result follows.

Step 2. Suppose that $P \in \Delta_{\Theta\times T}$ with conditionals $P_{T_{-i}}(\cdot \mid t_i) \in \Delta_{T_{-i}}$ for all i and $t_i \in T_i$. Next, define

$$\alpha_i(t_{-i} \mid t_i) = \frac{P_{T_{-i}}(t_{-i} \mid t_i)}{\|P_{T_{-i}}(\cdot \mid t_i)\|_2}$$

for each $t \in T$ (where $\|\cdot\|_2$ denotes the 2-norm) and note that

$$0 \leqslant \alpha_i(t_{-i} \mid t_i) \leqslant 1.$$

For each π, i, and t, there exists a number $\tau_i^\pi(t_{-i}, t_i) \geqslant 0$ such that

$$v_i\big(\big(1 + \tau_i^\pi(t_{-i}, t_i)\big)\beta_i(\pi)\zeta_i(\pi), \pi\big) - v_i\big(\beta_i(\pi)\zeta_i(\pi), \pi\big) = \eta\alpha_i(t_{-i} \mid t_i)$$

(this is possible because $\beta_i(\pi)\zeta_i(\pi) \neq 0$). Furthermore, $(1+\tau_i^\pi(t_{-i}, t_i))\beta_i(\pi) \leqslant 1$. (If $(1+\tau_i^\pi(t_{-i}, t_i))\beta_i(\pi) > 1$, then monotonicity implies that

$$\begin{aligned} &v_i\big(\big(1 + \tau_i^\pi(t_{-i}, t_i)\big)\beta_i(\pi)\zeta_i(\pi), \pi\big) - v_i\big(\beta_i(\pi)\zeta_i(\pi), \pi\big) > v_i\big(\zeta_i(\pi), \pi\big) - v_i\big(\beta_i(\pi)\zeta_i(\pi), \pi\big) \\ &\quad = \eta \geqslant \eta\alpha_i(t_{-i} \mid t_i) \end{aligned}$$

a contradiction.)

Defining

$$x_i(\pi \mid t) = \big(1 + \tau_i^\pi(t)\big)\beta_i(\pi)\zeta_i(\pi)$$

it follows that the collections $\{x_i(\pi \mid t)\}_{\pi\in\Delta_\Theta, t\in T}$ satisfy

$$x_i(\pi \mid t) \in \Re_+^\ell \quad \text{for each } i \quad \text{and} \quad \sum_{i\in N}\big(x_i(\pi \mid t) - w_i\big) \leqslant 0.$$

Furthermore,

$$v_i\big(x_i(\pi \mid t_{-i}, t_i), \pi\big) - v_i\big(\beta_i(\pi)\zeta_i(\pi), \pi\big) = \eta\alpha_i(t_{-i} \mid t_i)$$

for all $t \in T$. Therefore,

$$v_i\big(x_i(\pi \mid t_{-i}, t_i), \pi\big) = v_i\big(\zeta_i(\pi), \pi\big) + \eta\alpha_i(t_{-i} \mid t_i) - \eta$$

and

$$v_i\big(\zeta_i(\pi), \pi\big) \geqslant v_i\big(x_i(\pi \mid t_{-i}, t_i), \pi\big) = v_i\big(\beta_i(\pi)\zeta_i(\pi), \pi\big) + \eta\alpha_i(t_{-i} \mid t_i) \geqslant v_i\big(\zeta_i(\pi), \pi\big) - \eta.$$

Step 3. For each $t \in T^*$, let $q(t) = P_\Theta(\cdot \mid t)$ and define a PIE allocation $z(\cdot)$ as follows:

$$z_i(t) = \begin{cases} x_i(q(t) \mid t) & \text{if } t \in T^*, \\ 0 & \text{if } t \notin T^*. \end{cases}$$

We will now prove that $z(\cdot)$ has the desired properties.

Claim 1. *$z(\cdot)$ is a PIE allocation.*

Proof. This follows from the observation that

$$x_i(\pi \mid t) \in \Re^\ell_+ \quad \text{and} \quad \sum_{i \in N}\big(x_i(\pi \mid t) - w_i\big) \leqslant 0$$

for every $\pi \in \Delta_\Theta$ and $t \in T$. □

Claim 2. *For each $t \in T^*$*

$$f_i\big(P_\Theta(\cdot \mid t)\big) \geqslant v_i\big(z_i(t), P_\Theta(\cdot \mid t)\big) \geqslant f_i\big(P_\Theta(\cdot \mid t)\big) - \varepsilon.$$

Proof. This is an immediate consequence of the definition of $z_i(t)$ and the assumption that $\eta < \varepsilon$. □

Claim 3. *Let $B = 1/(2\sqrt{|T|})$ and choose δ so that*

$$0 < \delta < \min_i \left\{ \frac{B\eta}{3(K_i + M_i)} \right\}.$$

If

$$\max_i \nu_i^P \leqslant \delta \min_i \Omega_i^P,$$

then $z(\cdot)$ satisfies IC.

Proof. *Part 1.* Since

$$v_i\big(x_i(\pi \mid t_{-i}, t_i), \pi\big) = v_i\big(\zeta_i(\pi), \pi\big) + \eta\alpha_i(t_{-i} \mid t_i) - \eta$$

for each $\pi \in \Delta_\Theta$ and each $(t_{-i}, t_i) \in T$, it follows that

$$v_i\big(x_i\big(q(t_{-i}, t_i) \mid t_{-i}, t_i\big), q(t_{-i}, t_i)\big) = v_i\big(\zeta_i\big(q(t_{-i}, t_i)\big), q(t_{-i}, t_i)\big) + \eta\alpha_i(t_{-i} \mid t_i) - \eta$$

and

$$v_i\big(x_i\big(q(t_{-i}, t_i') \mid t_{-i}, t_i'\big), q(t_{-i}, t_i')\big) = v_i\big(\zeta_i\big(q(t_{-i}, t_i')\big), q(t_{-i}, t_i')\big) + \eta\alpha_i(t_{-i} \mid t_i') - \eta$$

whenever $(t_{-i}, t_i), (t_{-i}, t_i') \in T^*$. Therefore,

$$\begin{aligned} & v_i\big(x_i\big(q(t_{-i}, t_i) \mid t_{-i}, t_i\big), q\big(t_{-i}, t_i)\big) - v_i\big(x_i\big(q(t_{-i}, t_i') \mid t_{-i}, t_i')\big), q(t_{-i}, t_i')\big) \\ &\quad = v_i\big(\zeta_i\big(q(t_{-i}, t_i)\big), q(t_{-i}, t_i)\big) - v_i\big(\zeta_i\big(q(t_{-i}, t_i')\big), q(t_{-i}, t_i')\big) + \eta\big(\alpha_i(t_{-i} \mid t_i) - \alpha_i(t_{-i} \mid t_i')\big) \\ &\quad \geqslant \eta\big(\alpha_i(t_{-i} \mid t_i) - \alpha_i(t_{-i} \mid t_i')\big) - K_i\big\|P_\Theta\big(\cdot \mid t_{-i}, t_i) - P_\Theta(\cdot \mid t_{-i}, t_i')\big\|. \end{aligned}$$

Part 2. Applying the conclusion of part 1, we conclude that

$$\sum_{\substack{t_{-i}:\\(t_{-i},t_i)\in T^*\\(t_{-i},t_i')\in T^*}} \big[v_i\big(x_i\big(q(t_{-i},t_i)\mid t_{-i},t_i\big),q(t_{-i},t_i)\big)-v_i\big(x_i\big(q(t_{-i},t_i')\mid t_{-i},t_i'\big),q(t_{-i},t_i')\big)\big]P(t_{-i}\mid t_i)$$
$$\geqslant \sum_{\substack{t_{-i}:\\(t_{-i},t_i)\in T^*\\(t_{-i},t_i')\in T^*}} \big[\eta\big(\alpha_i(t_{-i}\mid t_i)-\alpha_i(t_{-i}\mid t_i')\big)-K_i\big\|P(\cdot\mid t_{-i},t_i)-P(\cdot\mid t_{-i},t_i')\big\|\big]P(t_{-i}\mid t_i)$$
$$\geqslant \sum_{\substack{t_{-i}:\\(t_{-i},t_i)\in T^*\\(t_{-i},t_i')\in T^*}} \eta\big(\alpha_i(t_{-i}\mid t_i)-\alpha_i(t_{-i}\mid t_i')\big)P(t_{-i}\mid t_i)-3K_i v_i^P,$$

where the last inequality follows from the observation that $\|P(\cdot\mid t_{-i},t_i)-P(\cdot\mid t_{-i},t_i')\|\leqslant 2$.

Part 3. The normalization assumption implies that $v_i(0,q(t))=0$ for each $t\in T^*$. Therefore,

$$\sum_{\substack{t_{-i}:\\(t_{-i},t_i)\in T^*\\(t_{-i},t_i')\notin T^*}} \big[v_i\big(x_i\big(q(t_{-i},t_i)\mid t_{-i},t_i\big),q(t_{-i},t_i)\big)-v_i\big(0,q(t_{-i},t_i)\big)\big]P(t_{-i}\mid t_i)$$
$$= \sum_{\substack{t_{-i}:\\(t_{-i},t_i)\in T^*\\(t_{-i},t_i')\notin T^*}} \big(v_i\big(\zeta_i\big(q(t_{-i},t_i)\big),q(t_{-i},t_i)\big)+\eta\alpha_i(t_{-i}\mid t_i)-\eta\big)P(t_{-i}\mid t_i)$$
$$\geqslant \sum_{\substack{t_{-i}:\\(t_{-i},t_i)\in T^*\\(t_{-i},t_i')\notin T^*}} \big(\eta\alpha_i(t_{-i}\mid t_i)-\eta+\lambda\big)P(t_{-i}\mid t_i)\geqslant \sum_{\substack{t_{-i}:\\(t_{-i},t_i)\in T^*\\(t_{-i},t_i')\notin T^*}} \eta\alpha_i(t_{-i}\mid t_i)P(t_{-i}\mid t_i).$$

Part 4. Finally, we again make the observation that $\|P(\cdot\mid t_{-i},t_i))-P(\cdot\mid t_{-i},t_i')\|\leqslant 2$ and conclude that

$$\sum_{\substack{t_{-i}:\\(t_{-i},t_i)\in T^*\\(t_{-i},t_i')\in T^*}} \big[v_i\big(x_i\big(q(t_{-i},t_i')\mid t_{-i},t_i'\big),q(t_{-i},t_i')\big)-v_i\big(x_i\big(q(t_{-i},t_i')\mid t_{-i},t_i'\big),q(t_{-i},t_i)\big)\big]P(t_{-i}\mid t_i)$$
$$= \sum_{\substack{t_{-i}:\\(t_{-i},t_i)\in T^*\\(t_{-i},t_i')\in T^*}} \sum_\theta u_i\big(x_i\big(q(t_{-i},t_i')\mid t_{-i},t_i'\big),\theta\big)\big[P(\theta\mid t_{-i},t_i')-P(\theta\mid t_{-i},t_i)\big]P(t_{-i}\mid t_i)$$
$$\geqslant -M_i \sum_{\substack{t_{-i}:\\(t_{-i},t_i)\in T^*\\(t_{-i},t_i')\in T^*}} \big\|P(\cdot\mid t_{-i},t_i)-P(\cdot\mid t_{-i},t_i')\big\|P(t_{-i}\mid t_i)$$
$$\geqslant -3M_i v_i^P.$$

Part 5. Let X be a finite set with cardinality k and let $p,q\in\Delta_X$. Then

$$\left[\frac{p}{\|p\|_2}-\frac{q}{\|q\|_2}\right]\cdot p=\frac{\|p\|_2}{2}\left\|\frac{p}{\|p\|_2}-\frac{q}{\|q\|_2}\right\|_2^2\geqslant\frac{1}{2\sqrt{k}}\left\|\frac{p}{\|p\|_2}-\frac{q}{\|q\|_2}\right\|_2^2.$$

To complete the proof of Claim 3, we combine parts 2, 3, and 4 to obtain

$$\sum_\theta\sum_{t_{-i}}\big[u_i\big(z_i(t_{-i},t_i),\theta\big)-u_i\big(z_i(t_{-i},t_i'),\theta\big)\big]P(\theta,t_{-i}\mid t_i)$$
$$= \sum_{\substack{t_{-i}:\\(t_{-i},t_i)\in T^*}} \big[v_i\big(x_i\big(q(t_{-i},t_i)\mid t_{-i},t_i\big),q(t_{-i},t_i)\big)-v_i\big(z_i(t_{-i},t_i'),q(t_{-i},t_i)\big)\big]P(t_{-i}\mid t_i)$$

$$= \sum_{\substack{t_{-i}: \\ (t_{-i},t_i)\in T^* \\ (t_{-i},t_i')\in T^*}} \big[v_i\big(x_i\big(q(t_{-i},t_i) \mid t_{-i},t_i\big), q(t_{-i},t_i)\big) - v_i\big(x_i\big(q(t_{-i},t_i') \mid t_{-i},t_i'\big), q(t_{-i},t_i')\big)\big] P(t_{-i} \mid t_i)$$

$$+ \sum_{\substack{t_{-i}: \\ (t_{-i},t_i)\in T^* \\ (t_{-i},t_i')\in T^*}} \big[v_i\big(x_i\big(q(t_{-i},t_i') \mid t_{-i},t_i'\big), q(t_{-i},t_i')\big) - v_i\big(x_i\big(q(t_{-i},t_i') \mid t_{-i},t_i'\big), q(t_{-i},t_i)\big)\big] P(t_{-i} \mid t_i)$$

$$+ \sum_{\substack{t_{-i}: \\ (t_{-i},t_i)\in T^* \\ (t_{-i},t_i')\notin T^*}} \big[v_i\big(x_i\big(q(t_{-i},t_i) \mid t_{-i},t_i\big), q(t_{-i},t_i)\big) - v_i\big(0, q(t_{-i},t_i)\big)\big] P(t_{-i} \mid t_i)$$

$$\geqslant \sum_{\substack{t_{-i}: \\ (t_{-i},t_i)\in T^* \\ (t_{-i},t_i')\in T^*}} \eta\big(\alpha_i(t_{-i} \mid t_i) - \alpha_i(t_{-i} \mid t_i')\big) P(t_{-i} \mid t_i) + \sum_{\substack{t_{-i}: \\ (t_{-i},t_i)\in T^* \\ (t_{-i},t_i')\notin T^*}} \eta\alpha_i(t_{-i} \mid t_i) P(t_{-i} \mid t_i) - 3(K_i + M_i)\nu_i^P.$$

Since $\alpha_i(t_{-i} \mid t_i') = 0$ if $(t_{-i}, t_i') \notin T^*$, it follows from part 5 that

$$\sum_{\substack{t_{-i}: \\ (t_{-i},t_i)\in T^* \\ (t_{-i},t_i')\in T^*}} \eta\big(\alpha_i(t_{-i} \mid t_i) - \alpha_i(t_{-i} \mid t_i')\big) P(t_{-i} \mid t_i) + \sum_{\substack{t_{-i}: \\ (t_{-i},t_i)\in T^* \\ (t_{-i},t_i')\notin T^*}} \eta\alpha_i(t_{-i} \mid t_i) P(t_{-i} \mid t_i)$$

$$= \sum_{\substack{t_{-i}: \\ (t_{-i},t_i)\in T^*}} \eta\big(\alpha_i(t_{-i} \mid t_i) - \alpha_i(t_{-i} \mid t_i')\big) P(t_{-i} \mid t_i)$$

$$= \sum_{\substack{t_{-i}: \\ (t_{-i},t_i)\in T}} \eta\big(\alpha_i(t_{-i} \mid t_i) - \alpha_i(t_{-i} \mid t_i')\big) P(t_{-i} \mid t_i)$$

$$\geqslant \eta B \Omega_i^P.$$

Therefore,

$$\sum_{\theta} \sum_{t_{-i}} \big[u_i\big(z_i(t_{-i},t_i),\theta\big) - u_i\big(z_i(t_{-i},t_i'),\theta\big)\big] P(\theta, t_{-i} \mid t_i) \geqslant \eta B \Omega_i^P - 3(K_i + M_i)\nu_i^P \geqslant 0.$$

This completes the proof of Claim 3. □

A.2. Proof of Lemma 2

For each $\pi \in \Delta_\Theta$, let $v(w,\pi) := (v_1(w_1,\pi), \ldots, v_n(w_n,\pi))$ and define

$$\mu(\pi) = \arg\max\big[\mu \mid v(w,\pi) + \mu e \in \Phi(\pi)\big].$$

Note that $\mu(\pi)$ is well defined since $\Phi(\pi)$ is compact and that $\mu(\pi) \geqslant 0$. For each $\pi \in \Delta_\Theta$, choose a feasible allocation $x(\pi)$ for $e(\pi)$ satisfying $v_i(x_i(\pi),\pi) = v_i(w_i,\pi) + \mu(\pi)e$ for each $i \in N$. Finally, let $f : \Delta_\Theta \to \Re^n$ be the map defined by $f_i(\pi) = v_i(x_i(\pi),\pi)$ for each i. Clearly, f satisfies individual rationality since $\mu(\pi) \geqslant 0$. To show that $f(\pi) \in \Phi^0(\pi)$, it suffices to show that $x(\pi)$ is efficient in $e(\pi)$. Suppose that $x(\pi)$ is not efficient in $e(\pi)$. Then there exists an allocation $y = (y_1, \ldots, y_n)$ satisfying $v_i(y_i,\pi) > v_i(x_i(\pi),\pi)$ for all i. Choose i_0 so that

$$v_{i_0}(y_{i_0},\pi) - v_{i_0}(w_{i_0},\pi) = \min_{i\in N}\big[v_i(y_i,\pi) - v_i(w_i,\pi)\big] := \sigma$$

and note that $\sigma > \mu(\pi)$. For each $i \neq i_0$, there exists a $\beta_i \in [0,1]$ such that $v_i(\beta_i y_i,\pi) - v_i(w_i,\pi) = \sigma$ (this follows from monotonicity, continuity and normalization). Defining $z_{i_0} = y_{i_0}$ and $z_i = \beta_i y_i$, otherwise, it follows that $(z_1, \ldots, z_n)$ is a feasible allocation for $e(\pi)$ and that

$$\big(v_1(z_1,\pi), \ldots, v_n(z_n,\pi)\big) = v(w,\pi) + \sigma e.$$

Since $\sigma > \mu(\pi)$, we arrive at a contradiction and, therefore, $x(\pi)$ is efficient in $e(\pi)$.

Next, suppose that the economy $e(\pi)$ satisfies λ-IR for some $\lambda > 0$. Then using precisely the same argument used above in the proof of efficiency, we can construct a feasible allocation $(z_1, \ldots, z_n)$ for $e(\pi)$ satisfying

$$\big(v_1(z_1, \pi), \ldots, v_n(z_n, \pi)\big) = v(w, \pi) + \sigma e.$$

and $\sigma \geqslant \lambda$. Therefore, $\mu(\pi) \geqslant \lambda$ and it follows that $f_i(\pi) - v_i(w_i, \pi) = \mu(\pi) \geqslant \lambda$.

Finally, we prove the following claim.

Claim. *Let*

$$M = \max_i \max_\theta u_i\left(\sum_{j\in N} w_j, \theta\right)$$

and note that $M > 0$. *For each* i, *the mapping* f_i *is uniformly Lipschitz on* Δ_Θ *with modulus* $K_i = 3M$.

Proof. Choose $\pi, \pi' \in \Delta_\Theta$ and w.l.o.g., suppose that $\mu(\pi) \leqslant \mu(\pi')$. To prove the claim, it is enough to show that $\mu(\pi') \leqslant \mu(\pi) + 2M\|\pi - \pi'\|$ since we would then conclude that, for each i,

$$\begin{aligned}\big|f_i(\pi) - f_i(\pi')\big| &= \big|\big(v_i(w_i, \pi) + \mu(\pi)\big) - \big(v_i(w_i, \pi') + \mu(\pi')\big)\big| \\ &\leqslant \big|v_i(w_i, \pi) - v_i(w_i, \pi')\big| + \big|\mu(\pi) - \mu(\pi')\big| \leqslant M\|\pi - \pi'\| + 2M\|\pi - \pi'\| \\ &= 3M\|\pi - \pi'\|.\end{aligned}$$

First, define $y = (v_1(x_1(\pi'), \pi), \ldots, v_n(x_n(\pi'), \pi))$ and observe that

$$\big|y_i - f_i(\pi')\big| = \big|v_i\big(x_i(\pi'), \pi\big) - v_i\big(x_i(\pi'), \pi'\big)\big| \leqslant M\|\pi - \pi'\|.$$

To complete the proof, suppose that $\mu(\pi') > \mu(\pi) + 2M\|\pi - \pi'\|$. We claim that $y_i > v_i(x_i(\pi), \pi)$ for each i, contradicting the efficiency of the allocation $x(\pi)$ in $e(\pi)$. To see this, suppose that $y_i \leqslant v_i(x_i(\pi), \pi) = v_i(w_i, \pi) + \mu(\pi)$ for some i. Using the fact that

$$v_i(w_i, \pi') - v_i(w_i, \pi) \geqslant -\big|v_i(w_i, \pi') - v_i(w_i, \pi)\big| \geqslant -M\|\pi - \pi'\|$$

we deduce that

$$\begin{aligned}f_i(\pi') - y_i &= v_i(w_i, \pi') + \mu(\pi') - y_i \\ &> v_i(w_i, \pi') + \mu(\pi) + 2M\|\pi - \pi'\| - y_i \\ &= \big(v_i(w_i, \pi) + \mu(\pi) - y_i\big) + v_i(w_i, \pi') - v_i(w_i, \pi) + 2M\|\pi - \pi'\| \\ &\geqslant v_i(w_i, \pi') - v_i(w_i, \pi) + 2M\|\pi - \pi'\| \\ &\geqslant 2M\|\pi - \pi'\| - M\|\pi - \pi'\| \\ &= M\|\pi - \pi'\|.\end{aligned}$$

This is impossible since $|y_i - f_i(\pi')| \leqslant M\|\pi - \pi'\|$. Therefore, $y_i > v_i(x_i(\pi), \pi)$ for each i. Hence, the hypothesis that $\mu(\pi') > \mu(\pi) + 2M\|\pi - \pi'\|$ leads to a contradiction and the proof of the claim is complete. □

A.3. Proof of Theorem 2

Let $\{(\{e^r(\theta)\}_{\theta\in\Theta}, \tilde\theta, \tilde t^r, P^r)\}_{r=1}^\infty$ be a conditionally independent sequence and suppose that each $u_i(\cdot, \theta)$ is concave. Since $P(\theta, t) > 0$ for each $(\theta, t) \in \Theta \times T$, it follows from the definition of a conditionally independent sequence that $T^r = (T^r)^*$ for all r. Choose $\varepsilon > 0$.

Step 1. From the claim in step 1 of the proof of Theorem 2 in McLean and Postlewaite (2002), it follows that, for every $\rho > 0$, there exists an integer $\hat{r}$ such that for all $r > \hat{r}$,

$$v_{i,s}^{P^r} \leqslant \rho.$$

Step 2. For the 'basic' PIE $(\{e(\theta)\}_{\theta\in\Theta}, \tilde{\theta}, \tilde{t}, P)$ (i.e., the PIE with $r = 1$), we can apply Lemma 2 and conclude that there exists a strictly individually rational Lipschitz selection f for Φ^0 with the property that $f_i(\pi) - v_i(w_i, \pi) > 0$ for each i and for each π. Since the function $\pi \to f_i(\pi) - v_i(w_i, \pi)$ is continuous and positive on Δ_Θ, and since Δ_Θ is compact, it follows that there exists a $\lambda > 0$ such that $f_i(\pi) - v_i(w_i, \pi) \geqslant \lambda$ for all i and π. Let $K_i > 0$ denote the modulus of continuity of f_i and define

$$M_i = u_i\left(\sum_{j\in N} w_j, \theta\right).$$

For each $\pi \in \Delta_\Theta$, choose an allocation $\{\zeta_i(\pi)\}_{i\in N}$ for $e(\pi)$ satisfying

$$v_i\big(\zeta_i(\pi), \pi\big) = f_i(\pi)$$

for each i and note that each $\zeta_i(\pi) \neq 0$. Finally, choose η such that $0 < \eta < \min\{\varepsilon, \lambda\}$.

Suppose that, for each $\pi \in \Delta_\Theta$, the allocation $\{\zeta_i(\pi)\}_{i\in N}$ is an allocation for $e(\pi)$ satisfying

$$\big(v_1\big(\zeta_1(\pi), \pi\big), \ldots, v_n\big(\zeta_n(\pi), \pi\big)\big) = f(\pi).$$

Duplicating verbatim the steps 1 and 2 in the proof of Proposition 3, we construct

$$x_i(\pi \mid t) = \big(1 + \tau_i^\pi(t)\big)\beta_i(\pi)\zeta_i(\pi)$$

for each π and each $t \in T$. From the construction, it follows that the collections $\{x_i(\pi \mid t)\}_{\pi\in\Delta_\Theta, t\in T}$ satisfy

(i) $x_i(\pi \mid t) \in \Re_+^\ell$ and $\sum_{i\in N}(x_i(\pi \mid t) - w_i) \leqslant 0$,

(ii) $v_i(\zeta_i(\pi), \pi) \geqslant v_i(x_i(\pi \mid t_{-i}, t_i), \pi) \geqslant v_i(\zeta_i(\pi), \pi) - \eta$, and

(iii) $v_i(x_i(\pi \mid t_{-i}, t_i), \pi) = v_i(\zeta_i(\pi), \pi) + \eta\alpha_i(t_{-i} \mid t_i) - \eta$,
where

$$\alpha_i(t_{-i} \mid t_i) = \frac{P_{T_{-i}}(t_{-i} \mid t_i)}{\|P_{T_{-i}}(\cdot \mid t_i)\|_2}.$$

Step 3. We now use this construction for the 'basic' PIE $(\{e(\theta)\}_{\theta\in\Theta}, \tilde{\theta}, \tilde{t}, P)$ to define a mechanism for the replica PIE $(\{e^r(\theta)\}_{\theta\in\Theta}, \tilde{\theta}, \tilde{t}^r, P^r)$. For each $r \geqslant 1$ and each $t \in T^r$, let $q(t^r) = P_\Theta^r(\cdot \mid t^r)$ and define a collection

$$z_{i,s}^r\big(t^r\big) = x_i\big(q\big(t^r\big) \mid t^r(s)\big) \quad \text{if } t^r \in T^r.$$

It follows from the construction of $x_i(\pi \mid t)$ in Step 2 above that

$$v_i\big(\zeta_i\big(q\big(t^r\big)\big), q\big(t^r\big)\big) \geqslant v_i\big(z_{i,s}^r\big(t^r\big), q\big(t^r\big)\big) \geqslant v_i\big(\zeta_i\big(q\big(t^r\big)\big), q\big(t^r\big)\big) - \eta.$$

To complete the proof, we will show that mechanism $z^r(\cdot) = \{z_{i,s}^r(\cdot)\}_{(i,s)\in N\times J_r}$ is an individually rational, incentive compatible, ε-efficient PIE allocation for $(\{e^r(\theta)\}_{\theta\in\Theta}, \tilde{\theta}, \tilde{t}^r, P^r)$ whenever r is sufficiently large.

Claim 1. *For each positive integer r, the mechanism $z^r(\cdot)$ is a PIE allocation for $(\{e^r(\theta)\}_{\theta\in\Theta}, \tilde{\theta}, \tilde{t}^r, P^r)$.*

Proof. Since

$$x_i(\pi \mid t) \in \Re_+^\ell \quad \text{and} \quad \sum_{i\in N}\big(x_i(\pi \mid t) - w_i\big) \leqslant 0$$

for every $\pi \in \Delta_\Theta$ and $t \in T$, it follows that

$$\sum_{s=1}^{r}\sum_{i\in N}\big(z_{i,s}^r\big(t^r\big) - w_i\big) = \sum_{s=1}^{r}\sum_{i\in N}\big(x_i\big(q\big(t^r\big) \mid t^r(s)\big) - w_i\big) \leqslant 0$$

whenever $t^r \in T^r$. □

Claim 2. *For each positive integer r, the mechanism $z^r(\cdot)$ is ex post IR and ex post ε-efficient for the PIE $(\{e^r(\theta)\}_{\theta\in\Theta}, \tilde{\theta}, \tilde{t}^r, P^r)$.*

Proof. Since $v_i(\zeta_i(q(t^r)), q(t^r)) - v_i(w_i, q(t^r)) = f(q(t^r)) - v_i(w_i, q(t^r)) \geqslant \lambda$ for each $t^r \in T^r$, it follows that

$$\begin{aligned} v_i\big(z^r_{i,s}\big(t^r\big), q\big(t^r\big)\big) - v_i\big(w_i, q\big(t^r\big)\big) &= v_i\big(z_{i,s}\big(t^r\big), q\big(t^r\big)\big) - v_i\big(\zeta_i\big(q\big(t^r\big)\big), q\big(t^r\big)\big) \\ &\quad + v_i\big(\zeta_i\big(q\big(t^r\big)\big), q\big(t^r\big)\big) - v_i\big(w_i, q\big(t^r\big)\big) \\ &\geqslant \lambda - \eta > 0 \end{aligned}$$

for each i so $z^r(t^r)$ is individually rational for each $t^r \in T^r$. That is, $z^r(\cdot)$ satisfies XIR for the replica PIE e^r.

To show that $z^r(\cdot)$ satisfies $\mathrm{X}_\varepsilon\mathrm{E}$ in the replica PIE e^r, suppose that $t^r \in T^r$ and that $y^r(\cdot)$ is a PIE allocation for e^r satisfying

$$v_i\big(y^r_{is}\big(t^r\big), q\big(t^r\big)\big) > v_i\big(z^r_{is}\big(t^r\big), q\big(t^r\big)\big) + \varepsilon$$

for each (i,s). For each i, let

$$\bar{y}_i = \frac{1}{r}\sum_{s=1}^{r} y^r_{is}\big(t^r\big)$$

and therefore,

$$\sum_{i=1}^{n} \bar{y}_i = \frac{1}{r}\sum_{i=1}^{n}\sum_{s=1}^{r} y^r_{is}\big(t^r\big) \leqslant \sum_{i=1}^{n} w_i.$$

Since each $v_i(\cdot, q(t^r))$ is concave and

$$v_i\big(z^r_{is}\big(t^r\big), q\big(t^r\big)\big) \geqslant v_i\big(\zeta_i\big(q\big(t^r\big)\big), q\big(t^r\big)\big) - \eta,$$

it follows that

$$\begin{aligned} v_i\big(\bar{y}_i, q\big(t^r\big)\big) &\geqslant \frac{1}{r}\sum_s v_i\big(y^r_{is}\big(t^r\big), q\big(t^r\big)\big) > \frac{1}{r}\sum_s v_i\big(z^r_{is}\big(t^r\big), q\big(t^r\big)\big) + \varepsilon \geqslant v_i\big(\zeta_i\big(q\big(t^r\big)\big), q\big(t^r\big)\big) - \eta + \varepsilon \\ &> v_i\big(\zeta_i\big(q\big(t^r\big)\big), q\big(t^r\big)\big) \end{aligned}$$

for each i and we conclude that $\{\zeta_i(q(t^r))\}_{i\in N}$ is not Pareto optimal in $e(q(t^r))$, a contradiction. □

Claim 3. *There exists a positive integer $\hat{r}$ such that, whenever $r > \hat{r}$, the mechanism $z^r(\cdot)$ satisfies IC.*

Proof. Let $K = \max_i K_i$, $M = \max_i M_i$ and define $B = 1/(2\sqrt{|T|})$. Assumption (d) in the definition of a conditionally independent sequence implies that $\Lambda_i^P > 0$. Hence, we can apply the result of step 1 and conclude that there exists a positive integer $\hat{r}$ such that $\eta B \Lambda_i^P - 3(K+M)v_i^{P^r} > 0$ whenever $r > \hat{r}$. We will show that the mechanism $z^r(\cdot)$ satisfies IC whenever $r > \hat{r}$. Choose $(i,s) \in N \times J_r$. Since

$$v_i\big(x_i(\pi \mid t_{-i}, t_i), \pi\big) = v_i\big(\zeta_i(\pi), \pi\big) + \eta\alpha_i(t_{-i} \mid t_i) - \eta$$

for each $\pi \in \Delta_\Theta$ and each $(t_{-i}, t_i) \in T$, it follows that

$$\begin{aligned} u_i\big(z^r_{i,s}\big(t^r_{-(i,s)}, t_i\big), \theta\big) &= v_i\big(x_i\big(q\big(t^r_{-(i,s)}, t_i\big) \bigm| t^r_{-i}(s), t_i\big), q\big(t^r_{-(i,s)}, t_i\big)\big) \\ &= v_i\big(\zeta_i\big(q\big(t^r_{-(i,s)}, t_i\big)\big), q\big(t^r_{-(i,s)}, t_i\big)\big) + \eta\alpha_i\big(t^r_{-i}(s) \mid t_i\big) - \eta \end{aligned}$$

and

$$\begin{aligned} u_i\big(z^r_{i,s}\big(t^r_{-(i,s)}, t'_i\big), \theta\big) &= v_i\big(x_i\big(q\big(t^r_{-(i,s)}, t'_i\big) \bigm| t^r_{-i}(s), t'_i\big), q\big(t^r_{-(i,s)}, t'_i\big)\big) \\ &= v_i\big(\zeta_i\big(q\big(t^r_{-(i,s)}, t'_i\big)\big), q\big(t^r_{-(i,s)}, t'_i\big)\big) + \eta\alpha_i\big(t^r_{-i}(s) \mid t'_i\big) - \eta \end{aligned}$$

for each $(t^r_{-(i,s)}, t_i), (t^r_{-(i,s)}, t'_i) \in T^r$. Therefore,

$$u_i\big(z^r_{i,s}\big(t^r_{-(i,s)}, t_i\big), \theta\big) - u_i\big(z^r_{i,s}\big(t^r_{-(i,s)}, t'_i\big), \theta\big)$$
$$\geqslant \eta\big(\alpha_i\big(t^r_{-i}(s) \mid t_i\big) - \alpha_i\big(t^r_{-i}(s) \mid t'_i\big)\big) - K_i \big\| P^r_\Theta\big(\cdot \mid t^r_{-(i,s)}, t_i\big) - P^r_\Theta\big(\cdot \mid t^r_{-(i,s)}, t'_i\big)\big\|$$

so that

$$\sum_\theta \sum_{\substack{t^r_{-(i,s)}:\\ (t_{-(i,s)}, t_i) \in T^r}} \big[u_i\big(z^r_{i,s}\big(t_{-(i,s)}, t_i\big), \theta\big) - u_i\big(z^r_{i,s}\big(t_{-(i,s)}, t'_i\big), \theta\big)\big] P^r\big(\theta, t^r_{-i,s} \mid t_i\big)$$
$$\geqslant \sum_{\substack{t^r_{-(i,s)}:\\ (t_{-(i,s)}, t_i) \in T^r}} \eta\big(\alpha_i\big(t^r_{-i}(s) \mid t_i\big) - \alpha_i\big(t^r_{-i}(s) \mid t'_i\big)\big) P^r\big(t^r_{-(i,s)} \mid t_i\big) - 3K_i \nu_i^{P^r}.$$

Since

$$\sum_{\substack{t^r_{-(i,s)}:\\ (t_{-(i,s)}, t_i) \in T^r}} \eta\big(\alpha_i\big(t^r_{-i}(s) \mid t_i\big) - \alpha_i\big(t^r_{-i}(s) \mid t'_i\big)\big) P^r\big(t^r_{-i,s} \mid t_i\big)$$
$$= \eta \sum_{t^r(1) \in T} \cdots \sum_{t^r_{-i}(s) \in T_{-i}} \cdots \sum_{t^r(r) \in T} \big(\alpha_i\big(t^r_{-i}(s) \mid t_i\big) - \alpha_i\big(t^r_{-i}(s) \mid t'_i\big)\big) P^r\big(t^r(1), \ldots, t^r_{-i}(s), \ldots, t^r(r) \mid t_i\big)$$
$$= \eta \sum_{t^r_{-i}(s) \in T_{-i}} \big(\alpha_i\big(t^r_{-i}(s) \mid t_i\big) - \alpha_i\big(t^r_{-i}(s) \mid t'_i\big)\big) P\big(t_{-i}(s) \mid t_i\big)$$
$$\geqslant \eta B \Omega^P_i,$$

we conclude that

$$\sum_\theta \sum_{t^r_{-i,s}} \big[u_i\big(z^r_{i,s}(t_{-i,s}, t_i), \theta\big) - u_i\big(z^r_{i,s}(t_{-i,s}, t'_i), \theta\big)\big] P^r\big(\theta, t^r_{-i,s} \mid t_i\big) \geqslant \eta B \Lambda^P_i - 3(K+M)\nu_i^{P^r} > 0$$

and the proof of incentive compatibility is complete. □

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