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Informational size, incentive compatibility, and the core of a game with incomplete information

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Abstract

We study the ex ante incentive compatible core, and provide conditions under which the ex ante incentive compatible core is nonempty when agents are informationally small in the sense of McLean and Postlewaite (2002a, Econometrica, 70, 2421–2453).

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1. Introduction

While most of the game theoretic literature dealing with asymmetric information has focused primarily on noncooperative games, there is an expanding literature that studies the core in the presence of incomplete information, most of which is surveyed in Forges et al. (2002). Several different definitions of the core in incomplete information environments are possible depending on whether incentive constraints are imposed and on whether coalitional decisions are made ex ante (before agents learn their types) or at the interim stage (after agents learn their types). Our analysis deals with the ex ante incentive compatible core which, as the name suggests, treats the case in which decisions are made at the ex ante stage and incentive constraints are taken to matter.

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While the core with complete information is nonempty under quite general circumstances, Vohra (1999) and Forges et al. (2002) have recently shown that the ex ante incentive compatible core may be empty in well-behaved exchange economies. One goal of this paper is to provide something akin to a continuity theorem: when the asymmetry of information among the agents in an exchange economy is small, the approximate ex ante incentive compatible core is nonempty.

We study the core of pure a exchange economy in which agents are asymmetrically informed. Specifically, the agents' utility functions will depend on an underlying but unobserved state of nature and each agent will receive a private signal that is correlated with the state of nature. Roughly speaking, this corresponds to a "common value" model in which signals do not directly affect the underlying payoff functions but do affect expected utilities.

Vohra (1999) has shown that the ex ante incentive compatible core is nonempty under strong conditions that limit the degree of informational asymmetry among agents. Postlewaite and Schmeidler (1986) introduced the notion of nonexclusive information, under which no single agent's information is necessary to identify the correct state of the world. Vohra shows that, if information is nonexclusive, then the ex ante incentive compatible core is nonempty. When an information structure is nonexclusive, then the information of one agent cannot affect the conditional probability distribution over states when the information of the other agents is known. For the common value model that we study in this paper, we use the concept of informational size that we developed in McLean and Postlewaite (2002a). This notion of informational size extends the nonexclusive information concept by quantifying the degree to which a given agent's information can affect the probability distribution over states, given all other agents' information. Roughly speaking, an agent will be informationally small if, given other agents' information, it is very likely that the given agent's information will have a small effect on the probability distribution over states.

Our theorem on the nonemptiness of the ex ante incentive compatible core depends on two other aspects of the information structure used in McLean and Postlewaite (2002a): aggregate uncertainty and the variability of agents' beliefs. Aggregate uncertainty quantifies the degree to which the aggregate of agents' information resolves all uncertainty regarding the state of nature. Roughly speaking, the variability of an agent's beliefs quantifies the difference in the conditional distributions on the state space induced by the different types he might be. We show that generically, the ex ante incentive compatible core is nonempty if all agents are informationally small relative to the variability of their beliefs and aggregate uncertainty. We further show that, in a replica economy, an agent's informational size goes to zero as the number of agents increases.

2. Basic notation

Throughout the paper, let $J_q = \{1, \ldots, q\}$ for each positive integer q and let $\|\cdot\|$ denote the 1-norm unless specified otherwise. Let $N = \{1, 2, \ldots, n\}$ denote the set of *economic agents*. Let $\Theta = \{\theta_1, \ldots, \theta_m\}$ denote the (finite) *state space* and let $T_1, T_2, \ldots, T_n$ be finite sets where T_i represents the set of possible *signals* that agent i might receive. For each

$S \subseteq N$, let $T_S \equiv \prod_{i\in S} T_i$. Elements of T_S will be written t_S. For notational simplicity, we will simply write T for T_N and t for t_N. If $t \in T$, then we will often write $t = (t_{N\setminus S}, t_S)$. If X is a finite set, define

$$\Delta_X := \left\{\rho \in \Re^{|X|} \;\middle|\; \rho(x) \geqslant 0,\ \sum_{x\in X} \rho(x) = 1\right\}$$

and

$$\Delta_X^0 := \left\{\rho \in \Re^{|X|} \;\middle|\; \rho(x) > 0,\ \sum_{x\in X} \rho(x) = 1\right\}.$$

In our model, nature chooses an element $\theta \in \Theta$. The state of nature is unobservable but each agent i receives a "signal" t_i that is correlated with nature's choice of θ. More formally, let $(\tilde{\theta}, \tilde{t}_1, \tilde{t}_2, \ldots, \tilde{t}_n)$ be an $(n+1)$-dimensional random vector taking values in $\Theta \times T$ with associated distribution $P \in \Delta_{\Theta\times T}$ where

$$P(\theta, t_1, \ldots, t_n) = \text{Prob}\{\tilde{\theta} = \theta, \tilde{t}_1 = t_1, \ldots, \tilde{t}_n = t_n\}.$$

We will make the following assumption regarding the marginal distributions:[1]

For each $\theta \in \Theta$,

$$P(\theta) = \text{Prob}\{\tilde{\theta} = \theta\} > 0$$

and for each $t = (t_1, \ldots, t_n) \in T$,

$$P(t) = \text{Prob}\{\tilde{t}_1 = t_1, \ldots, \tilde{t}_n = t_n\} > 0.$$

If $t \in T$, let $P_\Theta(\cdot|t) \in \Delta_\Theta$ denote the induced conditional probability measure on Θ. Let $\chi_\theta \in \Delta_\Theta$ denote the degenerate measure that puts probability one on state θ.

2.1. Economies

The *consumption set* of each agent is $\Re^\ell_+$ and for each $\theta \in \Theta$, $w_i \in \Re^\ell_{++}$ denotes the (state independent) initial endowment of agent i in state θ. The preferences of agent i are given by a utility function $u_i : \Re^\ell_+ \times \Theta \to \Re$ where $u_i(\cdot, \theta)$ is the utility function of agent i in state θ. The following assumptions are maintained throughout the paper:

(i) $u_i(\cdot, \theta)$ is continuous and concave;
(ii) $u_i(0, \theta) = 0$;
(iii) $u_i(\cdot, \theta)$ is (strongly) monotonic: if $x, y \in \Re^\ell_+$, $x \geqslant y$ and $x \neq y$, then $u_i(x, \theta) > u_i(y, \theta)$.

The collection $(\{u_i, w_i\}_{i\in N}, \tilde{\theta}, \tilde{t}, P)$ will be called a *private information economy* (PIE for short). It will be assumed that the data defining the PIE is common knowledge. A *private information economy allocation* $z = (z_1, z_2, \ldots, z_n)$ for the PIE is a collection of functions $z_i : T \to \Re^\ell_+$ satisfying $\sum_{i\in N}(z_i(t) - w_i) \leqslant 0$ for all $t \in T$. We will not

[1] The assumption that $P(t) > 0$ for all $t \in T$ is relaxed in McLean and Postlewaite (2002b).

distinguish between $w_i \in \Re^{\ell}_{++}$ and the constant allocation that assigns the bundle w_i to agent i for all $t \in T$.

For each $\pi \in \Delta_\Theta$, the collection $(\pi, \{u_i, w_i\}_{i\in N})$ defines an associated Arrow–Debreu economy with state contingent commodities. A commodity vector for agent i in this Arrow–Debreu economy is a vector of state contingent bundles in $\Re^{\ell m}_{+}$ and is written as

$$\big(x_i(\theta_1), \ldots, x_i(\theta_m)\big).$$

The initial endowment of agent i is the vector $\widehat{w}_i = (w_i, \ldots, w_i) \in R^{\ell m}_{++}$ and the utility of agent i is the function $v_i : \Re^{\ell m}_{+} \to \Re$ defined for each $(x_i(\theta_1), \ldots, x_i(\theta_m)) \in \Re^{\ell m}_{+}$ as follows:

$$v_i\big(x_i(\theta_1), \ldots, x_i(\theta_m)\big) := \sum_{k=1}^{m} u_i\big(x_i(\theta_k); \theta_k\big)\pi(\theta_k).$$

The Arrow–Debreu economy with commodity bundles, endowments and utilities defined in this manner will be denoted $E(\pi, \{u_i, w_i\}_{i\in N})$. The economy $E(\pi, \{u_i, w_i\}_{i\in N})$ will play an important technical role in our work and for this reason, we will refer to $E(\pi, \{u_i, w_i\}_{i\in N})$ as the *π-auxiliary economy*, or auxiliary economy for short. Finally, each PIE $(\{u_i, w_i\}_{i\in N}, \tilde{\theta}, \tilde{t}, P)$ gives rise to a natural auxiliary economy $(P_\Theta, \{u_i, w_i\}_{i\in N})$ where P_Θ is the marginal of P on Θ.

For each endowment profile $w = (w_1, \ldots, w_n)$ and $S \subseteq N$, the set of S-feasible allocations in $E(\pi, \{u_i, w_i\}_{i\in N})$ is the set

$$\Phi_S(w) = \left\{\big(x_i(\theta_1), \ldots, x_i(\theta_m)\big)_{i\in S} \in \big(\Re^{\ell m}_{+}\big)^{|S|} \;\Big|\; \sum_{i\in S} x_i(\theta) \leqslant \sum_{i\in S} w_i \text{ for each } \theta \in \Theta\right\}.$$

Throughout the remainder of this section, let $E = E(\pi, \{u_i, w_i\}_{i\in N})$ be an auxiliary economy. The economy E gives rise to an NTU game in a natural way by defining the set of attainable payoffs as

$$\begin{aligned} V^E(S) = \big\{(y_i)_{i\in S} \;\big|\; &\text{for some } \big(x_i(\theta_1), \ldots, x_i(\theta_m)\big)_{i\in S} \in \Phi_S(w), \\ &y_i \leqslant v_i\big(x_i(\theta_1), \ldots, x_i(\theta_m)\big) \text{ for all } i \in S\big\}. \end{aligned}$$

Since each u_i is concave, a standard argument establishes that this NTU game is balanced and, therefore, has a nonempty core.

An allocation $(\hat{x}_i(\theta_1), \ldots, \hat{x}_i(\theta_m))_{i\in N} \in \Phi_N(w)$ is a core allocation of E if

$$\big(v_1\big(\hat{x}_1(\theta_1), \ldots, \hat{x}_1(\theta_m)\big), \ldots, v_n\big(\hat{x}_n(\theta_1), \ldots, \hat{x}_n(\theta_m)\big)\big)$$

is a payoff vector in the core of the NTU game V^E.

For an allocation $x \in \Phi_N(w)$, denote the set of bundles weakly preferred by i to his component of the allocation by

$$\mathcal{P}_i(x) = \big\{y \in R^{\ell m}_{+} \;\big|\; v_i\big(y(\theta_1), \ldots, y(\theta_m)\big) \geqslant v_i\big(x_i(\theta_1), \ldots, x_i(\theta_m)\big)\big\}.$$

For each $S \subseteq N$, let $\mathcal{P}_S(x) = \sum_{i\in S} \mathcal{P}_i(x)$, $w_S = \sum_{i\in S} w_i$, and $x_S(\theta) = \sum_{i\in S} x_i(\theta)$. Define $\widehat{w}_S := (w_S, \ldots, w_S)$ to be the point $(y(\theta_1), \ldots, y(\theta_m)) \in R^{\ell m}_{+}$ with $y(\theta_k) = w_S$ for each k.

Definition 1. A core allocation $x = (x_i(\theta_1), \dots, x_i(\theta_m))_{i \in N}$ of the auxiliary economy E is a *strict* core allocation if

$$\widehat{w}_S \notin \mathcal{P}_S(x)$$

whenever $S \neq N$.

If $x = (x_i(\theta_1), \dots, x_i(\theta_m))_{i \in N}$ is a strict core allocation of E, then the agents in any proper subset $S \subset N$ cannot guarantee themselves the utility levels associated with x using only the resources available to S. Finally, an allocation $x = (x_i(\theta_1), \dots, x_i(\theta_m))_{i \in N} \in \Phi_N(w)$ is a Walras equilibrium of E if for some price vector $(p(\theta_1), \dots, p(\theta_m)) \in R_+^{\ell m}$, the bundle $(x_i(\theta_1), \dots, x_i(\theta_m))$ solves the problem

$$\max v_i\big(z_i(\theta_1), \dots, z_i(\theta_m)\big)$$
$$\text{s.t.} \sum_{\theta} p(\theta) \cdot \big[z_i(\theta) - w_i\big] \leqslant 0 \text{ and } z_i(\theta) \in \Re_+^{\ell} \text{ for all } \theta \in \Theta$$

for each $i \in N$.

3. Incentive compatible cores

3.1. Notions of blocking

Let $e = (\{u_i, w_i\}_{i \in N}, \tilde{\theta}, \tilde{t}, P)$ be a PIE. In order to define the core of an economy with incomplete information, it is necessary to define "improve upon" or "block." For each $S \subseteq N$, let the set of *S-feasible allocations* for the PIE e be defined as

$$A_S = \left\{ (z_i)_{i \in S} \;\middle|\; z_i : T_S \to \Re_+^{\ell} \text{ and } \sum_{i \in S} \big(z_i(t_S) - w_i\big) \leqslant 0 \text{ for all } t_S \in T_S \right\}.$$

An S-feasible allocation $(z_i)_{i \in S}$ is *incentive compatible* if

$$\sum_{\theta \in \Theta} \sum_{t_{S \setminus i} \in T_{S \setminus i}} u_i\big(z_i(t_{S \setminus i}, t_i), \theta\big) P(\theta, t_{S \setminus i} \mid t_i)$$
$$\geqslant \sum_{\theta \in \Theta} \sum_{t_{S \setminus i} \in T_{S \setminus i}} u_i\big(z_i\big(t_{S \setminus i}, t_i'\big), \theta\big) P(\theta, t_{S \setminus i} \mid t_i)$$

for each $t_i, t_i' \in T_i$ and $i \in S$.

The set of incentive compatible, S-feasible allocations will be denoted A_S^*.

Definition 2. Let $e = (\{u_i, w_i\}_{i \in N}, \tilde{\theta}, \tilde{t}, P)$ be a PIE and let $(z_i)_{i \in N} \in A_N$.

(i) (*Ex ante blocking*.) A coalition $S \subseteq N$ can X-block $(z_i)_{i \in N}$ if there exists $(x_i)_{i \in S} \in A_S$ satisfying the following condition:

$$\sum_{t_S\in T_S}\sum_{\theta\in\Theta}u_i\big(x_i(t_S),\theta\big)P(\theta,t_S)$$
$$>\sum_{t_S\in T_S}\sum_{t_{N\setminus S}\in T_{N\setminus S}}\sum_{\theta\in\Theta}u_i\big(z_i(t_{N\setminus S},t_S),\theta\big)P(\theta,t_{N\setminus S},t_S)$$

for all $i\in S$.

(ii) (*Ex ante incentive compatible blocking*.) A coalition $S\subseteq N$ can *ICX*-block $(z_i)_{i\in N}$ if there exists $(x_i)_{i\in S}\in A^*_S$ satisfying the following condition:

$$\sum_{t_S\in T_S}\sum_{\theta\in\Theta}u_i\big(x_i(t_S),\theta\big)P(\theta,t_S)$$
$$>\sum_{t_S\in T_S}\sum_{t_{N\setminus S}\in T_{N\setminus S}}\sum_{\theta\in\Theta}u_i\big(z_i(t_{N\setminus S},t_S),\theta\big)P(\theta,t_{N\setminus S},t_S)$$

for all $i\in S$.

(iii) (*Ex ante incentive compatible ε-blocking*.) Suppose $\varepsilon\geqslant 0$. A coalition $S\subseteq N$ can ε*ICX*-block $(z_i)_{i\in N}$ if there exists $(x_i)_{i\in S}\in A^*_S$ satisfying the following condition:

$$\sum_{t_S\in T_S}\sum_{\theta\in\Theta}u_i\big(x_i(t_S),\theta\big)P(\theta,t_S)$$
$$>\sum_{t_S\in T_S}\sum_{t_{N\setminus S}\in T_{N\setminus S}}\sum_{\theta\in\Theta}u_i\big(z_i(t_{N\setminus S},t_S),\theta\big)P(\theta,t_{N\setminus S},t_S)+\varepsilon$$

for all $i\in S$.

3.2. Incentive compatible cores

Definition 3. Let $e=(\{u_i,w_i\}_{i\in N},\tilde\theta,\tilde t,P)$ be a PIE.

(i) An N-feasible, incentive compatible allocation $(z_i)_{i\in N}\in A^*_N$ is an *Ex Ante Incentive Compatible Core Allocation* for e if $(z_i)_{i\in N}$ cannot be *ICX*-blocked by any $S\subseteq N$.

(ii) An N-feasible, incentive compatible allocation $(z_i)_{i\in N}\in A^*_N$ is an *Ex Ante Incentive Compatible ε-Core Allocation* for e if $(z_i)_{i\in N}$ cannot be *ICX*-blocked by N and $(z_i)_{i\in N}$ cannot be ε*ICX*-blocked by any $S\neq N$.

Obviously, every ex ante IC core allocation is an ex ante IC ε-core allocation for each $\varepsilon\geqslant 0$.

The model and definitions that we have described are similar to those in Vohra (1999) (see also Forges et al. (2002)). Vohra works with utilities $\hat u_i(\cdot;t)$ that depend on the type profile t but the two approaches are formally interchangeable. If we define

$$\hat u_i(\cdot;t)=\sum_{\theta\in\Theta}u_i(\cdot;\theta)P_\Theta(\theta|t)$$

then our model can be embedded in Vohra's. By defining $\Theta=T$ and $P(\theta|t)=1$ if $\theta=t$, then Vohra's model is a special case of ours. His work shows that, for this special case, the ex ante incentive compatible core can be empty under benign assumptions. When $\Theta=T$, agents are not informationally small in a sense defined precisely in the next section. Our goal is to consider those cases in which agents are informationally small and in which the ex ante incentive compatible core is nonempty as a result.

4. Informational size, aggregate uncertainty, and distributional variability

We will show that there exist incentive compatible core allocations for economies with asymmetric information when an agent's informational size is small relative to other properties of the information structure of the economy. In formulating the conditions under which the incentive compatible core is nonempty, we need the notions of informational size, aggregate uncertainty and distributional variability which we introduced in McLean and Postlewaite (2002a). We will define these concepts below, but refer the reader to that paper for a full discussion of the concepts.

4.1. Informational size

If $t \in T$, recall that $P_\Theta(\cdot|t) \in \Delta_\Theta$ denotes the induced conditional probability measure on Θ and $\chi_\theta \in \Delta_\Theta$ denotes the measure that puts probability one on θ. Any vector of agents' types $t = (t_{-i}, t_i) \in T$ induces a conditional distribution on Θ and, if agent i unilaterally changes his announced type from t_i to t_i', this conditional distribution will (in general) change. We consider agent i to be informationally small if, for each t_i, there is a "small" probability that he can induce a "large" change in the induced conditional distribution on Θ by changing his announced type from t_i to some other t_i'. We formalize this in the following definition.

Let

$$I_\varepsilon^i(t_i', t_i) = \{t_{-i} \in T_{-i} \mid \|P_\Theta(\cdot|t_{-i}, t_i) - P_\Theta(\cdot \mid t_{-i}, t_i')\| > \varepsilon\}.$$

The *informational size* of agent i is defined as

$$\nu_i^P = \max_{t_i \in T_i} \max_{t_i' \in T_i} \min\{\varepsilon \geqslant 0 \mid \text{Prob}\{\tilde{t}_{-i} \in I_\varepsilon^i(t_i', t_i) \mid \tilde{t}_i = t_i\} \leqslant \varepsilon\}.$$

Loosely speaking, we will say that agent i is *informationally small* with respect to P if his informational size ν_i^P is "small." If agent i receives signal t_i but reports $t_i' \neq t_i$, then the effect of this misreport is a change in the conditional distribution on Θ from $P_\Theta(\cdot|t_{-i}, t_i)$ to $P_\Theta(\cdot|t_{-i}, t_i')$. If $t_{-i} \in I_\varepsilon^i(t_i', t_i)$, then this change is "large" in the sense that $\|P_\Theta(\cdot|\hat{t}_{-i}, t_i) - P_\Theta(\cdot|\hat{t}_{-i}, t_i')\| > \varepsilon$. Therefore, $\text{Prob}\{\tilde{t}_{-i} \in I_\varepsilon^i(t_i', t_i) \mid \tilde{t}_i = t_i\}$ is the probability that i can have a "large" influence on the conditional distribution on Θ by reporting t_i' instead of t_i when his observed signal is t_i. An agent is informationally small if for each of his possible types t_i, he assigns small probability to the event that he can have a "large" influence on the distribution $P_\Theta(\cdot|t_{-i}, t_i)$, given his observed type.

4.2. Negligible aggregate uncertainty

We will next quantify aggregate uncertainty. Let

$$\mu_i^P = \max_{t_i \in T_i} \min\{\varepsilon \geqslant 0 \mid \text{Prob}\{\|P_\Theta(\cdot \mid \tilde{t}) - \chi_\theta\| > \varepsilon \text{ for all } \theta \in \Theta \mid \tilde{t}_i = t_i\} \leqslant \varepsilon\}.$$

If μ_i^P is small for each i, then we will say that P exhibits *negligible aggregate uncertainty*. In this case, each agent knows that, conditional on his own signal, the aggregate information of all agents will, with high probability, provide a good prediction of the true state.

4.3. Distributional variability

To define the measure of variability, we first define a metric d on Δ_Θ as follows: for each $\alpha, \beta \in \Delta_\Theta$, let

$$d(\alpha, \beta) = \left\| \frac{\alpha}{\|\alpha\|_2} - \frac{\beta}{\|\beta\|_2} \right\|_2$$

where $\|\cdot\|_2$ denotes the 2-norm. Hence, $d(\alpha, \beta)$ measures the Euclidean distance between the Euclidean normalizations of α and β. If $P \in \Delta_{\Theta \times T}$, let $P_\Theta(\cdot|t_i) \in \Delta_\Theta$ be the conditional distribution on Θ given that i receives signal t_i and define

$$\Lambda_i^P = \min_{t_i \in T_i} \min_{t_i' \in T_i \setminus t_i} d\big(P_\Theta(\cdot|t_i), P_\Theta(\cdot \mid t_i')\big)^2.$$

This is the measure of the "variability" of the conditional distribution $P_\Theta(\cdot|t_i)$ as a function of t_i. Let

$$\Delta^*_{\Theta \times T} = \big\{ P \in \Delta_{\Theta \times T} \bigm| \text{for each } i,\ P_\Theta(\cdot|t_i) \neq P_\Theta(\cdot \mid t_i') \text{ whenever } t_i \neq t_i' \big\}.$$

The set $\Delta^*_{\Theta \times T}$ is the collection of distributions on $\Theta \times T$ for which the induced conditionals are different for different types. Hence, $\Lambda_i^P > 0$ for all i whenever $P \in \Delta^*_{\Theta \times T}$.

It can be shown that for each i, t_i, and θ there exists a collection of numbers $z_i(\theta, t_i)$ satisfying

$$0 \leqslant z_i(\theta, t_i) \leqslant 1 \quad \text{and} \quad \sum_\theta \big[z_i(\theta, t_i) - z_i(\theta, t_i')\big] P_\Theta(\theta|t_i) > 0$$

for each $t_i, t_i' \in T_i$ if and only if $P \in \Delta^*_{\Theta \times T}$. This means that, if the posteriors $\{P_\Theta(\cdot|t_i)\}_{t_i \in T_i}$ are all distinct, then these "incentive compatibility" inequalities are strict. However, the expression on the left-hand side decreases as $\Lambda_i^P \to 0$.

The construction of an allocation in the ex ante IC core for a PIE $(\{u_i, w_i\}_{i \in N}, \tilde{\theta}, \tilde{t}, P)$ relies on a delicate balance between informational size, aggregate uncertainty and variability. We begin with a core allocation $(\zeta_i(\theta_1), \ldots, \zeta_i(\theta_m))_{i \in N}$ of the auxiliary economy $E(P_\Theta, \{u_i, w_i\}_{i \in N})$ where P_Θ is the marginal of P on Θ. We then use this core allocation to construct an incentive efficient PIE allocation $(z_i)_{i \in N} \in A_N^*$ in the ex ante incentive compatible core. Both aggregate uncertainty and informational size play a critical role in proving incentive compatibility. When agent i of type t_i lies and reports t_i', he may experience a utility gain by doing so. In order to give agent i the appropriate incentive to report truthfully, we must impose a utility loss on i when he lies that outweighs any gain from the lie. In the mechanism that we construct, the gain that i can experience from lying is bounded from above by i's informational size while the loss that we can impose is bounded from below by his measure of variability. Therefore, the mechanism will be incentive compatible if informational size is small enough relative to variability.

5. Existence results

5.1. The nonemptiness of the incentive compatible core

We now present two results concerning the nonemptiness of the core in the presence of incomplete information.

Theorem 1. *Let $\pi \in \Delta_\Theta$ and let*

$$\Delta_{\Theta\times T}(\pi) = \{P \in \Delta_{\Theta\times T} \mid P_\Theta = \pi\}.$$

For every $\varepsilon > 0$, there exists a $\delta > 0$ such that, whenever $P \in \Delta_{\Theta\times T}(\pi)$ and satisfies

$$\max_i \mu_i^P \leqslant \delta \min_i \Lambda_i^P, \qquad \max_i \nu_i^P \leqslant \delta \min_i \Lambda_i^P$$

the ex ante incentive compatible ε-core of the PIE $(\{u_i, w_i\}_{i\in N}, \tilde{\theta}, \tilde{t}, P)$ is nonempty.

Proof. Choose $\varepsilon > 0$. Let $(\zeta_i(\theta_1), \ldots, \zeta_i(\theta_m))_{i\in N}$ be a core allocation of the auxiliary economy $E(\pi, \{u_i, w_i\}_{i\in N})$. (The concavity assumption guarantees that a core allocation exists.) Let

$$M = \max_\theta \max_i u_i\left(\sum_{j=1}^n w_j; \theta\right),$$

and let

$$\eta = \frac{\varepsilon}{5M+2}.$$

The monotonicity and continuity assumptions imply that there exists an allocation $\zeta^\eta = (\zeta_i^\eta(\theta_1), \ldots, \zeta_i^\eta(\theta_m))_{i\in N}$ of the auxiliary economy such that, for each $i \in N$ and each k, $\zeta_i^\eta(\theta_k) \neq 0$ and

$$u_i\big(\zeta_i^\eta(\theta_k); \theta_k\big) \geqslant u_i\big(\zeta_i(\theta_k); \theta_k\big) - \eta.$$

Applying Theorem 1 in McLean and Postlewaite (2002a), we conclude that there exists a $\delta > 0$ such that, whenever $P \in \Delta_{\Theta\times T}(\pi)$ and satisfies

$$\max_i \mu_i^P \leqslant \delta \min_i \Lambda_i^P, \qquad \max_i \nu_i^P \leqslant \delta \min_i \Lambda_i^P,$$

there exists an incentive compatible PIE allocation $\hat{z}(\cdot)$ for the PIE $(\{u_i, w_i\}_{i\in N}, \tilde{\theta}, \tilde{t}, P)$ and a collection $B_1, \ldots, B_m$ of disjoint subsets of T such that

(i) $\mathrm{Prob}\{\tilde{t} \in \bigcup_{k=1}^m B_k\} \geqslant 1 - \eta$;
(ii) $\mathrm{Prob}\{\tilde{\theta} = \theta_k \mid \tilde{t} = t\} \geqslant 1 - \eta$ for all k and all $t \in B_k$;
(iii) for all $i \in N$, all k and all $t \in B_k$,

$$u_i\big(\hat{z}_i(t); \theta_k\big) \geqslant u_i\big(\zeta_i^\eta(\theta_k); \theta_k\big) - \eta.$$

Given the definition of ζ^η, it is clear that

$$u_i\big(\hat{z}_i(t);\theta_k\big) \geqslant u_i\big(\zeta_i(\theta_k);\theta_k\big) - 2\eta$$

when $t \in B_k$. Define $B_0 = T\backslash[\bigcup_{k=1}^m B_k]$. Since A_N^* is compact and each u_i is continuous, there exists $z(\cdot) \in A_N^*$ such that $z(\cdot)$ cannot be ICX blocked by N and

$$\sum_{t\in T}\sum_{\theta\in\Theta} u_i\big(z_i(t),\theta\big)P(\theta,t) \geqslant \sum_{t\in T}\sum_{\theta\in\Theta} u_i\big(\hat{z}_i(t),\theta\big)P(\theta,t)$$

for all $i \in N$. To show that $z(\cdot)$ is an ex ante incentive compatible ε-core allocation, let $S \subseteq N$, $S \neq N$, and suppose that there exists $(x_i)_{i\in S} \in A_S$ satisfying

$$\sum_{t_S\in T_S}\sum_{\theta\in\Theta} u_i\big(x_i(t_S),\theta\big)P(\theta,t_S) > \sum_{t\in T}\sum_{\theta\in\Theta} u_i\big(z_i(t),\theta\big)P(\theta,t) + \varepsilon$$

for all $i \in S$. We will show that this leads to a contradiction.

Step 1. For each $\theta \in \Theta$ and each $i \in S$, let

$$\xi_i(\theta) = \sum_{t_S\in T_S} x_i(t_S)P(t_S|\theta)$$

and note that

$$\sum_{i\in S}\xi_i(\theta) = \sum_{t_S\in T_S}\left[\sum_{i\in S} x_i(t_S)\right]P(t_S|\theta) \leqslant \sum_{t_S\in T_S}\left[\sum_{i\in S} w_i\right]P(t_S|\theta) \leqslant \sum_{i\in S} w_i.$$

Furthermore, for each $i \in S$,

$$\begin{aligned}
v_i\big(\xi_i(\theta_1),\ldots,\xi_i(\theta_k)\big) &= \sum_k u_i\big(\xi_i(\theta_k),\theta_k\big)\pi(\theta_k)\\
&= \sum_k u_i\left(\sum_{t_S\in T_S} x_i(t_S)P(t_S|\theta_k),\theta_k\right)\pi(\theta_k)\\
&\geqslant \sum_k \sum_{t_S\in T_S} u_i\big(x_i(t_S),\theta_k\big)P(t_S|\theta_k)\pi(\theta_k)\\
&= \sum_k \sum_{t_S\in T_S} u_i\big(x_i(t_S),\theta_k\big)P(t_S,\theta_k).
\end{aligned}$$

Step 2. Direct computation establishes that $\|\chi_{\theta_k} - P_\Theta(\cdot|t)\| \leqslant 2\eta$ whenever $t \in B_k$. Therefore, $\sum_k|\sum_{t\in B_k} P(t) - P(\theta_k)| \leqslant 3\eta$ and it follows that, for each $i \in S$,

$$\begin{aligned}
&\sum_{t\in T}\sum_{\theta\in\Theta} u_i\big(z_i(t),\theta\big)P(\theta,t)\\
&\quad\geqslant \sum_{t\in T}\sum_{\theta\in\Theta} u_i\big(\hat{z}_i(t),\theta\big)P(\theta,t)\\
&\quad= \sum_k\sum_{t\in B_k}\sum_{\theta\in\Theta} u_i\big(\hat{z}_i(t),\theta\big)P(\theta|t)P(t) + \sum_{t\in B_0}\sum_{\theta\in\Theta} u_i\big(\hat{z}_i(t),\theta\big)P(\theta,t)\\
&\quad\geqslant \sum_k\sum_{t\in B_k}\sum_j u_i\big(\hat{z}_i(t),\theta_j\big)P(\theta_j|t)P(t) \geqslant \sum_k\sum_{t\in B_k} u_i\big(\hat{z}_i(t),\theta_k\big)P(t) - 2M\eta
\end{aligned}$$

$$\geqslant \sum_k u_i\big(\zeta_i(\theta_k), \theta_k\big)\bigg[\sum_{t\in B_k} P(t)\bigg] - (2M+2)\eta$$
$$\geqslant \sum_k u_i\big(\zeta_i(\theta_k), \theta_k\big)P(\theta_k) - 5M\eta - 2\eta = v_i\big(\zeta_i(\theta_1), \dots, \zeta_i(\theta_m)\big) - (5M+2)\eta.$$

Step 3. Combining steps 1 and 2, we conclude that, for each $i \in S$,

$$v_i\big(\xi_i(\theta_1), \dots, \xi_i(\theta_m)\big)$$
$$\geqslant \sum_k \sum_{t_S \in T_S} u_i\big(x_i(t_S), \theta_k\big)P(t_S, \theta_k) > \sum_{t\in T}\sum_{\theta\in\Theta} u_i\big(z_i(t), \theta\big)P(\theta, t) + \varepsilon$$
$$\geqslant v_i\big(\zeta_i(\theta_1), \dots, \zeta_i(\theta_m)\big) - (5M+2)\eta + \varepsilon = v_i\big(\zeta_i(\theta_1), \dots, \zeta_i(\theta_k)\big)$$

contradicting the assumption that $\{(\zeta_i(\theta_1), \dots, \zeta_i(\theta_m))\}_{i\in N}$ is a core allocation of the auxiliary economy $E(\pi, \{u_i, w_i\}_{i\in N})$. □

The concavity assumption regarding utility functions guarantees that the core of the auxiliary economy $E(P_\Theta, \{u_i, w_i\}_{i\in N})$ is nonempty and this is enough to show that the ex ante incentive compatible ε-core is nonempty. However, we need the strict core of the auxiliary economy to be nonempty in order to prove the nonemptiness of the ex ante incentive compatible core.

Theorem 2. *Let $\pi \in \Delta_\Theta$ and let*

$$\Delta_{\Theta\times T}(\pi) = \{P \in \Delta_{\Theta\times T} \mid P_\Theta = \pi\}.$$

Furthermore, suppose that the strict core of the auxiliary economy $E(\pi, \{u_i, w_i\}_{i\in N})$ is nonempty. Then there exists a $\delta > 0$ such that, whenever $P \in \Delta_{\Theta\times T}(\pi)$ and satisfies

$$\max_i \mu_i^P \leqslant \delta \min_i \Lambda_i^P, \qquad \max_i \nu_i^P \leqslant \delta \min_i \Lambda_i^P$$

the ex ante incentive compatible core of the PIE $(\{u_i, w_i\}_{i\in N}, \tilde{\theta}, \tilde{t}, P)$ is nonempty.

Proof. Let $\pi \in \Delta_\Theta$ and suppose that ζ is a strict core allocation of $E(\pi, \{u_i, w_i\}_{i\in N})$.

Step 1. There exists $\varepsilon > 0$ such that the following condition holds: there is no coalition $S \neq N$ such that, for some $(y_i(\theta_1), \dots, y_i(\theta_m))_{i\in S} \in \Phi_S(w)$,

$$v_i\big(y_i(\theta_1), \dots, y_i(\theta_m)\big) \geqslant v_i\big(\zeta_i(\theta_1), \dots, \zeta_i(\theta_m)\big) - \varepsilon$$

for all $i \in S$.

To see this, suppose that, for every k, there is an $S^k \neq N$ and an allocation $(y_i^k(\theta_1), \dots, y_i^k(\theta_m))_{i\in S^k} \in \Phi_{S^k}$ such that

$$v_i\big(y_i^k(\theta_1), \dots, y_i^k(\theta_m)\big) \geqslant v_i\big(\zeta_i(\theta_1), \dots, \zeta_i(\theta_m)\big) - 1/k$$

for each $i \in S^k$. Since there are only finitely many coalitions, since each $\Phi_S(w)$ is compact and each v_i is continuous, it follows that there exists an $S \neq N$ and $(y_i(\theta_1), \dots, y_i(\theta_m))_{i\in S} \in \Phi_S(w)$ such that

$$v_i\big(y_i(\theta_1), \dots, y_i(\theta_m)\big) \geqslant v_i\big(\zeta_i(\theta_1), \dots, \zeta_i(\theta_m)\big)$$

for each $i \in S$. In particular, $\widehat{w}_S = (w_S, \dots, w_S) \in \mathcal{P}_S(\zeta)$ contradicting the assumption that ζ is a strict core allocation of $E(\pi, \{u_i, w_i\}_{i \in N})$.

Step 2. Let

$$M = \max_\theta \max_i u_i\left(\sum_{j=1}^n w_j; \theta\right),$$

and let

$$\eta = \frac{\varepsilon}{5M+2}.$$

Applying the same argument used in the proof of Theorem 1 above, there exists a $\delta > 0$ such that, whenever $P \in \Delta_{\Theta \times T}(\pi)$ and satisfies

$$\max_i \mu_i^P \leqslant \delta \min_i \Lambda_i^P, \qquad \max_i \nu_i^P \leqslant \delta \min_i \Lambda_i^P,$$

there exists an incentive compatible PIE allocation $\hat{z}(\cdot)$ for the PIE $(\{e(\theta)\}_{\theta \in \Theta}, \tilde{\theta}, \tilde{t}, P)$ and a collection $B_1, \dots, B_m$ of disjoint subsets of T such that

(i) $\text{Prob}\{\tilde{t} \in \bigcup_{k=1}^m B_k\} \geqslant 1 - \eta$;
(ii) $\text{Prob}\{\tilde{\theta} = \theta_k | \tilde{t} = t\} \geqslant 1 - \eta$ for all k and all $t \in B_k$;
(iii) For all $i \in N$, all k and all $t \in B_k$,

$$u_i\big(\hat{z}_i(t); \theta_k\big) \geqslant u_i\big(\zeta_i(\theta_k); \theta_k\big) - 2\eta.$$

Since A_N^* is compact and each u_i is continuous, there exists $z(\cdot) \in A_N^*$ such that $z(\cdot)$ cannot be ICX blocked by N and

$$\sum_{t \in T} \sum_{\theta \in \Theta} u_i\big(z_i(t), \theta\big) P(\theta, t) \geqslant \sum_{t \in T} \sum_{\theta \in \Theta} u_i\big(\hat{z}_i(t), \theta\big) P(\theta, t)$$

for all $i \in N$. To show that $z(\cdot)$ is an ex ante incentive compatible core allocation, let $S \subseteq N$, $S \neq N$, and suppose that there exists $(x_i)_{i \in S} \in A_S$ satisfying

$$\sum_{t_S \in T_S} \sum_{\theta \in \Theta} u_i\big(x_i(t_S), \theta\big) P(\theta, t_S) > \sum_{t \in T} \sum_{\theta \in \Theta} u_i\big(z_i(t), \theta\big) P(\theta, t)$$

for all $i \in S$. We will show that this leads to a contradiction.

Defining $(\xi_i(\theta_1), \dots, \xi_i(\theta_m))_{i \in S}$ as in the proof of Theorem 1 and using the same steps found there, we conclude that, for each $i \in S$,

$$\begin{aligned} &v_i\big(\xi_i(\theta_1), \dots, \xi_i(\theta_m)\big) \\ &\quad \geqslant \sum_k \sum_{t_S \in T_S} u_i\big(x_i(t_S), \theta_k\big) P(t_S, \theta_k) > \sum_{t \in T} \sum_{\theta \in \Theta} u_i\big(z_i(t), \theta\big) P(\theta, t) \\ &\quad \geqslant v_i\big(\zeta_i(\theta_1), \dots, \zeta_i(\theta_m)\big) - (5M+2)\eta \geqslant v_i\big(\zeta_i(\theta_1), \dots, \zeta_i(\theta_m)\big) - \varepsilon. \end{aligned}$$

This contradicts the conclusion of step 1 and it follows that the ex ante IC core is nonempty. □

5.2. Strict cores of auxiliary economies

In order to show that the incentive compatible core is nonempty, we assumed in Theorem 2 that the strict core of the auxiliary economy was nonempty. In the first result of this section, we provide conditions under which this assumption holds.

We begin with a definition.

Definition 4. A Walras equilibrium allocation $x = (x_i(\theta_1), \ldots, x_i(\theta_m))_{i \in N}$ of the auxiliary economy $E(\pi, \{u_i, w_i\}_{i \in N})$ with associated price vector $(p(\theta_1), \ldots, p(\theta_m)) \in R_+^{\ell m}$ is a *strict Walras equilibrium allocation* if for each $i \in N$, $(x_i(\theta_1), \ldots, x_i(\theta_m))$ is the unique solution to the problem:

$$\max v_i\big(z_i(\theta_1), \ldots, z_i(\theta_m)\big)$$
$$\text{s.t.} \sum_\theta p(\theta) \cdot \big[z_i(\theta) - w_i\big] \leqslant 0 \text{ and } z_i(\theta) \in \Re_+^\ell \text{ for all } \theta \in \Theta.$$

Not every strict Walras equilibrium is a strict core allocation but the two notions are related as the next result demonstrates.

Proposition 1. *Suppose $\pi \in \Delta_\Theta^0$ and that $x = (x_i(\theta_1), \ldots, x_i(\theta_m))_{i \in N}$ is a strict Walras equilibrium of the auxiliary economy $E(\pi, \{u_i, w_i\}_{i \in N})$ with associated price vector $(p(\theta_1), \ldots, p(\theta_m)) \in R_+^{\ell m}$. If $S \neq N$ implies that $w_S \neq x_S(\theta)$ for some $\theta \in \Theta$, then x is a strict core allocation for $E(\pi, \{u_i, w_i\}_{i \in N})$.*

Proof. First, we show that, for each S,

$$\big(y(\theta_1), \ldots, y(\theta_m)\big) \in \mathcal{P}_S(x),$$
$$y(\theta) \neq x_S(\theta) \text{ for some } \theta \Rightarrow \sum_\theta p(\theta) \cdot \big[y(\theta) - w_S\big] > 0.$$

To see this, choose $(y(\theta_1), \ldots, y(\theta_m)) \in \mathcal{P}_S(x)$ and suppose that $y(\theta') \neq x_S(\theta')$. From the definitions, it follows that there exist $(z_i(\theta_1), \ldots, z_i(\theta_m))_{i \in S}$ such that $z_S(\theta) = y(\theta)$ for all θ and $(z_i(\theta_1), \ldots, z_i(\theta_m)) \in \mathcal{P}_i(x)$ for each $i \in S$. If $\sum_\theta p(\theta) \cdot z_i(\theta) < \sum_\theta p(\theta) \cdot w_i$ for some i, then the monotonicity assumption implies that z_i does not maximize i's utility on his budget set (recall that $\pi \in \Delta_\Theta^0$ and $(p(\theta_1), \ldots, p(\theta_m)) \in R_+^{\ell m}$). This in turn means that x is not a Walras equilibrium. Therefore, $\sum_\theta p(\theta) \cdot z_i(\theta) \geqslant \sum_\theta p(\theta) \cdot w_i$ for every $i \in S$. Since $y(\theta') \neq x_S(\theta')$, it follows that $z_i(\theta') \neq x_i(\theta')$ for some $i \in S$, say i'. Since x is a strict Walras equilibrium, we conclude that $\sum_\theta p(\theta) \cdot z_{i'}(\theta) > \sum_\theta p(\theta) \cdot w_{i'}$ and, consequently,

$$\sum_\theta p(\theta) \cdot y(\theta) = \sum_\theta p(\theta) \cdot z_S(\theta) > \sum_\theta p(\theta) \cdot w_S.$$

To complete the proof of the proposition, suppose that $S \neq N$. Then $w_S \neq x_S(\theta)$ for some $\theta \in \Theta$. If $\widehat{w}_S \in \mathcal{P}_S(x)$, then $\sum_\theta p(\theta) \cdot w_S > \sum_\theta p(\theta) \cdot w_S$, an impossibility. Therefore, $\widehat{w}_S \notin \mathcal{P}_S(x)$. $\square$

The proof of Proposition 1 does not use the assumption that utility functions are concave. However, every Walras equilibrium is strict if utilities are strictly concave. Hence, Proposition 1 does imply the following corollary: if

(i) $\pi \in \Delta^0_\Theta$,
(ii) $x = (x_i(\theta_1), \ldots, x_i(\theta_m))_{i \in N}$ is a Walras equilibrium of the auxiliary economy $E(\pi, \{u_i, w_i\}_{i\in N})$,
(iii) the utilities $u_i(\cdot, \theta)$ are strictly concave, and
(iv) for each $S \neq N$ there exists a $\theta \in \Theta$ such that $w_S \neq x_S(\theta)$, then x is a strict core allocation for $E(\pi, \{u_i, w_i\}_{i\in N})$.

We now turn to the question of the nonemptiness of the strict core of an auxiliary economy. We begin with two results that will lead to a "genericity" theorem.

Proposition 2. *Suppose $\pi \in \Delta^0_\Theta$ and suppose that the strict core of the auxiliary economy $E(\pi, \{u_i, w_i\}_{i\in N})$ is nonempty. Then there exists an $\varepsilon > 0$ such that the strict core of the auxiliary economy $E(\pi, \{u_i, a_i\}_{i\in N})$ is nonempty whenever $\|w_i - a_i\| < \varepsilon$ for all i.*

Proof. Suppose that $x = (x_i(\theta_1), \ldots, x_i(\theta_m))_{i\in N}$ is a strict core allocation of the auxiliary economy $E(\pi, \{u_i, w_i\}_{i\in N})$.

Claim. *There exists an $\varepsilon^* > 0$ such that, for each $a = (a_1, \ldots, a_n)$ satisfying $\|w_i - a_i\| < \varepsilon^*$ for all i, there exists an $\xi \in \Phi_N(a)$ such that $\hat{a}_S \notin \mathcal{P}_S(\xi)$ whenever $S \neq N$.*

Proof of Claim. Suppose instead that there exists a sequence $a^k = (a^k_1, \ldots, a^k_n) \to w = (w_1, \ldots, w_n)$ such that, for each $\xi \in \Phi_N(a^k)$, there exists an $S_\xi \neq N$ such that $\hat{a}^k_{S_\xi} \in \mathcal{P}_{S_\xi}(\xi)$. Since $a^k \to w$, $x \in \Phi_N(w)$, and $\Phi_N(\cdot)$ is a lower hemi-continuous correspondence, it follows that there exists a sequence $\{\xi^k\}$ such that $\xi^k \in \Phi_N(a^k)$ and $\xi^k \to x$. Therefore, there exists for each k an $S_k \neq N$ such that $\hat{a}^k_{S_k} \in \mathcal{P}_{S_k}(\xi^k)$. Since there are only finitely many coalitions, we may (choosing a subsequence if necessary) assume that S is independent of k. In particular, there exists an $S \neq N$ such that for each k, $\hat{a}^k_S \in \mathcal{P}_S(\xi^k)$. Since $\hat{a}^k_S \to \widehat{w}_S$, $\xi^k \to x$, and $\mathcal{P}_S(\cdot)$ is a correspondence with closed graph, it follows that $\widehat{w}_S \in \mathcal{P}_S(x)$. This contradicts the assumption that x is a strict core allocation, and the proof of the claim is complete. □

To complete the proof of the theorem, choose $a = (a_1, \ldots, a_n)$ satisfying $\|w_i - a_i\| < \varepsilon^*$ for all i. Applying the claim, there exists $\xi \in \Phi_N(a)$ such that $\hat{a}_S \notin \mathcal{P}_S(\xi)$ whenever $S \neq N$. Next, choose an efficient allocation ξ' for the auxiliary economy $E(\pi, \{u_i, a_i\}_{i\in N})$ satisfying $\xi'_i \in \mathcal{P}_i(\xi)$ for each $i \in N$. Since $\mathcal{P}_S(\xi') \subseteq \mathcal{P}_S(\xi)$, it follows that $\hat{a}_S \notin \mathcal{P}_S(\xi')$ and we conclude that ξ' is a strict core allocation for $E(\pi, \{u_i, a_i\}_{i\in N})$. □

Proposition 3. *Suppose $\pi \in \Delta^0_\Theta$ and suppose that $x = (x_i(\theta_1), \ldots, x_i(\theta_m))_{i\in N}$ is a strict Walras equilibrium of the auxiliary economy $E(\pi, \{u_i, w_i\}_{i\in N})$ with associated price vector $(p(\theta_1), \ldots, p(\theta_m)) \in \Re^{\ell m}_+$. If there exists $S \neq N$ such that $w_S = x_S(\theta)$ for all θ,* *then for every $\delta > 0$ there exists a vector $a = (a_1, \ldots, a_n)$ such that $a_i \in R^{\ell}_{++}$ for all i, $\|w_i - a_i\| < \delta$ for all i, and x is a strict core allocation of the auxiliary economy $\{\pi, \{u_i, a_i\}_{i \in N}\}$.*

Proof. Let

$$\mathcal{S} = \bigl\{S \subseteq N \bigm| S \neq N \text{ and } w_S = x_S(\theta) \text{ for all } \theta \in \Theta\bigr\}$$

and suppose that $\mathcal{S} \neq \emptyset$. Let $y \neq 0$ be a net trade vector such that

$$\left[\sum_{\theta} p(\theta)\right] \cdot y = 0$$

where $(p(\theta_1), \ldots, p(\theta_m)) \in \Re^{\ell m}_{+}$ is the equilibrium price vector. Now consider the following perturbations to the endowment vector w:

$$w'_i(\varepsilon) = w_i + \varepsilon y \quad \text{for } i \neq 1,$$
$$w'_1(\varepsilon) = w_1 - (n-1)\varepsilon y.$$

Since each $w_i \in \Re^{\ell}_{++}$, the vector $w'_i(\varepsilon)$ will have positive components for sufficiently small ε. Also, note that

$$w'_N(\varepsilon) = w_N$$

while

$$w'_S(\varepsilon) \neq w_S$$

whenever $S \neq N$. If $S \in \mathcal{S}$, then $w'_S(\varepsilon) \neq x_S(\theta)$ for all θ and for all $\varepsilon > 0$. Since there are only finitely many coalitions and since $w'_S(\varepsilon) \to w_S$ as $\varepsilon \to 0$, it follows that $w'_S(\varepsilon) \neq x_S(\theta)$ for at least one θ whenever $S \notin \mathcal{S}$, $S \neq N$, and ε is small enough. Summarizing, there exists $\hat{\varepsilon} > 0$ such that, whenever $0 < \varepsilon < \hat{\varepsilon}$, the following statement holds: for each $S \neq N$, there exists $\theta \in \Theta$ such that $w'_S(\varepsilon) \neq x_S(\theta)$.

To complete the proof, choose $\delta > 0$ and choose ε^* satisfying $0 < \varepsilon^* < \hat{\varepsilon}$ and $\|w_i - w'_i(\varepsilon^*)\| < \delta$ for all i. Next, define $a_i = w'_i(\varepsilon^*)$. Since

$$\left[\sum_{\theta} p(\theta)\right] \cdot a_i = \left[\sum_{\theta} p(\theta)\right] \cdot w'_i(\varepsilon^*) = \left[\sum_{\theta} p(\theta)\right] \cdot w_i$$

for each i, it follows that x is a strict Walras equilibrium of the auxiliary economy $E(\pi, \{u_i, a_i\}_{i \in N})$ with associated price vector $(p(\theta_1), \ldots, p(\theta_m))$. The proof is now completed by applying Proposition 1 to the auxiliary economy $E(\pi, \{u_i, a_i\}_{i \in N})$. □

We can now combine Propositions 2 and 3 and provide a genericity result for strict cores of auxiliary economies.

Corollary. *Suppose that $\pi \in \Delta^0_{\Theta}$ and define*

$$X = \Bigl\{(w_1, \ldots, w_n) \in \bigl(\Re^{\ell}_{++}\bigr)^n \Bigm| \textit{the strict core of } \bigl\{\pi, \bigl\{u_i(\cdot, \theta), w_i\bigr\}_{\theta \in \Theta}\bigr\} \textit{ is nonempty}\Bigr\}.$$

If each $u_i(\cdot, \theta)$ is strictly concave, then X is an open dense subset of $(\Re^{\ell}_{++})^n$.

Proof. Proposition 2 implies that X is open. If $w \notin X$, then the strict concavity assumption, together with monotonicity and positivity of w_i, implies that there exists a strict Walras equilibrium of the auxiliary economy. Applying Propositions 1 and 3, we conclude that X is dense in $(\Re^{\ell}_{++})^n$. $\square$

6. The replica problem

In the presence of a large number of agents, we might expect any single agent to be informationally small and replica economies provide a natural framework in which to investigate this conjecture.

6.1. Notation and definitions

Recall that $J_r = \{1, 2, \ldots, r\}$ and define $N_r = N \times J_r$. Given the collection $\{w_i, u_i\}_{i\in N}$ and a positive integer r, let $\{w_{is}, u_{is}\}_{(i,s)\in N_r}$ denote the r replication of $\{w_i, u_i\}_{i\in N}$ satisfying:

(1) $w_{is} = w_i$ for all $i \in N$ and all $s \in J_r$;
(2) $u_{is}(z, \theta) = u_i(z, \theta)$ for all $z \in \Re^{\ell}_{+}$, $i \in N$ and $s \in J_r$.

For any positive integer r, let $T^r = T \times \cdots \times T$ denote the r-fold Cartesian product and let $t^r = (t^r_{\cdot 1}, \ldots, t^r_{\cdot r})$ denote a generic element of T^r where $t^r_{\cdot s} = (t^r_{1s}, \ldots, t^r_{ns})$. If $P^r \in \Delta_{\Theta\times T^r}$, then $e^r = (\{w_{is}, u_{is}\}_{(i,s)\in N_r}, \tilde{\theta}, \tilde{t}^r, P^r)$ is a PIE with *nr* agents.

Definition 5. A sequence of replica economies $\{(\{w_{is}, u_{is}\}_{(i,s)\in N_r}, \tilde{\theta}, \tilde{t}^r, P^r)\}_{r=1}^{\infty}$ is a *conditionally independent sequence* if there exists a $P \in \Delta^*_{\Theta\times T}$ such that

(a) for each r, each $s \in J_r$ and each $(\theta, t_1, \ldots, t_n) \in \Theta \times T$,

$$\mathrm{Prob}\big\{\tilde{\theta} = \theta, \tilde{t}^r_{1s} = t_1, \tilde{t}^r_{2s} = t_2, \ldots, \tilde{t}^r_{ns} = t_n\big\} = P(\theta, t_1, t_2, \ldots, t_n);$$

(b) for each r and each θ, the random vectors

$$\big(\tilde{t}^r_{11}, \tilde{t}^r_{21}, \ldots, \tilde{t}^r_{n1}\big), \ldots, \big(\tilde{t}^r_{1r}, \tilde{t}^r_{2r}, \ldots, \tilde{t}^r_{nr}\big)$$

are independent given $\tilde{\theta} = \theta$;

(c) for every $\theta, \hat{\theta}$ with $\theta \neq \hat{\theta}$, there exists a $t \in T$ such that $P(t|\theta) \neq P(t|\hat{\theta})$.

6.2. The replica theorem

A conditionally independent sequence is a sequence of PIE's with *nr* agents containing r "copies" of each agent $i \in N$. Each copy of an agent i is identical, i.e., has the same endowment and the same utility function. Furthermore, the realizations of type profiles across cohorts are independent given the true value of $\tilde{\theta}$. As r increases each agent is becoming "small" in the economy in terms of endowment, and we can show that each agent is also becoming informationally small. Note that, for large r, an agent may have a small amount of private information regarding the preferences of everyone through his information about $\tilde{\theta}$. We now state an analogue of Theorem 1 for replica economies.

Theorem 3. *Let* $\{(\{w_{is}, u_{is}\}_{(i,s)\in N_r}, \tilde{\theta}, \tilde{t}^r, P^r)\}_{r=1}^{\infty}$ *be a conditionally independent sequence. For every* $\varepsilon > 0$*, there exists an integer* $\hat{r} > 0$ *such that for all* $r > \hat{r}$*, the ex ante IC* ε*-core of the PIE* $(\{w_{is}, u_{is}\}_{(i,s)\in N_r}, \tilde{\theta}, \tilde{t}^r, P^r)$ *is nonempty.*

Proof. Let $\{(\{w_{is}, u_{is}\}_{(i,s)\in N_r}, \tilde{\theta}, \tilde{t}^r, P^r)\}_{r=1}^{\infty}$ be a conditionally independent sequence and let P^r_Θ denote the marginal of P^r on Θ. From the definition, it follows that there exists $\pi \in \Delta_\Theta$ such that $P^r_\Theta = \pi$ for all r. Since each $u_i(\cdot;\theta)$ is concave and monotonic and each $w_i \in \Re^\ell_{++}$, it follows that the auxiliary economy $E(\pi, \{w_i, u_i\}_{i\in N})$ has a Walras equilibrium $\zeta = (\zeta_i(\theta_1), \ldots, \zeta_i(\theta_m))_{i\in N}$. Defining $\zeta^r_{i,s}(\theta_k) = \zeta_i(\theta_k)$ for all r, i, s, and k, it follows that the allocation $(\zeta^r_{i,s}(\theta_1), \ldots, \zeta^r_{i,s}(\theta_m))_{(i,s)\in N_r}$ (the "r-replica" of ζ) is a core allocation of the r-replicated auxiliary economy $E(\pi, \{w_{is}, u_{is}\}_{(i,s)\in N_r})$ for every r. Let

$$M = \max_\theta \max_i u_i\left(\sum_{j=1}^n w_j; \theta\right).$$

Choose $\varepsilon > 0$ and let

$$\eta = \frac{\varepsilon}{5M + 2}.$$

There exists an $\hat{r} > 0$ such that, for all $r > \hat{r}$, there exists an incentive compatible PIE allocation $\hat{z}^r(\cdot)$ for the PIE $(\{w_{is}, u_{is}\}_{(i,s)\in N_r}, \tilde{\theta}, \tilde{t}^r, P^r)$ and a collection $B^r_1, \ldots, B^r_m$ of disjoint subsets of T^r such that

(i) $\text{Prob}\{\tilde{t}^r \in \bigcup_{k=1}^m B^r_k\} \geqslant 1 - \eta$,
(ii) $\text{Prob}\{\tilde{\theta} = \theta_k \mid \tilde{t}^r = t^r\} \geqslant 1 - \eta$ for all $k \in J_m$ and all $t^r \in B^r_k$,
(iii) For all $i \in N$, for all $k \in J_m$ and all $t^r \in B^r_k$,

$$u_i\big(\hat{z}^r_{is}(t^r); \theta_k\big) \geqslant u_i\big(\zeta_i(\theta_k); \theta_k\big) - 2\eta.$$

(To see this, apply the claim in step 3 in the proof of Theorem 2 in McLean and Postlewaite (2002a) to a perturbed allocation $(\zeta^\eta_i(\theta_1), \ldots, \zeta^\eta_i(\theta_m))_{i\in N}$ as defined in the proof of Theorem 1 above.)

If $C \subseteq N_r$, let A^r_C denote the C-feasible allocations for the PIE e^r and let A^{r*}_C denote the incentive compatible elements of A^r_C. Since A^{r*}_N is compact and each u_i is continuous, there exists $z^r(\cdot) \in A^{r*}_N$ such that $z^r(\cdot)$ cannot be ICX blocked by N_r and

$$\sum_{t^r\in T^r}\sum_{\theta\in\Theta} u_i\big(z^r_{i,s}(t^r), \theta\big) P^r(\theta, t^r) \geqslant \sum_{t^r\in T^r}\sum_{\theta\in\Theta} u_i\big(\hat{z}^r_{i,s}(t^r), \theta\big) P^r(\theta, t^r)$$

for all $(i,s) \in N_r$. Now choose $r > \hat{r}$. To show that $z^r(\cdot)$ is an ex ante incentive compatible ε-core allocation for e^r, let $C \subseteq N_r$, $C \neq N_r$, and suppose that there exists $(x^r_{i,s})_{(i,s)\in C} \in A^r_C$ satisfying

$$\sum_{t^r_C\in T^r_C}\sum_{\theta\in\Theta} u_i\big(x^r_{i,s}(t^r_C), \theta\big) P^r\big(\theta, t^r_C\big) > \sum_{t\in T^r}\sum_{\theta\in\Theta} u_i\big(z^r_{i,s}(t^r), \theta\big) P^r\big(\theta, t^r\big) + \varepsilon$$

for each $(i,s) \in C$. We will show that this leads to a contradiction. Let $C_i = \{s \in J_r \mid (i,s) \in C\}$ and let $I = \{i \mid C_i \neq \emptyset\}$.

In the remaining steps, we suppress the dependence of T^r, T^r_C, t^r, t^r_C, $t^r_{i,s}$, A^r_k, P^r, z^r, $\hat{z}^r$, etc. on r.

Step 1. For each $\theta \in \Theta$ and each $i \in I$, let

$$\xi_i(\theta) = \frac{1}{|C_i|}\sum_{s\in C_i}\left[\sum_{t_C\in T_C} x_{i,s}(t_C)P(t_C|\theta)\right]$$

and note that

$$\begin{aligned}\sum_{i\in I}|C_i|\xi_i(\theta) &= \sum_{i\in I}\sum_{s\in C_i}\left[\sum_{t_C\in T_C} x_{i,s}(t_C)P(t_C|\theta)\right] \leqslant \sum_{t_C\in T_C}\left[\sum_{(i,s)\in C} w_{i,s}\right]P(t_C|\theta)\\ &= \sum_{(i,s)\in C} w_{i,s}.\end{aligned}$$

Furthermore, for each $i \in I$,

$$\begin{aligned}v_i\bigl(\xi_i(\theta_1),\ldots,\xi_i(\theta_k)\bigr) &= \sum_k u_i\bigl(\xi_i(\theta_k),\theta_k\bigr)\pi(\theta_k)\\ &= \sum_k u_i\left(\frac{1}{|C_i|}\sum_{s\in C_i}\left[\sum_{t_C\in T_C} x_{i,s}(t_C)P(t_C|\theta_k)\right],\theta_k\right)\pi(\theta_k)\\ &\geqslant \sum_k \frac{1}{|C_i|}\sum_{s\in C_i}\sum_{t_C\in T_C} u_i\bigl(x_{i,s}(t_C),\theta_k\bigr)P(t_C|\theta_k)\pi(\theta_k)\\ &= \frac{1}{|C_i|}\sum_{s\in C_i}\left[\sum_k\sum_{t_C\in T_C} u_i\bigl(x_{i,s}(t_C),\theta_k\bigr)P(t_C,\theta_k)\right].\end{aligned}$$

Step 2. Duplicating the argument in step 2 of the proof of Theorem 1, we conclude that for each $(i,s) \in C$,

$$\sum_{t\in T}\sum_k u_i\bigl(z_{i,s}(t),\theta_k\bigr)P(\theta_k,t) \geqslant v_i\bigl(\zeta_i(\theta_1),\ldots,\zeta_i(\theta_m)\bigr) - (5M+2)\eta.$$

Step 3. Combining steps 1 and 2, it follows that for each $i \in I$,

$$\begin{aligned}v_i\bigl(\xi_i(\theta_1),\ldots,\xi_i(\theta_m)\bigr) &\geqslant \frac{1}{|C_i|}\sum_{s\in C_i}\left[\sum_k\sum_{t_C\in T_C} u_i\bigl(x_{i,s}(t_C),\theta_k\bigr)P(t_C,\theta_k)\right]\\ &> \frac{1}{|C_i|}\sum_{s\in C_i}\left[\sum_{t\in T}\sum_k u_i\bigl(z_{i,s}(t),\theta_k\bigr)P(\theta_k,t)+\varepsilon\right]\\ &\geqslant \frac{1}{|C_i|}\sum_{s\in C_i}\bigl[v_i\bigl(\zeta_i(\theta_1),\ldots,\zeta_i(\theta_m)\bigr)-(5M+2)\eta+\varepsilon\bigr]\\ &= v_i\bigl(\zeta_i(\theta_1),\ldots,\zeta_i(\theta_m)\bigr).\end{aligned}$$

To complete the proof, define an allocation $(\xi_{i,s}(\theta_1), \ldots, \xi_{i,s}(\theta_m))_{(i,s)\in C}$ where $\xi_{i,s}(\theta_k) = \xi_i(\theta_k)$ for each k and each $(i,s) \in C$. Step 1 establishes that

$$\sum_{(i,s)\in C} \xi_{i,s}(\theta_k) = \sum_{i\in I} |C_i| \xi_i(\theta) \leqslant \sum_{(i,s)\in C} w_{i,s}$$

so $(\xi_{i,s}(\theta_1), \ldots, \xi_{i,s}(\theta_m))_{(i,s)\in C}$ is C-feasible. Step 3 establishes that

$$\begin{aligned} v_{i,s}\big(\xi_{i,s}(\theta_1), \ldots, \xi_{i,s}(\theta_m)\big) &= v_i\big(\xi_i(\theta_1), \ldots, \xi_i(\theta_m)\big) > v_i\big(\zeta_i(\theta_1), \ldots, \zeta_i(\theta_m)\big) \\ &= v_{i,s}\big(\zeta^r_{i,s}(\theta_1), \ldots, \zeta^r_{i,s}(\theta_m)\big) \end{aligned}$$

for each $(i,s) \in C$. Therefore, the coalition C can improve upon the allocation $(\zeta^r_{i,s}(\theta_1), \ldots, \zeta^r_{i,s}(\theta_m))_{(i,s)\in N_r}$ contradicting the assumption that $(\zeta^r_{i,s}(\theta_1), \ldots, \zeta^r_{i,s}(\theta_m))_{(i,s)\in N_r}$ is a core allocation of the replicated auxiliary economy $E(\pi, \{w_{is}, u_{is}\}_{(i,s)\in N_r})$. □

7. Discussion

1. In independent work, Krasa and Shafer (2001) investigate a question similar to that addressed in this paper. They consider economies with asymmetric information in which agents receive noisy signals of the state of the world. They then study a sequence of economies with incomplete information that converges to an economy with complete information in the sense that the agents' signals are asymptotically perfect signals of the state of the world. Using the notion of strict core, Krasa and Shafer show that, for a sequence of asymmetric information economies that converge to a complete information economy for which the strict core is nonempty, the incentive compatible core will be nonempty for economies close to the limit (see their Theorem 2).

There is a close relationship between their notion of convergence to complete information and our concept of informational size. If the accuracy of all agents' information is increased uniformly, the agents' informational size will go to zero. It can be the case, however, that if the accuracy of one agent's information increases much faster than the accuracy of other agents, then that agent will not become informationally small. In McLean and Postlewaite (2002a), it is discussed how our arguments can be extended to this case.

While uniformly increasing the accuracy of agents' signals of the state necessarily makes the agents informationally small, the converse is *not* true. This is demonstrated by our replica theorem. In this case, the agents' information is not becoming increasingly accurate, but they are nevertheless becoming informationally small. The relevant informational consideration for assuring a nonempty incentive compatible core is not that each agent necessarily have very accurate information about the environment. Rather, the relevant consideration is that no agent have a *monopoly* on information about the environment.

2. Forges et al. (2002) construct an example of a three person asymmetric information economy in which the incentive compatible core is empty. It is straightforward to show that agents are informationally large, as they must be given our results. One can parametrize the information structure in their example so that the parameter controls the information size of agents. In the parametrized version, one can determine the limits on the informational size of the agents that would guarantee that the incentive compatible core is nonempty.

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