Journal of Economic Theory **101**, 333–373 (2001)
doi:10.1006/jeth.2001.2797, available online at http://www.idealibrary.com on IDEAL®

Efficient Non-Contractible Investments in Large Economies[1]

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Received June 1, 1999; final version received January 26, 2001; published online August 17, 2001

Do investors making complementary investments face the correct incentives, especially when they cannot contract with each other prior to their decisions? We present a two-sided matching model in which buyers and sellers make investments prior to matching. Once matched, buyer and seller bargain over the price, taking into account outside options. Efficient decisions can always be sustained in equilibrium. We characterize the inefficiencies that can arise in equilibrium and show that equilibria will be constrained efficient. We also show that the degree of diversity in a large market has implications for the extent of any inefficiency. *Journal of Economic Literature* Classification Numbers: C78, D41, D51.

[1] This paper draws heavily from two earlier papers of ours, "Bargaining and Investment in a Two-Sided Matching Model" and "Efficient Non-Contractible Investments." We thank Daron Acemoglu, Jere Behrman, John Moore, Mark Rosenzweig, Randall Wright, a referee, an associate editor, and seminar audiences at numerous universities for helpful discussion and comments. The research reported here was conducted while Cole was a member of the Research Department of the Federal Reserve Bank of Minneapolis. The views expressed herein are those of the authors and not necessarily those of the Federal Reserve Bank of Minneapolis or the Federal Reserve System. Mailath and Postlewaite thank the Economics Department at the University of Bonn and the Minneapolis FRB for their hospitality. They also acknowledge financial support from the NSF, the Deutsche Forschungsgemeinschaft (SFB 303), and the Minneapolis FRB.

0022-0531/01 $35.00

1. INTRODUCTION

Complementary investments are often made by different individuals; for example, a worker may invest in human capital while a firm invests in machinery that utilizes that human capital. Do investors making complementary investments face the correct incentives, especially when they cannot contract with each other prior to their decisions? The traditional answer is no (Williamson [22] and Grossman and Hart [12]). An agent's investment is a sunk cost by the time the agents bargain over the split of the surplus that results from the investment. Since bargaining typically allocates part of the surplus generated by an agent's investment to the other party, the failure of that agent to capture the full benefit of his investment leads to underinvestment.

In the analysis of this holdup problem, the degree to which an agent cannot capture the benefits of his or her investment is related to asset specificity. An agent's outside options will put a lower bound on the share of the surplus that he or she gets in any plausible bargaining process. A worker whose skills are nearly as valuable on a machine other than that owned by the person he or she is currently bargaining with can play off the two owners against each other. In many circumstances, competition between potential partners provides protection against the holdup problem, and agents capture the bulk of the benefits of their investments and, consequently, have incentives to invest efficiently. The polar extreme to this case is that an agent's investment is only of value to a single individual, for example, a worker who becomes an expert on a unique machine. The value of his or her investment is specific to the match with the owner of that machine. Intuitively, the lack of outside options for such an agent should lead him or her to expect a smaller share of the surplus generated by his or her investment than when there is potential competition for his or her services.

While there is a large literature that analyzes the effect of asset specificity on investment, the degree to which investments are specific is typically taken to be exogenous. That analysis considers a single pair in isolation, taking as given other agents' investments and the outside options inherent in those investments. The difficulty with analyzing investments of a single pair is that those investments determine (at least in part) the outside options of other pairs. Consider a matching problem in which there are a number of people on each side who might make investments in hopes of subsequently pairing with someone who has made a complementary investment. The return any individual can expect from investing will be the outcome of the bargaining with his or her future partner, which will depend on the outside options of both individuals. These outside options, of course, are determined precisely by the investment decisions of the agents involved.

Our aim is to analyze the investment decisions of agents who, subsequent to investing, pair off, produce a surplus, and share that surplus through some bargaining process. We treat the agents' investment decisions as a noncooperative game, with each agent's decision depending on the (equilibrium) investment choices of other agents like him or her and of the agents with whom he or she can potentially match. In this way the asset specificity of agents' investments is endogenously determined, rather than exogenously assumed. We are particularly interested in comparing the investments agents make when they can contract prior to investing and those they make when they cannot. If agents can contract over the investment levels they make, investments will be efficient. We take those investments as a benchmark to which we compare investments when ex ante contracting is impossible. When ex ante contracting is impossible, there will always be an equilibrium in which agents invest efficiently, but there may be additional equilibria characterized by inefficient investments. The analysis also suggests that, in many situations, the efficient investment equilibrium is implausible.

In order to focus on the efficiency of investment choices and bargaining over the resulting surplus, we label the two sides in the relationship "buyers" and "sellers." There is, of course, nothing important about this, and we could have used the terms "workers" and "firms."

In the next section, we present two simple examples that illustrate the investment and matching process, one with a finite number of agents and another with an infinite number of agents. The second example illustrates the possibilities of both equilibrium under- and over-investment. Section 3 then discusses related literature. We are interested in the case of many agents, when any single agent's behavior does not affect other agents' possibilities. Toward this end, we introduce in Section 4 a model with a continuum of agents. We characterize the payoffs to agents, conditional on their investment decisions, and show that a version of Makowski and Ostroy's [19] full appropriation condition holds, in that almost all agents receive the marginal social value of their investment decisions (since we are dealing with a continuum of agents, the appropriate notion of marginal social value is, of course, delicate). Section 5 compares the equilibria when ex ante contracting is possible with those when it is not possible. We provide a version of the neoclassical second welfare theorem: The ex ante efficient outcome is always an equilibrium outcome even when ex ante contracting is not possible. However, as the examples of Section 2 indicate, when ex ante contracting is impossible, inefficient equilibria typically also exist. Section 6 characterizes the types of inefficiencies that can arise in equilibrium, and in particular, shows that equilibria will be, in a natural sense, constrained efficient. We also show that the degree of diversity of agents' exogenous characteristics has implications for the extent of any inefficiency.

2. TWO MOTIVATING EXAMPLES

We begin by illustrating several issues with a simple finite example. There are two buyers, $\{1, 2\}$, and two sellers, $\{1, 2\}$. For now, we fix the attributes of the buyers and sellers as in Table I. The surplus generated by a pair (b, s) is given by the product of their attributes, $b \cdot s$. Table I displays one particular outcome for this environment with each of the two columns representing a matched pair and the split of the surplus for that pair. Total surplus is maximized by the indicated matching, and the split of the surplus for the pairs is unique if the sharing rule is symmetric with respect to buyers and sellers.

Suppose now that attributes are not fixed, but are chosen from the set $\{2, 3\}$. We focus on the behavior of seller 1, with the attributes of the other agents unchanging.[2] If the surplus is always divided equally and seller 1 chose instead $s = 3$, then the matching and surplus division are as in Table II. For these attributes, equal division violates equal treatment: The two sellers have the same attribute but receive different payoffs. Such a specification of payoffs is not *stable*, however, since seller 1 could make buyer 2 a marginally better offer than he or she gets when matched with seller 2.

In addition to violating equal treatment, equal division may also prevent efficient attribute choices. If, for example, the cost of attribute 2 to seller 1 is 0, while the cost of attribute 3 is $\frac{3}{2}$, then the increase in surplus when seller 1 chooses attribute 3 rather than attribute 2 is 2, while the increased cost to seller 1 of choosing the higher attribute is only $\frac{3}{2}$. This is, of course, a simple consequence of having a sharing rule that gives part of the increase in output that results from seller 1's investment to the buyer that is matched with seller 1.

There are sharing rules that satisfy equal treatment (and so are stable); Table III gives one such rule. While we obtain equal treatment here, the incentive for inefficient choice remains. For example, if the cost of attribute 3 to seller 1 is $2\frac{1}{4}$, then seller 1 chooses $s = 3$, even though it is inefficient to do so. The problem now is that the payoff to the buyer who is matched with seller 1 falls in response to the higher attribute of the seller.

There does exist a specification of payoffs for this vector of buyers' and sellers' attribute choices that satisfies equal treatment, is stable, and implies efficient choices by seller 2; Table IV gives one. When seller 1 changes his or her attribute, the surplus division between buyer 2 and seller 2 changes even though the characteristics of that match did not change.

[2] We can choose cost functions for the two buyers and for seller 2 to ensure (assuming the bargaining is monotonic) that their optimal choices of attributes is as in Table I.

TABLE I

An Example with Two Buyers and Sellers

Buyer's share (x_i)	2	$4\frac{1}{2}$
Buyer's attribute (b_i)	2	3
Buyer (i)	1	2
Seller (j)	1	2
Seller's attribute (s_j)	2	3
Seller's share (p_j)	2	$4\frac{1}{2}$

TABLE II

Seller 1 with Attribute $s=3$

x_i	3	$4\frac{1}{2}$
b_i	2	3
i	1	2
j	1	2
s_j	3	3
p_j	3	$4\frac{1}{2}$

TABLE III

Equal Treatment and Inefficiency

x_i	$1\frac{1}{2}$	$4\frac{1}{2}$
b_i	2	3
i	1	2
j	1	2
s_j	3	3
p_j	$4\frac{1}{2}$	$4\frac{1}{2}$

TABLE IV

Equal Treatment and Efficiency

x_i	2	5
b_i	2	3
i	1	2
j	1	2
s_j	3	3
p_j	4	4

With a large population, one might expect that a single buyer (or seller) changing attribute would not change the division of those matches that are unaffected by the attribute change. Furthermore, if a particular agent's partner changes his or her attribute, this agent has the option of matching with other agents in the economy. In many situations with large numbers of agents, the existence of alternative partners should eliminate any change in the payoff received by the partner. The violation of both of these properties in the example is due to the finiteness of the set of agents.

The possibility of inefficient ex ante investment, however, is not a consequence of the finiteness of the set of agents. We now present an example with a continuum of agents that has the property that a single agent's attribute choice will not affect the payoffs of any other agent. Nonetheless, there is an equilibrium with underinvestment in attributes. In addition, there also exists an equilibrium with an overinvestment in attributes, as well as an equilibrium with efficient investment. We describe the example here somewhat crudely; the details of the example can be found in Appendix D.

Buyers are indexed by i and sellers by j, with i and j uniformly distributed on an interval $[\underline{i}, \bar{\imath}]$. We begin with the surplus function of the previous example, $v(b, s) = bs$, where b and s are respectively the buyer's and seller's attributes. The cost functions for acquiring attributes are ψ and c, where $\psi(b, i) = b^5/(5i)$ is the cost to buyer i of attribute b and $c(s, j) = s^5/(5j)$ is the cost to seller j of attribute s.

Aggregate net surplus, $v(b, s) - \psi(b, i) - c(s, j)$, is maximized by matching buyer i with seller $j = i$, and setting $b = s = \sqrt[3]{i}$. Suppose agents choose these joint maximizing attributes and share equally the surplus generated by these attributes. It can be shown that this assortative matching (matching buyer and sellers with the same attribute), along with equal sharing of the surplus is *stable*: there is no unmatched buyer and seller that would be better off matching.

We are interested in whether agents have an incentive to choose the surplus maximizing attributes; toward this end, we formulate agents' attribute choices as a noncooperative game. Let β and σ be strictly increasing attribute choice functions for buyers and sellers, and consider a stable matching and sharing of the resulting surpluses. Given the sharing rule, $\tilde{x}(b)$ denotes the payoff that a buyer who chose attribute b in the range of β receives. Similarly, $\tilde{p}(s)$ denotes the payoff to any seller choosing attribute s in the range of σ.

We assume that any buyer who chooses attribute b in the range of β will receive payoff $\tilde{x}(b)$, capturing the idea that in a continuum economy an agent who mimics another agent will receive the same (gross) payoff as the imitated agent. Similarly, a seller choosing attribute s in the range of σ will receive payoff $\tilde{p}(s)$. But what is the payoff to a buyer choosing an attribute b *not* in the range of β? With supermodular surplus functions, stable matchings must be positively assortative. For any attribute b a buyer might choose, stability thus determines the attribute of the seller he or she will be matched with, denoted $\tilde{s}(b)$. The buyer's (gross) payoff, $\tilde{x}(b)$, when he or she chooses attribute b will then be $v(b, \tilde{s}(b)) - \tilde{p}(\tilde{s}(b))$, again capturing the idea that in a continuum economy, individual agents' choices don't affect the payoffs to other agents. Note that for b in the range of β, this agrees with our earlier definition. The seller's gross payoff, $\tilde{p}(s)$, is similarly extended. We call $\{(\beta, \sigma), (\tilde{x}, \tilde{p})\}$ an *ex post contracting equilibrium* if for all i, $\beta(i)$ maximizes $\tilde{x}(b) - \psi(b, i)$ and $\sigma(i)$ maximizes $\tilde{p}(s) - c(s, i)$.

It is straightforward to argue that choosing attributes that maximize aggregate net surplus, and dividing the consequent surplus equally, constitutes an ex post contracting equilibrium: Buyer i's problem is to choose the attribute that results in a match for which i's share of the surplus less his cost of that attribute is highest, that is

$$\max_b \tfrac{1}{2} v(b, \tilde{s}(b)) - \psi(b, i).$$

The first order condition is

$$\frac{1}{2}\frac{\partial v(b, \tilde{s}(b))}{\partial b} + \frac{1}{2}\frac{\partial v(b, \tilde{s}(b))}{\partial s}\frac{d\tilde{s}}{db} - \frac{\partial \psi(b, i)}{\partial b} = 0.$$

By the symmetry between buyers and sellers, $\tilde{s}(b) = b$ and $\partial v(b, \tilde{s}(b))/\partial b = \partial v(b, \tilde{s}(b))/\partial s$. Hence, the first order condition can be rewritten as

$$\frac{\partial v(b, b)}{\partial b} - \frac{\partial \psi(b, i)}{\partial b} = 0,$$

which is equivalent to the first order condition from jointly maximizing the total net surplus. Note that this argument works for any symmetric surplus function v.

The observation that agents have an incentive to choose their joint maximizing attribute choice when all other agents are doing so is particularly straightforward in the presence of symmetry.[3] We show in Section 5 that this property does *not* depend on symmetry.

We next show that not every ex post equilibrium yields efficient investment. We modify the surplus function to

$$v^*(b, s) = \begin{cases} bs, & \text{if } bs \leqslant \frac{1}{2}, \\ 2(bs)^2, & \text{if } bs > \frac{1}{2}. \end{cases}$$

[3] A similar argument can be found in Kremer [17].

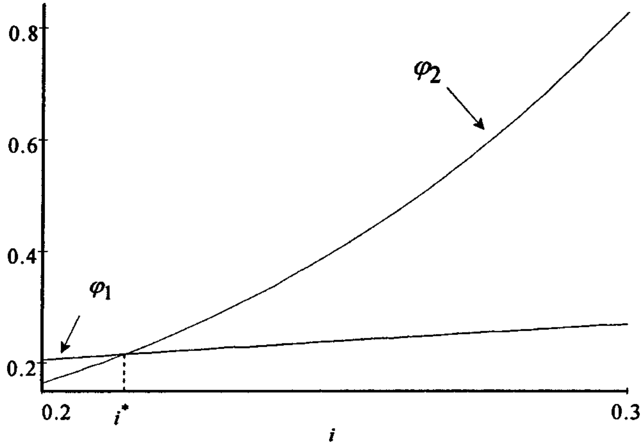

FIG. 1. The net surplus functions corresponding to bs and $2(bs)^2$, where $\varphi_1(i) = \max_{b,s} bs - \psi(b,i) - c(s,i)$ and $\varphi_2(i) = \max_{b,s} 2(bs)^2 - \psi(b,i) - c(s,i)$.

The cost functions are unchanged. Note that bs and $2(bs)^2$ are both strictly supermodular on $\Re^2_+$, and moreover, $2(bs)^2 < bs$ if and only if $bs < \frac{1}{2}$. Hence, this new surplus function is supermodular as well. The joint maximizing choices for some agent pairs will clearly be higher under v^* than under v, since the marginal product of attribute is higher.

Figure 1 shows the net surplus functions for agents with index in the interval $[\underline{i}, \bar{\imath}] = [.2, .3]$ that correspond to efficient attribute choice under surplus functions bs and $2(bs)^2$. In Fig. 2, we show the buyers' efficient attribute choices $\hat{\beta}_1(i)$ and $\hat{\beta}_2(i)$ for the surplus functions bs and $2(bs)^2$ respectively. (The sellers' efficient attribute choices are the same as those of the buyers'.)

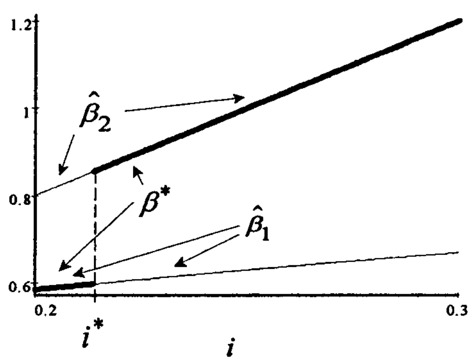

FIG. 2. The attribute choice functions.

For the "hybrid" surplus function v^*, efficient attribute choices are given by $\hat{\beta}_1(i)$ for agents with index below i^*, and by $\hat{\beta}_2(i)$ for agents with index above i^*, where i^* is the agent index for which $(\hat{\beta}_1(i))^2 - 2(\hat{\beta}_1(i))^5/(5i) = 2(\hat{\beta}_2(i))^4 - 2(\hat{\beta}_2(i))^5/(5i)$. There is a discontinuity in agents' attribute choice at i^*, the point at which the net surplus under $v = 2(bs)^2$ overtakes $v = bs$.[4] Of course, the definition of i^* ensures that there is no discontinuity in net payoffs there.

Since v^* is symmetric, there is an ex post contracting equilibrium in which the efficient attributes are chosen. However, for the case $[\underline{i}, \bar{i}] = [.2, .3]$, there will be an inefficient underinvestment equilibrium as well. If all agents' attribute choices are given by $b_i = \hat{\beta}_1(i)$, and $s_j = \hat{\sigma}_1(j)$, no agent will have an incentive to alter his choice. It is not profitable for buyer $\bar{i} = .3$, for example, to deviate because the maximum attribute available among all sellers is so low. Essentially, there is a coordination failure in this equilibrium: For all matched pairs with index $i > i^*$, increasing both b_i and s_i increases the net payoffs to both buyer and seller, but an increase by only one of the agents will decrease that agent's net payoff. However, for $\bar{i} > .63$, the diversity in attribute choices under $\hat{\beta}_1$ is so large that the underinvestment outcome is no longer an equilibrium (see Section 6).

Besides these two equilibria—efficient attribute investment and underinvestment—there is, again for the case $[\underline{i}, \bar{i}] = [.2, .3]$, a third equilibrium with overinvestment. Appendix D verifies that $b_i = \hat{\beta}_2(i)$ and $s_i = \hat{\sigma}_2(i)$ for all $i \in [\underline{i}, \bar{i}]$ constitutes an equilibrium. This is clearly overinvestment for all matched pairs with index $i < i^*$. It is not profitable for buyer $\underline{i} = .2$, for example, to deviate because the marginal reward of increasing attribute is so high, even matching with the lowest attribute seller. On the other hand, if $\underline{i} = .1$, the overinvestment outcome is not an equilibrium (again, see Section 6).

This example demonstrates that moving to a continuum of agents eliminates several undesirable effects of a finite population, but the possibility of either overinvestment or underinvestment remains. The discussion of the example, however, glossed over a number of substantial technical issues. After discussing the related literature, we deal with these technical issues and investigate in more detail the extent of the possible inefficiencies.

3. RELATED LITERATURE

Hart [13, 14] and Makowski and Ostroy [19] are conceptually close to our paper. Hart [13] analyzed a model of monopolistic competition in

[4] It is important to note that this is *not* because of the kink in v^*, but rather because of the supermodularity of v^*. Similar examples could be constructed with smooth surplus functions.

which firms simultaneously decide whether to pay a setup cost that will enable them to produce goods, and subsequently to choose what goods to produce (this is analogous to our attribute choice) and the prices they will charge for those goods. A finite number of firms will choose to enter the market, and they will choose unique goods to produce, earning monopolistic profits because of the uniqueness.

An important assumption in Hart [13] is that consumers' preferences are convex and differentiable. Hart [14] drops the assumption on preferences and shows that if there are complementary goods, there may be inefficient equilibria due to coordination failure even in large economies. While equilibria in Hart [14] may be inefficient, they are constrained efficient in the sense that if a given equilibrium is inefficient, it cannot be Pareto dominated by an equilibrium allocation using only the goods produced in the equilibrium. We prove a similar result (Proposition 4): Pareto gains necessitate changing the ex ante decisions of multiple agents.

Makowski and Ostroy [19] consider a finite population model in which individuals choose occupations, and those occupations determine the goods that can be consumed. The aim of their paper is to demonstrate that when each individual's benefit from an occupational choice coincides with the social contribution of that choice (*full appropriation*), and there are no complementarities among occupational choices, equilibria will be efficient. The condition that there are no complementarities rules out the coordination-failure inefficiency treated in Hart [14] and which can arise in our model. A version of full appropriation holds in our model (see Section 4.1), and we focus on the implications of the existence of complementarities.

Unlike Hart [13, 14] and Makowski and Ostroy [19], we work in a matching-bargaining environment that permits a more transparent modeling of complementarities in production, as well as the equilibrium determination of the division of the resulting surplus. This setting also allows us to obtain more informative results on the scope and nature of the inefficiencies that can arise in equilibrium.

Subsequent to our work, there have been several other papers that study the case in which contracting at the time investments are made is ruled out. Felli and Roberts [10] analyze a finite agent model in which Bertrand competition among workers for jobs leads to efficient investment. It is worth noting that, for the finite version of our model (analyzed in Cole, Mailath, and Postlewaite [5]), the Bertrand competition they study is a noncooperative selection from the set of stable payoffs in the ex post contracting game. DeMeza and Lockwood [7] and Chatterjee and Chiu [4] analyze models in which both sides of a market can undertake investments prior to matching. Both, however, analyze models that are constructed to generate inefficient investment, with the aim to understanding how different ownership structures affect the inefficiency. Peters and Siow [20] analyze a model in which utility is not transferable between parties (the marriage problem) and demonstrate conditions under which investments will be efficient.

Besides these papers, there are several other papers that are related, but less closely. Acemoglu [1] studies a model with two-sided investments in a matching setting, but with costly bilateral search to obtain pecuniary externalities. Acemoglu [2] analyzes a worker-firm model in which there may be inefficient underinvestment in human capital. The inefficiency in that model stems from costly search if a worker-firm match is dissolved. Acemoglu and Shimer [3] use a matching model with one-sided investments to investigate the hold-up problem. Their focus, however, is on the role of search frictions and the non-investing partner's ability to direct search on the efficiency of investments. Cole and Prescott [6] and Ellickson, Grodal, Scotchmer, and Zame [8], [9] analyze models that take agents' characteristics as given. When agents differ in ability, coalitions are inefficiently small. MacLeod and Malcomson [18] study the hold-up problem associated with investment decisions taken prior to contracting and provide, in a specific model, the idea that ex ante investments will be efficient, as long as the investments are general and there are outside options. That investments in their model are general leads to competition for the individual making the investment, assuring him or her of the incremental surplus that results from the investment. This is similar to the effect of competition from agents with attributes that are close in our model. Their model, however, doesn't give rise to coordination inefficiencies.

4. THE EX POST ASSIGNMENT GAME

There is a continuum of buyers and of sellers, with the population of each described by Lebesgue measure on $[0, 1]$. Buyer $i \in [0, 1]$ can choose attribute $b \in \Re_+$ at a cost $\psi(b, i)$, and seller $j \in [0, 1]$ can choose attribute $s \in \Re_+$ at a cost $c(s, j)$.[5] A buyer of attribute b who matches with a seller of attribute s generates a (gross) surplus of size $v(b, s)$.

The surplus function v is $\mathscr{C}^2$ and displays strict complementarities in attributes (v is supermodular): for $b < b'$ and $s < s'$, $v(b', s) + v(b, s') < v(b, s) + v(b', s')$. Since v is $\mathscr{C}^2$, this is equivalent to $\partial^2 v / \partial b\, \partial s > 0$. We also assume v is strictly increasing in b and in s.

Buyers and sellers first simultaneously choose attributes and, subsequent to the choice of attributes, match and share the surplus generated by these matches. Denote buyers' and sellers' behavior by the respective functions

[5] Some attributes may be infinitely costly.

$\beta: [0, 1] \to \mathfrak{R}_+$ and $\sigma: [0, 1] \to \mathfrak{R}_+$. We model the bargaining and matching process that follows the attribute choices (β, σ) as a cooperative game. Given a fixed distribution of buyers' and sellers' attributes, the resulting cooperative game is an *assignment game*: there are two populations of agents (here, buyers and sellers), with each pair of agents (one from each population) generating some value if matched. We call this assignment game the *ex post assignment game* (indicating that attribute choices are taken as fixed). We describe later the ex ante assignment game. An *outcome* in the assignment game is a *matching* (intuitively, each buyer matching with no more than one seller and each seller matching with no more than one buyer) and a *bargaining outcome* (a division of the value generated by each matched pair between members of that pair). We denote buyer i's return from the surplus by $x(i) \geqslant 0$ and seller j's return by $p(j) \geqslant 0$.

In a model with finite populations of buyers and sellers, a bargaining outcome is feasible if in all matched pairs (i, j), $x(i) + p(j) \leqslant v(\beta(i), \sigma(j))$, and unmatched agents receive zero. A continuum of agents presents some complications in defining feasible bargaining outcomes. We first define feasible bargaining outcomes; we discuss the definition in some detail after the definition of stability.

Definition 1. Suppose β and σ are strictly increasing. A bargaining outcome (x, p) is *feasible* if, for all $i, j \in [0, 1]$,

$$x(i) \leqslant \max\{\limsup_{j' \to i} [v(\beta(i), \sigma(j')) - p(j')], 0\}$$

and

$$p(j) \leqslant \max\{\limsup_{i' \to j} [v(\beta(i'), \sigma(j)) - x(i')], 0\}.$$

To capture the idea that the division of the surplus within any match should respect outside options, we require that the bargaining outcome, with its associated matching, be *stable*: there are no pairs of agents who, by matching and sharing the resulting surplus, can make themselves strictly better off.[6]

[6] While the definition of stability has a cooperative feel, the notion is not inherently cooperative. Equilibrium outcomes of almost any noncooperative game with frictionless matching will be stable. See, for example, Felli and Roberts [10].

The set of stable bargaining outcomes coincides with the core. Moreover, the core of any assignment game is nonempty and coincides with the set of Walrasian allocations (see Kaneko [16] and Quinzii [21] for the finite population case and Gretsky, Ostroy, and Zame [11] for the continuum population case).

DEFINITION 2. A bargaining outcome (x, p) is *stable* if it is feasible and for all $i, j \in [0, 1]$,

$$x(i) + p(j) \geqslant v(\beta(i), \sigma(j)). \tag{1}$$

A matching associated with a stable bargaining outcome is a *stable matching*.

In a model with finite populations of buyers and sellers, a bargaining outcome is feasible if in all matched pairs (i, j), $x(i) + p(j) \leqslant v(\beta(i), \sigma(j))$, and unmatched agents receive zero. Moreover, stability implies that matching is positively assortative in attributes (this is an immediate implication of v being supermodular). In our case, with a continuum of buyers and sellers, it would then be natural to specify that when the attribute functions, β and σ, are strictly increasing, i matches with $j = i$.[7] Indeed, as we will see, when the attribute functions are continuous (as well as strictly increasing) and matching is positively assortative in index, feasibility is adequately captured by the finite population pairwise feasibility requirement: $x(i) + p(i) \leqslant v(\beta(i), \sigma(i))$ for all i. However, as we saw in Section 2, there is no reason to believe that endogenous attribute choices will necessarily be continuous functions of agent characteristics. Indeed, efficient attribute choices may *preclude* continuity. Feasible bargaining outcomes must then be defined when attribute functions are increasing, but not necessarily continuous.

We illustrate the issues through an example: Suppose first that $v(b, s) = bs$, $\beta(i) = 1 + i$ for all i, $\sigma(j) = 1 + j$ for all j, and matching is positively assortative by index (equivalently, by attribute). Then the bottom pair generates a surplus of 1, and equal division of the surplus for each pair is feasible under the pairwise feasibility requirement and stable. Suppose now the bottom buyer's attribute is 0 rather than 1 (i.e., $\beta(0) = 0$). The pairwise feasibility requirement forces $p(0) = 0$. However, the point of modeling the population of agents as a continuum is to capture the idea that a single agent's actions do not adversely affect the feasible returns available to other agents (since the other agents can avoid this agent).

Consider now the sequence of matchings $\{m_n\}_{n=2}^{\infty}$ where i matches with $j = i$, except that buyers 0 and $\frac{1}{n}$ exchange partners.[8] If returns under m_n are determined by equal division of the induced surpluses, then the returns for all agents, *except* buyer 0, converge to the returns they receive under equal

[7] The feature that buyer i matches with seller i is a labeling convention. Suppose buyers are distributed on an interval $[\underline{i}, \bar{\imath}]$ according to a strictly increasing distribution F_1 with attribute cost function $\tilde{\psi}$. Assign to buyer $\tilde{\imath}$ the index $F_1(\tilde{\imath})$. Buyers are now uniformly distributed on $[0, 1]$, with buyer i facing a cost of $\psi(b, i) = \tilde{\psi}(b, F_1^{-1}(i))$ for attribute b. A similar relabeling applies to sellers.

[8] That is, $m_n \colon [0, 1] \to [0, 1]$ is given by $m_n(0) = \frac{1}{n}$, $m_n(\frac{1}{n}) = 0$, and $m_n(i) = i$ for all $i \neq 0, \frac{1}{n}$. Note that m_n is one-to-one and preserves measure.

division when $\beta(0)=1$. This *includes* seller 0. Thus, there is a sequence of matchings that yield returns that satisfy the pairwise feasibility requirement, and yet their limit does not. Note, moreover, that in the case $\beta(0)=0$, the pairwise feasibility requirement with stability forces $p(j)\to 0$ as $j\to 0$. At an intuitive level, we would like the bargaining outcome (x^*, p^*), where $x^*(0)=0$, $x^*(i)=(1+i)^2/2$ for $i>0$, and $p^*(j)=(1+j)^2/2$ for all j, to be feasible and stable.[9] Our definition of feasibility accomplishes this.

Another possibility is to require pairwise feasibility only almost everywhere, rather than everywhere. The drawback with this notion for our purposes is that the payoffs to a single agent when he or she deviates in choice of attribute is not determined. Our definition has the essential feature that when combined with stability, it uniquely determines a single agent's return as a function of the other agents' returns. This is necessary if an agent is to compare returns from different attribute choices.

Rather than give a complete treatment of feasibility in all assignment games with a continuum of agents and arbitrary attribute choice functions, we have defined feasibility in the simple case of strictly increasing attribute choice functions with positively assortative matching on index effectively imposed. Almost everywhere positive assortative matching by attribute can be deduced from stability and the notion of feasibility used by Gretsky, Ostroy, and Zame [11] or that used by Kamecke [15].[10] In Section 4.1, we will be concerned with the total social surplus of nonincreasing attribute choice functions, and in that case, we use the feasibility notion of Gretsky, Ostroy, and Zame [11]. The indeterminacy of individual payoffs is not an issue when we are concerned only with total social surplus.

[9] It is not critical in this example that the bottom buyer has chosen an isolated attribute. The same issue arises whenever there is a discontinuity in the attribute choice functions. Suppose, for example, that the buyer attribute choice function is discontinuous. We would like the set of sellers' feasible returns to be the same when the buyer attribute choice function only differs in whether it is continuous from the left or from the right.

[10] Our notion of feasibility differs from that in Gretsky, Ostroy, and Zame [11] and in Kamecke [15]. Since our definition applies only to positively assortative matchings, we have not described feasibility for "most" matchings. The measure-theoretic notion of feasibility in Gretsky, Ostroy, and Zame [11], when combined with stability, does not force isolated attributes to have unique returns (when other agents' returns are fixed). The notion of feasibility in Kamecke [15] effectively requires that the attribute functions be continuous. Kamecke defines a bargaining outcome to be feasible if it can be approximated, in the sense of uniform convergence, by payoffs that are pairwise feasible. In our example, (x^*, p^*) would not be feasible under this notion. Simply requiring pointwise convergence, on the other hand, is too weak, since under this notion of feasibility, there are feasible and stable returns that violate equal treatment. Consider again the example, but with $\beta(i)=\sigma(j)=1$ for all i and j. Let m_n be the matching described in footnote 8. The payoff (x_n, p_n) given by $x_n(0)=\frac{3}{4}$, $x_n(i)=\frac{1}{2}$, $p_n(\frac{1}{n})=\frac{1}{4}$, and $p_n(j)=\frac{1}{2}$ is feasible for m_n. Moreover, it converges pointwise to the stable returns $(\tilde{x}, \tilde{p})$, where $\tilde{x}(0)=\frac{3}{4}$, $\tilde{x}(i)=\frac{1}{2}$, and $\tilde{p}(j)=\frac{1}{2}$.

There are several things to note about our definition of feasibility. First, if all the relevant functions (β, σ, x, and p) are continuous and the non-negativity constraints are not binding, this reduces to the pairwise feasibility definition for positively assortative matching by index. Second, the role of the nonnegativity constraint (which, we show below, cannot bind almost everywhere) is to describe agents like buyer 0 in the example above. Finally, as in the example, with a continuum of agents, an agent i may not be matching with precisely $j=i$. Rather, he or she may be matching with agents arbitrarily close to $j=i$. Moreover, these matches may yield higher returns. Taking the lim sup captures these possibilities.[11]

It is immediate that the definition of stability implies that the inequalities in the definition of feasibility hold as equalities for stable bargaining outcomes. In the finite case, equal treatment implies that if stable returns have been fixed for all but one buyer (similar statements hold for sellers) and if that buyer has the same attribute as a second buyer, then that buyer's return is determined by the second buyer's return. There is a similar result for the continuum agent case. Suppose that stable returns have been fixed for all but one buyer. Then that buyer's return is determined by that of any other buyers whose attributes are arbitrarily close.

Lemma 1. *Suppose β and σ are strictly increasing. For any stable bargaining outcome (x, p), x and p are strictly increasing (and so their left hand and right hand limits exist). Moreover, x and p inherit the continuity properties of β and σ, respectively (i.e., if β is continuous from the left at i', then x is continuous from the left at i', etc.).*

Proof. See Appendix A.

Let $C(\beta, \sigma)$ be the set of common continuity points of β and σ. By Lemma 1, for $i' \in C(\beta, \sigma)$, stable x and p are both continuous at i', and so $x(i') \leqslant \max\{v(\beta(i'), \sigma(i')) - p(i'), 0\}$ and $p(i') \leqslant \max\{v(\beta(i'), \sigma(i')) - x(i'), 0\}$. Hence, $x(i'), p(i') \leqslant v(\beta(i'), \sigma(i'))$ and so $x(i') + p(i') = v(\beta(i'), \sigma(i'))$. We can thus assume that buyer i with attribute $b = \beta(i)$ is matching with precisely seller $j = i$ with attribute $s = \sigma(i)$. This allows us to define the function $\tilde{s}\colon \beta(C(\beta, \sigma)) \to S$ given by $\tilde{s}(b) = \sigma(\beta^{-1}(b))$ and the function $\tilde{b}\colon \sigma(C(\beta, \sigma)) \to B$ given by $\tilde{b}(s) = \beta(\sigma^{-1}(s))$. For $b \in \beta(C(\beta, \sigma))$, $\tilde{s}(b)$ is the attribute of the seller that the buyer with attribute b matches with.

It is also helpful to have specific notation for the return that a particular attribute receives in a stable bargaining outcome (x, p). Suppose β and σ are strictly increasing. Define

$$\tilde{x}(b) \equiv x(\beta^{-1}(b)) \tag{2}$$

[11] We need to take the lim sup, rather than simply taking limits, because the limit does not exist when the attribute functions are discontinuous.

and

$$\tilde{p}(s) \equiv p(\sigma^{-1}(s)). \tag{3}$$

Equivalently, $(x, p) = (\tilde{x} \circ \beta, \tilde{p} \circ \sigma)$. We say the return vector $(\tilde{x}, \tilde{p})$ is *stable* if $(\tilde{x}(\beta), \tilde{p}(\sigma))$ is stable.

The special case in which there are isolated attribute choices is straightforward to analyze and does not add anything substantive. To simplify notation, we accordingly rule out isolated attribute choices in the statement of our characterization result.

DEFINITION 3. A function is *well-behaved* if it is strictly increasing, discontinuous at only a finite number of points, Lipschitz on every interval of continuity points, and has no isolated values.

We use repeatedly the following property of well-behaved functions. Suppose $f: [0, 1] \to \Re$ is well-behaved. There is then a finite set of points $D \equiv \{i_1, i_2, ..., i_T\}$ at which f is discontinuous, and f is continuous for all $i \notin D$. Define $I_t = (i_t, i_{t+1})$ (with the obvious modification for $t = 0$ and $t = T$). Since f is monotone, f is differentiable almost everywhere. Since f is Lipschitz on each I_t, f is absolutely continuous on I_t, and so f is the indefinite integral of its derivative on each I_t.

We now characterize the stable bargaining outcomes of the assignment game for well-behaved attribute-choice functions. Kamecke [15] has previously shown that stability implies part 3 of the proposition for surpluses that need not be supermodular, when β and σ are differentiable everywhere. As usual, $f(x+)$ denotes the right hand limit ($f(x+) = \lim_{\varepsilon \downarrow 0} f(x+\varepsilon)$) and $f(x-)$ denotes the left hand limit ($f(x-) = \lim_{\varepsilon \downarrow 0} f(x-\varepsilon)$).

PROPOSITION 1. *Suppose β and σ are both well-behaved. Stable bargaining outcomes (x, p) exist. The bargaining outcome (x, p) is stable if and only if the following hold*:

1. *No waste*:

$$x(i) + p(i) = v(\beta(i), \sigma(i)) \qquad \forall i \in C(\beta, \sigma); \tag{4}$$

2. *x and p are continuous at all $i \in C(\beta, \sigma)$*;

3. *$\tilde{x}$ and $\tilde{p}$ are differentiable on $\beta(C(\beta, \sigma))$ and $\sigma(C(\beta, \sigma))$, respectively, with derivatives*

$$\tilde{x}'(b) = \frac{\partial v(b, \tilde{s}(b))}{\partial b} \qquad \textit{for all} \quad b \in \beta(C(\beta, \sigma)), \textit{ and} \tag{5}$$

$$\tilde{p}'(s) = \frac{\partial v(\tilde{b}(s), s)}{\partial s} \qquad \textit{for all} \quad s \in \sigma(C(\beta, \sigma)); \tag{6}$$

and

4. *at any point of discontinuity i,*

$$x(i+)+p(i+)=v(\beta(i+),\sigma(i+)),$$
$$x(i+)-x(i-)\geqslant v(\beta(i+),\sigma(i-))-v(\beta(i-),\sigma(i-)),\ \textit{and} \tag{7}$$
$$p(i+)-p(i-)\geqslant v(\beta(i-),\sigma(i+))-v(\beta(i-),\sigma(i-)).$$

Proof. See Appendix A.

The two inequalities in (7) are written to be symmetric with (5) and (6). They are equivalent to

$$x(i+)+p(i-)\geqslant v(\beta(i+),\sigma(i-))$$

and

$$x(i-)+p(i+)\geqslant v(\beta(i-),\sigma(i+)).$$

It should be clear from Proposition 1 that there may be multiple stable bargaining outcomes, where the multiplicity is due to the indeterminacy of the division of the surplus $v(\beta(0),\sigma(0))$ for the bottom pair of agents (in the event that this surplus is positive), as well as the indeterminacy (constrained only by (7)) at any discontinuities of the attribute functions.

Let $\mathscr{F}$ be the set of well-behaved attribute-choice functions and $\mathscr{P}$ be the set of possible bargaining outcomes. A *bargaining outcome function* $g: \mathscr{F} \to \mathscr{P}$ is a mapping that selects a stable bargaining outcome $(x, p) = g(\beta, \sigma)$ for every pair of well-behaved attribute-choice functions. Note that if two well-behaved functions agree almost everywhere, then their continuity points agree. Suppose $(x, p) = g(\beta, \sigma)$ and $(\bar{x}, \bar{p}) = g(\bar{\beta}, \bar{\sigma})$. We require that if $\beta = \bar{\beta}$ a.e. and $\sigma = \bar{\sigma}$ a.e., then $x(i) = \bar{x}(i)$ and $p(i) = \bar{p}(i)$ for all $i \in C(\beta, \sigma) = C(\bar{\beta}, \bar{\sigma})$. The values of the returns for $i \notin C(\beta, \sigma)$ are then uniquely determined (Lemma 1).

4.1. *An Interpretation of the Marginal Condition*

Equations (5) and (6) clearly have bearing on whether agents have incentives to efficiently invest in a stable bargaining outcome. Equation (5) states that at any stable bargaining outcome, the marginal return to each buyer to increasing his or her attribute is equal to the marginal change in the surplus in his or her match. The question of whether this guarantees efficient investments, however, is subtle. In a world with a finite number of agents, when a buyer changes his or her attribute, he or her may well end up matched with a different seller since stable matchings must maintain positive assortative matching. But if the consequence of a buyer increasing his attribute is that the seller he or her matches with has a higher attribute, the buyer with whom that seller was initially matched must find himself or herself matched with a lower attribute seller. At the same time, the original partner is also now matching with a buyer with a higher attribute.

In other words, a buyer who changes his or her attribute creates an externality on other buyers and sellers. In principle, there can be a large set of other agents, both buyers and sellers, who find themselves in different matches as a result of a single buyer's attribute change. Indeed, our modeling choice of a continuum of agents was motivated precisely by this fact. While a continuum of agents obviates this difficulty, we should be cautious about what is meant by the impact on social surplus from a marginal change in a buyer's attribute, since these externalities are suppressed with a continuum of agents. We now introduce a notion of marginal social surplus that takes into account external effects and show that the right hand side of (5) is, in fact, the appropriate notion of marginal social surplus.

Suppose β and σ are well-behaved. Then, total surplus is maximized by matching buyer i with seller i, for almost all i. The total surplus of (β, σ) is then

$$V(\beta, \sigma) = \int_0^1 v(\beta(i), \sigma(i))\, di.$$

We consider the change in social surplus when an interval of buyers containing a particular buyer all increase their attribute by a given amount, δ, and take the limit as both the measure of the set of buyers who are changing their attribute goes to 0, and the amount by which the buyers increase their attribute goes to 0.

Fix $\bar{\imath} \in C(\beta, \sigma) \cap (0, 1)$ and $\varepsilon > 0$ such that $(\bar{\imath} - \varepsilon, \bar{\imath} + \varepsilon) \subset C(\beta, \sigma)$ (recall that $C(\beta, \sigma)$ is a union of open intervals, except for the subintervals including 0 and 1). Fix δ and consider the attribute choice function in which the ε neighborhood of agent $\bar{\imath}$ increase their attribute by δ,

$$\beta^{\delta, \varepsilon}(i) = \begin{cases} \beta(i) + \delta, & i \in [\bar{\imath} - \varepsilon, \bar{\imath} + \varepsilon], \\ \beta(i), & i \notin [\bar{\imath} - \varepsilon, \bar{\imath} + \varepsilon]. \end{cases}$$

Consider the assignment game with the population of buyers described by $\beta^{\delta, \varepsilon}$ and of sellers described by σ, i.e., the assignment game when the buyers in $[\bar{\imath} - \varepsilon, \bar{\imath} + \varepsilon]$ have changed their attribute by δ.[12] (See Fig. 3.) Let $\Delta_{\delta, \varepsilon}(\bar{\imath})$ be the change in social surplus due to the increase in the buyers' attributes, taking into account externalities,

$$\Delta_{\delta, \varepsilon}(\bar{\imath}) = V(\beta^{\delta, \varepsilon}, \sigma) - V(\beta, \sigma).$$

[12] Note that $\beta^{\delta, \varepsilon}(i)$ is not well-behaved. As we mentioned earlier, we use the feasibility notion of Gretsky, Ostroy, and Zame [11] to calculate $V(\beta^{\delta, \varepsilon}, \sigma)$.

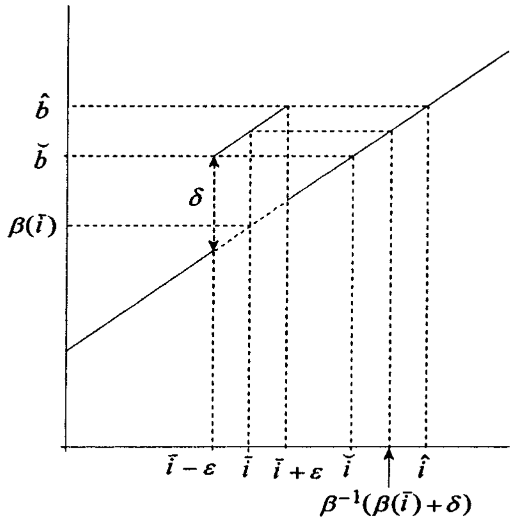

FIG. 3. The modified attribute choice function, $\beta^{\delta,\varepsilon}$.

We then have the following proposition (proved in Appendix B):[13]

PROPOSITION 2. *Suppose* β *and* σ *are well behaved. For* $\bar{\imath} \in C(\beta, \sigma) \cap (0, 1)$,

$$\lim_{\delta \to 0} \frac{1}{\delta} \left\{ \lim_{\varepsilon \to 0} \frac{\Delta_{\delta,\varepsilon}}{2\varepsilon} \right\} = \frac{\partial v(\beta(\bar{\imath}), \sigma(\bar{\imath}))}{\partial b}.$$

Hence, the right hand side of (5) is, in fact, the marginal social value of a change in buyer i's attribute. Thus, in stable bargaining outcomes, agents receive their marginal social value, or, in the language of Makowski and Ostroy [19], full appropriability holds.

5. EX POST AND EX ANTE CONTRACTING EQUILIBRIA

We now turn to attribute investment decisions. In *the economy with ex post contracting*, agents noncoooperatively invest in attributes and then receive a payoff from the resulting ex post assignment game. In *the economy with ex ante contracting*, agents can contract over attributes,

[13] The analogous statement for sellers obviously holds as well.
We believe that the order of limits is not important. If the derivatives of the attribute choice functions are bounded away from zero, then a similar argument to that in Appendix B yields the result directly.

matches, and the division of the resulting surplus prior to investing in attributes. The economy with ex post contracting is a combination of a noncooperative investment game followed by a cooperative assignment game. On the other hand, the economy with ex ante contracting is solely a cooperative assignment game.

We first determine the change in return to an agent who unilaterally changes attribute. After such a deviation by buyer i', say, the resulting attribute choice function is no longer well-behaved. However, it will fail to be well-behaved only because of a single agent's choice of attribute. Accordingly, we *assume* that all agents' returns, except for buyer i', are determined as if β and σ are well-behaved. Since this only involves altering a single agent's attribute choice, this implies that a single agent changing attribute does not change other agents' returns. Let (x, p) be the bargaining outcome when the attribute-choice functions (β, σ) are well-behaved, and suppose i' chooses an attribute $b \neq \beta(i')$. For $b \leqslant \beta(1)$, define $i_b \equiv \inf \{i: b \leqslant \beta(i)\}$, and for $b > \beta(1)$, define $i_b \equiv 1$. Then, the return to attribute b is

$$\tilde{x}(b) = \max\{v(b, \sigma(i_b-)) - p(i_b-), v(b, \sigma(i_b+)) - p(i_b+), 0\}, \quad (8)$$

where $i_b- \;=0$ when $i_b = 0$, and $i_b+ \;=1$ when $i_b = 1$. Note that if $b \in \beta([0, 1])$, then the definition of $\tilde{x}(b)$ coincides with (2). A similar construction applies to sellers.

Given a well-behaved pair of attribute-choice functions (β, σ), and an associated bargaining outcome $g(\beta, \sigma) = (x, p)$, we thus have attribute returns $\tilde{x}: \Re_+ \to \Re_+$ and $\tilde{p}: \Re_+ \to \Re_+$ that are well-defined on all possible attribute choices. We are now in a position to define an equilibrium of the economy with ex post contracting.

DEFINITION 4. An *ex post contracting equilibrium* is a quadruple $\{(\beta, \sigma), (x, p)\}$ where β and σ are well-behaved, such that

1. (x, p) is a stable *bargaining outcome* for the attribute choices (β, σ), and

2. for each $i \in [0, 1]$ and $b \in \Re_+$,

$$\tilde{x}(\beta(i)) - \psi(\beta(i), i) \geqslant \tilde{x}(b) - \psi(b, i),$$

and for each $j \in [0, 1]$ and $s \in \Re_+$,

$$\tilde{p}(\sigma(j)) - c(\sigma(j), j) \geqslant \tilde{p}(s) - c(s, j).$$

We compare the investments taken in an ex post contracting equilibrium with the investments agents would make if buyers and sellers could contract with each other over matches, the investments to be undertaken, and the sharing of the resulting surplus. If a buyer i and seller j agree to match and make investments b and s respectively, then the total surplus so generated is $v(b, s) - \psi(b, i) - c(s, j) \equiv W(b, s; i, j)$. In a world of ex ante contracting, investments maximize this total surplus. Thus, if buyer i and seller j are considering matching, they are bargaining over the net surplus $\varphi(i, j) = \max_{b,s} W(b, s; i, j)$. (We make assumptions below that guarantee that φ is well-defined.) The *ex ante assignment game* is the assignment game with the value function φ. Just as we considered stable outcomes for the ex post assignment, we impose stability on outcomes of the ex ante assignment game. If φ is supermodular, then (as for the ex post contracting assignment game) total net surplus is maximized by positive assortative matching over index. Feasibility is in fact simpler, since (from the Maximum Theorem) φ is a continuous function of the indices, which themselves form a connected set. Thus, in the following definition of an equilibrium of the economy with ex ante contracting, we can assume that buyer i matches with seller i.

DEFINITION 5. Suppose φ is supermodular. The outcome of the ex ante assignment game $\{(\beta^*, \sigma^*), (x^*, p^*)\}$ is an *ex ante contracting equilibrium* if

1. $x^*(i) + p^*(i) \leqslant v(\beta^*(i), \sigma^*(i))$ for all i; and
2. for all $i, j \in [0, 1]$,

$$x^*(i) - \psi(\beta^*(i), i) + p^*(j) - c(\sigma^*(j), j) \geqslant \varphi(i, j).$$

We make some standard assumptions on the surplus and cost functions that imply that φ is well-defined and strictly supermodular.

Assumption 1. The surplus function $v: \Re^2_+ \to \Re_+$ is $\mathscr{C}^2$ with $\partial v(b, s)/\partial b > 0$, $\partial v(b, s)/\partial s > 0$, and $\partial^2 v(b, s)/\partial b \partial s > 0$ for all $(b, s) \in \Re^2_+$. The cost functions satisfy:

1. for each $i \in [0, 1]$ there exists $\bar{B}(i) > 0$ such that $\lim_{b \to \bar{B}(i)} \psi(b, i) = \infty$;
2. ψ is continuous on $\{(b, i): i \in [0, 1], b \in [0, \bar{B}(i))\}$ and $\mathscr{C}^2$ on its interior;
3. $\psi(0, i) = 0$, $\lim_{b' \to 0} \partial \psi(b', i)/\partial b = 0$, for all $i \in (0, 1)$;
4. $\partial \psi(b, i)/\partial b > 0$, $\partial^2 \psi(b, i)/\partial b^2 > 0$ and $\partial^2 \psi/\partial b\, \partial i < 0$ for $b \in (0, \bar{B}(i))$, $i \in (0, 1)$;

5. for each $j \in [0, 1]$ there exists $\bar{S}(j) > 0$ such that $\lim_{s \to \bar{S}(j)} c(b, j) = \infty$;

6. c is continuous on $\{(s, j): j \in [0, 1], s \in [0, \bar{S}(j))\}$ and $\mathscr{C}^2$ on its interior;

7. $c(0, j) = 0$, $\partial c(0, j)/\partial s = 0$, for all $j \in (0, 1)$; and

8. $\partial c(s, j)/\partial s > 0$, $\partial^2 c(s, j)/\partial s^2 > 0$ and $\partial^2 c/\partial s \partial j < 0$ for $s \in (0, \bar{S}(j))$, $j \in (0, 1)$.

With this assumption, we can apply Proposition 1 to conclude that ex ante contracting equilibria exist. Furthermore, the assumption implies that the problem

$$\max_{b, s} v(b, s) - \psi(b, i) - c(s, i) \tag{9}$$

has an interior solution for all $i \in [0, 1]$. We use the notation $W(b, s; i)$ for $W(b, s; i, i) = v(b, s) - \psi(b, i) - c(s, i)$. For the analysis that follows, it is convenient to assume:

Assumption 2. There is a well-behaved pair of attribute choice functions, (β^*, σ^*), such that $(\beta^*(i), \sigma^*(i))$ maximizes $W(b, s; i)$ for all i.

While this is a direct assumption on efficient attribute choice functions, it is one that is typically satisfied.[14]

Our first result is a counterpart of the second welfare theorem of neoclassical economics.

PROPOSITION 3. *Under Assumptions* 1 *and* 2, *there exists a bargaining outcome function* g^* *such that* $(g^*, (\beta^*, \sigma^*))$ *is an ex post contracting equilibrium.*

Proof. See Appendix C.

By construction, any change of attribute by a single agent leaves all other payoffs unchanged, and a single agent's attribute choice has no impact on social value. Nonetheless, as we saw in Section 4.1, there is a sense in which, at least for continuous attribute choice functions, β^* and σ^*,

[14] Suppose $(\hat{b}, \hat{s})$ is a local maximizer of $W(b, s; i) \equiv v(b, s) - \psi(b, i) - c(s, i)$ at $\hat{\imath}$ and the Hessian of $W(b, s; i)$ with respect to b and s is invertible at $(\hat{b}, \hat{s})$. Then (applying the implicit function theorem), there is a neighborhood of $\hat{\imath}$, I, for which $(\hat{b}, \hat{s}): I \to \Re^2_+$ describes a locally unique, differentiable, strictly increasing selection from the set of local maximizers.

The global maximizer of $W(b, s; i)$ is also typically strictly increasing in i from Assumption 1: Let $(\hat{b}, \hat{s})$, $(\tilde{b}, \tilde{s})$: $I' \to \Re^2_+$ describe local maximizers on some interval I, and suppose the Hessian of W is invertible on the graph of $(\hat{b}, \hat{s})$ and $(\tilde{b}, \tilde{s})$. If $W(\hat{b}(\bar{\imath}), \hat{s}(\bar{\imath}), \bar{\imath}) = W(\tilde{b}(\bar{\imath}), \tilde{s}(\bar{\imath}), \bar{\imath})$ for some $\bar{\imath} \in I$, and $\hat{b}(\bar{\imath}) < \tilde{b}(\bar{\imath})$, then supermodularity implies that $\hat{s}(\bar{\imath}) < \tilde{s}(\bar{\imath})$. Moreover, for $\bar{\imath} < i \in I$, $W(\hat{b}(i), \hat{s}(i), i) < W(\tilde{b}(i), \tilde{s}(i), i)$ (from the envelope theorem).

all agents are receiving the correct marginal incentives. Stable payoffs are determined completely by the division for the bottom pair of attributes and (5) and (6). The two marginal conditions, (5) and (6), essentially assert that each attribute is paid its marginal social value, and so it is not surprising that Proposition 3 holds in this case. Moreover, the definition of g^* is trivial, since it is given by the division for the bottom pair of attributes and (5) and (6), and by (8) for deviating attributes outside the range of β^* and σ^*.

The case of discontinuous attribute choice functions is more interesting. As we noted at the beginning of the previous paragraph, any change of attribute by a single agent leaves all other payoffs unchanged, and so there is no problem in determining stable payoffs for the other agents. Suppose β^* (and so σ^*) is discontinuous at i. From (7), at i, there is a range of possible divisions that is consistent with stability. However, *only one* division is consistent with (β^*, σ^*) being an ex post contracting equilibrium, namely, the division that makes the buyer indifferent between the choices $\beta^*(i-)$ and $\beta^*(i+)$ and, at the same time, makes the seller indifferent between $\sigma^*(i-)$ and $\sigma^*(i+)$:

$$x(i+) - \psi(\beta^*(i+), i) = x(i-) - \psi(\beta^*(i-), i)$$

and

$$p(i+) - c(\sigma^*(i+), i) = p(i-) - c(\sigma^*(i-), i).$$

(This division is feasible because the total net surplus at $i-$ equals that at $i+$.) There is thus a sense in which the appropriate g^* is "special." Moreover, given (β^*, σ^*), the bargaining outcome function depends on the cost functions directly, as well as through their determination of (β^*, σ^*).

Thus, nothing precludes an ex post contracting equilibrium from generating incentives for efficient ex ante investments, since there is a bargaining-outcome function g^* which supports efficient choices. At the same time, this proposition does not imply that we should necessarily expect an ex post contracting equilibrium to have efficient attribute choices.

6. INEFFICIENT EX POST CONTRACTING EQUILIBRIA

In both of the inefficient equilibria of the continuous example described in Section 2, there is an absence of agents on the other side of the market with the attributes that would induce efficient investment. In this section, we provide a series of results that illustrate how the existence of inefficient ex post contracting equilibria is affected by alterations of the populations of buyers and sellers. In particular, it is an implication of these results that for the continuum example in Section 2, the underinvestment outcome is not an equilibrium for populations with large values of $\bar{\imath}$, while the overinvestment outcome is not consistent with equilibrium for small values of $\underline{i}$. Intuitively, if $\bar{\imath}$ is large, the marginal benefit of investment for high buyers is so large that they will invest in a significant way even if sellers underinvest. By complementarity, the sellers then have an incentive not to underinvest and this then induces other buyers not to underinvest (a "trickle-down" effect). Similarly, a "trickle-up" effect precludes overinvestment for $\underline{i}$ small.

While ex post contracting equilibria need not be efficient (in the sense of not being ex ante contracting equilibria), they are efficient in a constrained sense:

Lemma 2. *Suppose $\{(\hat{\beta}, \hat{\sigma}), (\hat{x}, \hat{p})\}$ is an ex post contracting equilibrium. If $(\hat{\beta}, \hat{\sigma})$ is not consistent with any ex ante contracting equilibrium, then for any blocking coalition $(i^\dagger, j^\dagger)$ with attribute choices $(b^\dagger, s^\dagger)$, there does not exist i' such that $b^\dagger = \hat{\beta}(i')$, nor does there exist j' such that $s^\dagger = \hat{\sigma}(j')$.*

Proof. Suppose $(i^\dagger, j^\dagger)$ is a blocking coalition with attribute choices $(b^\dagger, s^\dagger)$ and shares $(x^\dagger, p^\dagger)$. Then,

$$x^\dagger + p^\dagger = v(b^\dagger, s^\dagger),$$

$$x^\dagger - \psi(b^\dagger, i^\dagger) > \hat{x}(\hat{\beta}(i^\dagger)) - \psi(\hat{\beta}(i^\dagger), i^\dagger), \text{ and}$$

$$p^\dagger - c(s^\dagger, j^\dagger) > \hat{p}(\hat{\sigma}(j^\dagger)) - c(\hat{\sigma}(j^\dagger), j^\dagger).$$

The proof is by contradiction. Suppose there exists j' such that $s^\dagger = \hat{\sigma}(j')$. Since $p^\dagger - c(s^\dagger, j^\dagger) > \hat{p}(\hat{\sigma}(j^\dagger)) - c(\hat{\sigma}(j^\dagger), j^\dagger) \geqslant \hat{p}(s^\dagger) - c(s^\dagger, j^\dagger)$, we have $p^\dagger > \hat{p}(s^\dagger)$, and so $x^\dagger - \psi(b^\dagger, i^\dagger) = v(b^\dagger, s^\dagger) - p^\dagger - \psi(b^\dagger, i^\dagger) < v(b^\dagger, s^\dagger) - \hat{p}(s^\dagger) - \psi(b^\dagger, i^\dagger)$. But stability, the hypothesis that stable payoffs to non-deviating players are unchanged, and the fact that $(\hat{\beta}, \hat{\sigma})$ is part of an ex post contracting equilibrium imply that $v(b^\dagger, s^\dagger) - \hat{p}(s^\dagger) - \psi(b^\dagger, i^\dagger)$ is a lower bound on buyer $i^\dagger$'s payoff in equilibrium, and so we have a contradiction. An identical argument, mutatis mutandis, shows that there cannot exist an i' such that $b^\dagger = \hat{\beta}(i')$. ∎

We use this lemma in the next proposition.

Proposition 4. *Suppose $(\hat{\beta}, \hat{\sigma})$ is a pair of ex post contracting equilibrium attribute-choice functions. If for some buyer i', $(\hat{\beta}(i'), s)$ does not maximize $W(b, s; i')$ for any s, then there is no seller j such that $\hat{\sigma}(j) = s^*$, for any (b^*, s^*) maximizing $W(b, s; i')$. Similarly, if for some seller j',*

$(b, \hat{\sigma}(j'))$ does not maximize $W(b, s; j')$ for any b, then there is no buyer i such that $\hat{\beta}(i) = b^*$, for any (b^*, s^*) maximizing $W(b, s; j')$.

Proof. We prove the buyer case; the seller case is identical. Since $\hat{\beta}$ and $\hat{\sigma}$ are both (weakly) increasing in index without loss of generality, in (8), we can take $i_{\hat{\beta}(i')}$ to be equal to i'. Consider first $i' \in (0, 1)$. Then, from (8), either

$$x(i') = v(\hat{\beta}(i'), \sigma(i'-)) - p(i'-) \geqslant v(\hat{\beta}(i'), \sigma(i'+)) - p(i'+)$$

$$\text{or } x(i') = v(\hat{\beta}(i'), \sigma(i'+)) - p(i'+) > v(\hat{\beta}(i'), \sigma(i'-)) - p(i'-).$$

Suppose it is the former. Since $(\hat{\beta}(i'), s)$ does not maximize $v(b, s) - \psi(b, i') - c(s, i')$ for any s, there is a pair of attributes (b^*, s^*) and returns (x', p') such that

$$x' - \psi(b^*, i') > x(i') - \psi(\hat{\beta}(i'), i'),$$

$$p' - c(s^*, i') > p(i'-) - c(\sigma(i'-), i'),$$

and

$$x' + p' = v(b^*, s^*).$$

Moreover, the pair of attributes (b^*, s^*) can be chosen to maximize $v(b, s) - \psi(b, i') - c(s, i')$. Since $(\hat{\beta}, \hat{\sigma})$ are part of an ex post contracting equilibrium,

$$p(i') - c(\sigma(i'), i') = p(i'-) - c(\sigma(i))$$

(otherwise either seller i' or sellers arbitrarily close to i' would deviate). Thus, (i', i') forms a blocking coalition using the attribute choices (b^*, s^*). Applying Lemma 2 yields the result.

This argument also covers the other possibilities. ∎

This proposition allows us to conclude that in the continuous example of Section 2, the attribute choices $(\hat{\beta}_1, \hat{\sigma}_1)$ are *not* consistent with any ex post contracting equilibrium, for $\bar{\imath}$ large.[15] In particular, if $\bar{\imath}$ is large enough that $\hat{\sigma}_1(\bar{\imath}) = \sqrt[3]{\bar{\imath}} > 4i^* = \sigma^*(\bar{\imath})$ (i.e., $\bar{\imath} > .63$), then a buyer with index i above, but just near i^*, can profitably deviate to $4i$ and match with seller j' with attribute s' (see Fig. 4). It is worth noting that the requirement that $\bar{\imath}$ not deviate is not binding for $\bar{\imath} < 2^{3/10}(5 - 2^{5/3})^{-9/10}] \approx .71$.

Continuity implies the following stronger result (which we state only for the buyer case).

[15] It is an implication of Proposition 6 that when $\bar{\imath}$ is large, the only underinvestment equilibrium is the trivial one in which $\beta(i) = \sigma(i) = 0$ for all i.

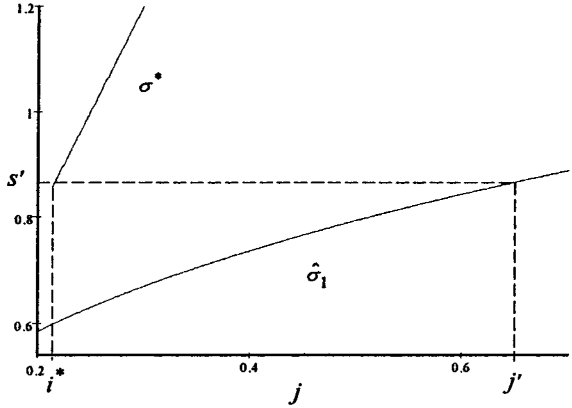

FIG. 4. The inefficient outcome is inconsistent with equilibrium when $\bar{\imath} > .63$.

COROLLARY 1. *Suppose* $(\hat{\beta}, \hat{\sigma})$ *is a pair of ex post contracting equilibrium attribute-choice functions, and that there is some buyer* $\bar{\imath}$ *for which* $(\hat{\beta}(\bar{\imath}), s)$ *does not maximize* $W(b, s; \bar{\imath})$ *for any* s. *Suppose* (b^*, s^*) *maximizes* $W(b, s; \bar{\imath})$. *Then there is a neighborhood,* $\mathcal{O}$, *of* s^* *such that, for all sellers* j, $\hat{\sigma}(j) \notin \mathcal{O}$.

We also have the following local version of Proposition 4. Recall that ex post contracting equilibrium choice functions have no isolated values, so that if $i \notin C(\hat{\beta}, \hat{\sigma})$, either $\hat{\beta}(i+) = \hat{\beta}(i)$ or $\hat{\beta}(i-) = \hat{\beta}(i)$ (with a similar statement holding for $\hat{\sigma}$).

PROPOSITION 5. *Suppose* $(\hat{\beta}, \hat{\sigma})$ *is a pair of ex post contracting equilibrium attribute-choice functions. Then, for all* $i \in C(\hat{\beta}, \hat{\sigma})$, $(\hat{\beta}(i), \hat{\sigma}(i))$ *is a local maximizer of* $W(b, s; i)$. *Moreover, for* $i \notin C(\hat{\beta}, \hat{\sigma})$, *if* $\hat{\beta}(i+) = \hat{\beta}(i)$, *then* $(\hat{\beta}(i), \hat{\sigma}(i+))$ *is a local maximizer of* $W(b, s; i)$ (*with similar statements holding for the other cases*).

Proof. We prove this by contradiction. Suppose there is an $i \in C(\hat{\beta}, \hat{\sigma})$ for which $(\hat{\beta}(i), \hat{\sigma}(i))$ is not a local maximizer of $W(b, s; i) = v(b, s) - \psi(b, i) - c(s, i)$. Since the attribute choice functions are strictly increasing and continuous on a neighborhood of i, there exists an $\varepsilon > 0$ such that $(\hat{\beta}(i) - \varepsilon, \hat{\beta}(i) + \varepsilon) \subset \hat{\beta}([0, 1])$ and $(\hat{\sigma}(i) - \varepsilon, \hat{\sigma}(i) + \varepsilon) \subset \hat{\sigma}([0, 1])$. Since $(\hat{\beta}(i), \hat{\sigma}(i))$ is not a local maximizer of $W(b, s; i)$, there exists (b', s') yielding a higher value of $W(b, s; i)$ with $|b' - \hat{\beta}(i)| < \varepsilon$ and $|s' - \hat{\sigma}(i)| < \varepsilon$. Thus, the coalition (i, i) can block $(\hat{\beta}, \hat{\sigma})$ using the attribute choices (b', s'). From Lemma 2, there is no i' such that $\hat{\beta}(i') = b'$, yielding a contradiction.

Continuity and the Maximum Theorem imply the result for $i \notin C(\hat{\beta}, \hat{\sigma})$. ∎

We can use this proposition to show that the discontinuities are not themselves a source of inefficiency.

PROPOSITION 6. *Suppose* $(\hat{\beta}, \hat{\sigma})$ *is a pair of ex post contracting equilibrium attribute-choice functions and that there is a discontinuity in* $\hat{\beta}$ *(and so* $\hat{\sigma}$*) at* i'*. Then,* $W(\hat{\beta}(i'+), \hat{\sigma}(i'+); i') = W(\bar{\beta}(i'-), \bar{\sigma}(i'-); i')$*, and the discontinuity is efficiency enhancing. That is, suppose* (β_1, σ_1) *and* (β_2, σ_2) *are two continuous attribute choice functions defined on a neighborhood of* i' *such that* $(\beta_1(i), \sigma_1(i))$ *agrees with* $(\hat{\beta}(i), \hat{\sigma}(i))$ *for* $i<i'$*,* $(\beta_2(i), \sigma_2(i))$ *agrees with* $(\hat{\beta}(i), \hat{\sigma}(i))$ *for* $i>i'$*, and for all* i *in the neighborhood, both* $(\beta_1(i), \sigma_1(i))$ *and* $(\beta_2(i), \sigma_2(i))$ *describe local maxima of* $W(b, s; i)$*. Suppose that the Hessian of* W *is well-defined and nonsingular on the graphs of* (β_1, σ_1) *and* (β_2, σ_2)*. Then,* $W(\beta_1(i), \sigma_1(i); i) > W(\beta_2(i), \sigma_2(i); i)$ *for* $i<i'$*, and* $W(\beta_1(i), \sigma_1(i); i) < W(\beta_2(i), \sigma_2(i); i)$ *for* $i>i'$*.*

Proof. At the discontinuity, buyer i' must be indifferent between attributes $\hat{\beta}(i'-)$ and $\hat{\beta}(i'+)$, and seller i' must be indifferent between attributes $\hat{\sigma}(i'-)$ and $\hat{\sigma}(i'+)$. Since pairwise feasibility holds for $i \in C(\hat{\beta}, \hat{\sigma})$, we then have $W(\hat{\beta}(i'+), \hat{\sigma}(i'+); i') = W(\bar{\beta}(i'-), \bar{\sigma}(i'-); i')$.

The remainder of the proposition is an implication of the inequality $dW(\beta_1(i), \sigma_1(i); i)/di|_{i=i'} < dW(\beta_2(i), \sigma_2(i); i)/di|_{i=i'}$, which follows from the envelope theorem and the single-crossing assumptions on costs, $\partial^2\psi/didb < 0$ and $\partial^2 c/\partial i \partial s < 0$. ∎

Since any ex post contracting equilibrium attribute choices must be local maxima (from Proposition 5), and nonsingularity of the Hessian implies that β_1 is the only candidate extension of $\hat{\beta}$ that can be consistent with any ex post contracting equilibrium, the discontinuity results in an increase in net surplus. As the example of Section 2 illustrates, however, not all inefficiencies arise from too little investments.

Returning to the continuous example of Section 2, it should be clear that increasing $\bar{i}$ cannot destabilize the overinvestment equilibrium and reducing $\underline{i}$ cannot destabilize the underinvestment equilibrium. On the other hand, overinvestment is inconsistent with equilibrium for populations $[.1, \bar{i}]$. Note first that for the lowest buyer and seller, attribute choices of $\underline{b} = \underline{s} = 4 \times .1 = .4$ imply $\underline{b}\underline{s} < \frac{1}{2}$, and so the pair $(\underline{b}, \underline{s})$ is not even a local maximizer of $W(b, s; .1) \equiv v(b, s) - \psi(b, .1) - c(s, .1)$. Thus, if there is to be an overinvestment equilibrium, it must have the lowest buyers and sellers choosing attributes in accordance with $\sqrt[3]{i}$. Proposition 6 implies that, in any ex post contracting equilibrium, if there is a discontinuity in attribute choices, it must occur at i^*. But this will yield ex ante efficiency.

APPENDIX A. PROOFS FOR SECTION 4

Proof of Lemma 1. We first argue that x and p are strictly increasing. Suppose there exists $i' < i$ such that $x(i') \geqslant x(i)$. For $\eta > 0$ small, let $\varepsilon = \frac{1}{2}\{v(\beta(i), \sigma(i'-\eta)) - v(\beta(i'), \sigma(i'-\eta))\}$. Since β is strictly increasing, $\varepsilon > 0$. Moreover, since σ is also strictly increasing and v is strictly supermodular, $v(\beta(i), \sigma(j)) - v(\beta(i'), \sigma(j)) > 2\varepsilon$ for all $j > i' - \eta$. Feasibility implies that there exists $j \in (i'-\eta, i'+\eta)$ such that

$$x(i') \leqslant v(\beta(i'), \sigma(j)) - p(j) + \varepsilon,$$

and so

$$\begin{aligned} x(i) + p(j) &\leqslant x(i') + p(j) \leqslant v(\beta(i'), \sigma(j)) + \varepsilon \\ &\leqslant v(\beta(i), \sigma(j)) - \varepsilon, \end{aligned}$$

contradicting the stability of (x, p), and so x is strictly increasing. A similar argument applies to p.

Consider the case of β continuous from the left at i', and $x(i') > \liminf_{i \uparrow i'} x(i)$. Let $\varepsilon = [x(i') - \liminf_{i \uparrow i'} x(i)]/4$. Suppose $\limsup_{j \to i'} [v(\beta(i'), \sigma(j)) - p(j)] > 0$. (If the reverse weak inequality holds, $x(i') = 0$, contradicting the assumption that x jumps up at i'.) There exists j close to i' such that $x(i') + p(j) < v(\beta(i'), \sigma(j)) + \varepsilon$. Moreover, for i close to (but less than) i', $v(\beta(i'), \sigma(j)) \leqslant v(\beta(i), \sigma(j)) + \varepsilon$ and $x(i) + 3\varepsilon \leqslant x(i')$. Thus,

$$\begin{aligned} x(i) + p(j) &\leqslant x(i') + p(j) - 3\varepsilon \\ &< v(\beta(i'), \sigma(j)) - 2\varepsilon \\ &< v(\beta(i), \sigma(j)) - \varepsilon < v(\beta(i), \sigma(j)). \end{aligned}$$

But this contradicts stability, and so $x(i') \leqslant \liminf_{i \uparrow i'} x(i)$.

Since x is strictly increasing, $x(i') \geqslant x(i)$, $i' \geqslant i$. But this implies $x(i') \geqslant \limsup_{i \uparrow i'} x(i)$, and so $x(i') = \lim_{i \uparrow i'} x(i)$.

The other possibilities are covered similarly. ∎

Proof of Proposition 1. Let $\{i_1, i_2, ..., i_T\}$ be the discontinuity points of β and σ, and define $I_t = (i_t, i_{t+1})$ for $t = 1, ..., T-1$, $I_0 = [0, i_1)$, and $I_T = (i_T, 1]$. Then, $C(\beta, \sigma) = \bigcup_{t=0}^{T} I_t$.

Existence of stable payoffs is addressed after the characterization. We have already argued that the no waste and continuity conditions must hold for any stable payoffs. These in turn imply at any point of discontinuity

i_t, $x(i_t-)+p(i_t-)=v(\beta(i_t-), \sigma(i_t-))$ and $x(i_t+)+p(i_t+)=v(\beta(i_t+), \sigma(i_t+))$. The two inequalities in (7) are then equivalent to the local stability conditions:

$$x(i_t+)+p(i_t-) \geqslant v(\beta(i_t+), \sigma(i_t-)), \text{ and} \tag{A.1}$$

$$x(i_t-)+p(i_t+) \geqslant v(\beta(i_t-), \sigma(i_t+)), \tag{A.2}$$

which (from continuity) are clearly necessary. The local condition (6) follows from the observation that since the payoffs are stable, for $b' \in \beta(C(\beta, \sigma))$ and all $s \in \sigma(C(\beta, \sigma))$,

$$v(b', \tilde{s}(b')) - \tilde{p}(\tilde{s}(b')) = \tilde{x}(b') \geqslant v(b', s) - \tilde{p}(s), \tag{A.3}$$

while (5) follows from fixing $s' \in \sigma(C(\beta, \sigma))$ in the same inequality and considering the value to the seller of matching with different buyers.

Now we turn to sufficiency. Fix a pair of nonnegative payoffs $(x(0), p(0))$ that satisfies

$$x(0)+p(0)=v(\beta(0), \sigma(0)).$$

Since any stable payoff must satisfy (5) and (6), we have

$$x(i)=x(0)+\int_{\beta(0)}^{\beta(i)} \frac{\partial v(b, \tilde{s}(b))}{\partial b}\, db, \qquad \text{for} \quad i \in I_0 \tag{A.4}$$

and

$$p(j)=p(0)+\int_{\sigma(0)}^{\sigma(j)} \frac{\partial v(\tilde{b}(s), s)}{\partial s}\, ds, \qquad \text{for} \quad j \in I_0. \tag{A.5}$$

Note that these equations determine $x(i_1-)$ and $p(i_1-)$. (We show below that (4), (A.4), and (A.5) are consistent.) It remains to extend x and p to the rest of $[0, 1]$. As on I_0, (5) and (6) determine x and p on I_t once the initial values, $x(i_t+)$ and $p(i_t+)$, have been determined. Let $(x(i_t+), p(i_t+))$ be any pair of payoffs satisfying (7). If, for example, β is continuous at i_t, then $x(i_t+)=x(i_t-)$, and there is only one choice for $(x(i_t+), p(i_t+))$. The payoff for buyer i_t is then determined by the continuity property of β: if β is continuous from the left, then $x(i_t)=\beta(i_t-)$, while if β is continuous from the right, $x(i_t)=\beta(i_t+)$ (the same considerations apply for seller i_t).

We next verify feasibility for $i \in C(\beta, \sigma)$. Suppose $i \in I_t$. By assumption, $x(i_t+)+p(i_t+)=v(\beta(i_t+), \sigma(i_t+))$, and for $i \in I_t$,

$$x(i)+p(i)=x(i_t+)+\int_{\beta(i_t+)}^{\beta(i)} \frac{\partial v(b,\tilde{s}(b))}{\partial b}\,db$$
$$+p(i_t+)+\int_{\sigma(i_t+)}^{\sigma(i)} \frac{\partial v(\tilde{b}(s),s)}{\partial s}\,ds$$
$$=v(\beta(i_t+),\sigma(i_t+))+\int_{i_t}^{i} \frac{dv(\beta(i),\sigma(i))}{di}\,di$$
$$=v(\beta(i),\sigma(i)),$$

so each pair efficiently shares the surplus.

We now verify stability. First note that (A.1) and (A.2) imply

$$x(i_t-)+p(i_{t+k}+)\geqslant v(\beta(i_t-),\sigma(i_{t+k}+)) \qquad \text{for all } k.$$

Suppose there exists a $k>1$ such that $x(i_t-)+p(i_{t+k}+)<v(\beta(i_t-),\sigma(i_{t+k}+))$. Then

$$x(i_t+)+p(i_{t+k}+)$$
$$<x(i_t+)+v(\beta(i_t-),\sigma(i_{t+k}+))-x(i_t-)$$
$$\leqslant x(i_t+)+v(\beta(i_t-),\sigma(i_{t+k}+))-v(\beta(i_t-),\sigma(i_t+))+p(i_t+)$$
$$=v(\beta(i_t+),\sigma(i_t+))+v(\beta(i_t-),\sigma(i_{t+k}+))-v(\beta(i_t-),\sigma(i_t+))$$
$$<v(\beta(i_t+),\sigma(i_{t+k}+)),$$

where the last inequality holds because v is strictly supermodular. Induction then yields a contradiction.

If (x,p) is not stable, then there is a pair i and j satisfying $x(i)+p(j)<v(\beta(i),\sigma(j))$. Suppose $i\in I_t$ and $j\in I_{t+k}$, $k\geqslant 1$ (the case of i and j in the same continuity interval is an obvious modification of the following, as is the case in which i and j are reversed). Then,

$$x(i_{t+1}-)+p(j)<x(i_{t+1}-)+v(\beta(i),\sigma(j))-x(i)$$
$$=v(\beta(i),\sigma(j))+\int_{\beta(i)}^{\beta(i_{t+1}-)} \frac{\partial v(b,\tilde{s}(b))}{\partial b}\,db$$
$$<v(\beta(i),\sigma(j))+\int_{\beta(i)}^{\beta(i_{t+1}-)} \frac{\partial v(b,\sigma(j))}{\partial b}\,db$$
$$=v(\beta(i_{t+1}-),\sigma(j)),$$

where the second inequality comes from the strict supermodularity of v. But then,

$$\begin{aligned} x(i_{t+1}-)+p(i_{t+k}+) &< v(\beta(i_{t+1}-),\sigma(j))-p(j)+p(i_{t+k}+) \\ &= v(\beta(i_{t+1}-),\sigma(j))-\int_{\sigma(i_{t+k}+)}^{\sigma(j)} \frac{\partial v(\tilde{b}(s),s)}{\partial s}\,ds \\ &< v(\beta(i_{t+1}-),\sigma(j))-\int_{\sigma(i_{t+k}+)}^{\sigma(j)} \frac{\partial v(\beta(i_{t+1}-),s)}{\partial s}\,ds \\ &= v(\beta(i_{t+1}-),\sigma(i_{t+k}+)), \end{aligned}$$

a contradiction. Thus, (x, p) is stable. ∎

APPENDIX B. PROOF FOR SECTION 4.1

Proof of Proposition 2. The distribution of buyer attributes is given by

$$F(b)=\lambda\{i:\beta^{\delta,\varepsilon}(i)\leqslant b\},$$

where λ is Lebesgue measure on $[0, 1]$, while the distribution on seller attributes is given by

$$G(s)=\lambda\{i:\sigma(i)\leqslant s\}=\sigma^{-1}(s).$$

In order to calculate F, define $\hat{b}=\beta(\bar{\imath}+\varepsilon)+\delta$, $\hat{\imath}=\beta^{-1}(\beta(\bar{\imath}+\varepsilon)+\delta)$, $\check{b}=\beta(\bar{\imath}-\varepsilon)+\delta$, and $\check{\imath}=\beta^{-1}(\beta(\bar{\imath}-\varepsilon)+\delta)$. These are illustrated in Fig. 3 preceding the statement of the proposition. Note that for $b<\beta(\bar{\imath}-\varepsilon)$ and $b>\hat{b}$, the distribution of attributes is unaffected. Moreover, for ε small, $\check{b}>\beta(\bar{\imath}+\varepsilon)$. Thus, $F(b)=\lambda\{i\colon \beta(i)\leqslant b\}=\beta^{-1}(b)$ for $b<\beta(\bar{\imath}-\varepsilon)$ and for $b>\hat{b}$. For $b\in[\beta(\bar{\imath}-\varepsilon),\beta(\bar{\imath}+\varepsilon)]$, $F(b)=\lambda\{i\colon \beta(i)\leqslant\beta(\bar{\imath}-\varepsilon)\}=\bar{\imath}-\varepsilon$. For $b\in[\beta(\bar{\imath}+\varepsilon),\check{b}]$, $F(b)=\beta^{-1}(b)-2\varepsilon$. Finally, for $b\in[\check{b},\hat{b}]$,

$$\begin{aligned} F(b) &= F(\check{b})+\lambda\{i\in[\bar{\imath}-\varepsilon,\bar{\imath}+\varepsilon]:\beta(i)\leqslant b-\delta\}+\lambda\{i\in[\check{\imath},\hat{\imath}]:\beta(i)\leqslant b\} \\ &= \check{\imath}-2\varepsilon+\beta^{-1}(b-\delta)-(\bar{\imath}-\varepsilon)+\beta^{-1}(b)-\check{\imath} \\ &= \beta^{-1}(b-\delta)+\beta^{-1}(b)-\bar{\imath}-\varepsilon. \end{aligned}$$

We calculate total surplus here assuming matching is positively assortative on attribute. The matching on buyers and sellers that supports this matching is described as follows. Following Gretsky, Ostroy, and Zame [11], a matching is a measure μ on $[0, 1]^2$ such that $\mu(A\times[0,1])=\lambda(A)$ and $\mu([0,1]\times B)=\lambda(B)$, for all Borel subsets A and B of $[0, 1]$, where λ

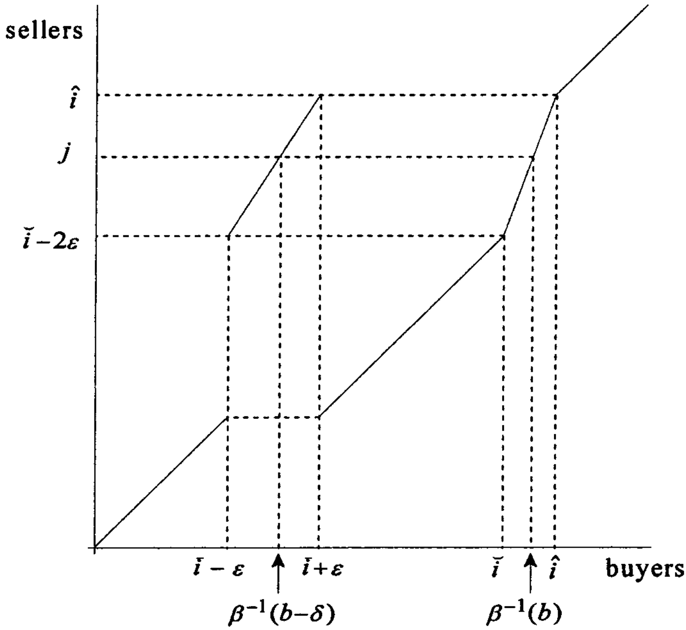

FIG. 5. The solid lines describe the support of the matching μ. The buyer attribute b is given by $b = (\tilde{s}^{\delta,\varepsilon})^{-1}(\sigma(j))$. The lines describing the matching for buyers in $[\bar{\imath} - \varepsilon, \bar{\imath} + \varepsilon]$ and in $[\check{\imath}, \hat{\imath}]$ need not be straight.

is Lebesgue measure on $[0, 1]$. The matching measure underlying the attribute matching can be calculated as follows: Since buyer $i \notin [\bar{\imath} - \varepsilon, \hat{\imath}]$ is matched with seller $j = i$, we have for all $C \subset ([0, \bar{\imath} - \varepsilon] \times [0, 1]) \cup ([0, 1] \times [0, \bar{\imath} - \varepsilon]) \cup ([\hat{\imath}, 1] \times [0, 1]) \cup ([0, 1] \times [\hat{\imath}, 1])$, $\mu(C) = \lambda\{i: (i, i) \in C\}$. A similar specification describes the matching of buyer $i \in [\bar{\imath} + \varepsilon, \check{\imath}]$ with seller $j = i - 2\varepsilon$.

It remains to describe the matching of buyer $i \in [\bar{\imath} - \varepsilon, \bar{\imath} + \varepsilon] \cup [\check{\imath}, \hat{\imath}]$ with seller $j \in [\check{\imath} - 2\varepsilon, \hat{\imath}]$ according to attribute. See Fig. 5. Seller $j \in [\check{\imath} - 2\varepsilon, \hat{\imath}]$ has attribute $s = \sigma(j)$ and "matches" with a buyer with attribute $b = (\tilde{s}^{\delta,\varepsilon})^{-1}(s)$. Two buyers have this attribute, $i = \beta^{-1}(b)$ and $i = \beta^{-1}(b - \delta)$. Thus, for $C \subset ([0, 1] \times [\check{\imath} - 2\varepsilon, \hat{\imath}]) \cup ([\bar{\imath} - \varepsilon, \bar{\imath} + \varepsilon] \times [0, 1]) \cup ([\check{\imath}, \hat{\imath}] \times [0, 1])$,

$$\mu(C) = \lambda\{i \in [\bar{\imath} - \varepsilon, \bar{\imath} + \varepsilon] \cup [\check{\imath}, \hat{\imath}] : (i, j) \in C \text{ such that either}$$

$$j = \sigma^{-1}(\tilde{s}^{\delta,\varepsilon}(\beta(i) + \delta)) \text{ or } j = \sigma^{-1}(\tilde{s}^{\delta,\varepsilon}(\beta(i)))\}.$$

Letting $\tilde{s}^{\delta,\varepsilon}(b)$ denote the attribute of the seller who is matched with a buyer with attribute b, positive assortative matching on attributes implies $F(b) = G(\tilde{s}^{\delta,\varepsilon}(b))$, i.e., $\tilde{s}^{\delta,\varepsilon}(b) = \sigma(F(b))$. Therefore, for b in the range of $\beta^{\delta,\varepsilon}$ we have

$$\tilde{s}^{\delta,\varepsilon}(b) = \begin{cases} \sigma(\beta^{-1}(b)), & b < \beta(\bar{\imath}-\varepsilon), \\ \sigma(\beta^{-1}(b)-2\varepsilon), & b \in [\beta(\bar{\imath}+\varepsilon), \check{b}], \\ \sigma(\beta^{-1}(b-\delta)+\beta^{-1}(b)-\bar{\imath}-\varepsilon), & b \in [\check{b}, \hat{b}], \\ \sigma(\beta^{-1}(b)), & b > \hat{b}. \end{cases}$$

We can interpret this attribute matching as arising from buyer $i \in [\bar{\imath}+\varepsilon, \check{\imath}]$ matching with seller $j = i - 2\varepsilon$, buyer $i \in [\bar{\imath}-\varepsilon, \bar{\imath}+\varepsilon] \cup [\check{\imath}, \hat{\imath}]$ matching with seller $j \in [\check{\imath}-2\varepsilon, \hat{\imath}]$ positively assortatively in attribute, and all other buyers i matching with sellers $j = i$. The surplus due to the changed matchings is

$$\int_{\bar{\imath}+\varepsilon}^{\check{\imath}} v(\beta(i), \sigma(i-2\varepsilon))\, di + \int_{\bar{\imath}-\varepsilon}^{\bar{\imath}+\varepsilon} v(\beta(i)+\delta, \tilde{s}^{\delta,\varepsilon}(\beta(i)+\delta))\, di$$
$$+ \int_{\check{\imath}}^{\hat{\imath}} v(\beta(i), \tilde{s}^{\delta,\varepsilon}(\beta(i)))\, di.$$

Thus the change in total surplus that arises from a buyer attribute choice function of $\beta^{\delta,\varepsilon}$ rather than β is

$$\begin{aligned} \Delta_{\delta,\varepsilon}(\bar{\imath}) = & \int_{\bar{\imath}+\varepsilon}^{\check{\imath}} v(\beta(i), \sigma(i-2\varepsilon)) - v(\beta(i), s(i))\, di \\ & + \int_{\bar{\imath}-\varepsilon}^{\bar{\imath}+\varepsilon} v(\beta(i)+\delta, \tilde{s}^{\delta,\varepsilon}(\beta(i)+\delta)) - v(\beta(i), s(i))\, di \\ & + \int_{\check{\imath}}^{\hat{\imath}} v(\beta(i), \tilde{s}^{\delta,\varepsilon}(\beta(i))) - v(\beta(i), s(i))\, di. \end{aligned} \quad \text{(B.1)}$$

We want to calculate $\lim_{\delta \to 0} \{\lim_{\varepsilon \to 0} \Delta_{\delta,\varepsilon}/2\varepsilon\}/\delta$. Define $\tilde{\imath} = \beta^{-1}(\beta(\bar{\imath}) + \delta)$. We proceed term by term: Dividing the first term in (B.1) by 2ε and taking limits as $\varepsilon \to 0$ yields (by Lebesgue's dominated convergence theorem)

$$\int_{\bar{\imath}}^{\tilde{\imath}} -\frac{\partial v(\beta(i), \sigma(i))}{\partial s} \sigma'(i)\, di.$$

Substituting for $\tilde{s}^{\delta,\varepsilon}$, the second term in (B.1) is

$$\int_{\bar{\imath}-\varepsilon}^{\bar{\imath}+\varepsilon} v(\beta(i)+\delta, \sigma(i+\beta^{-1}(\beta(i)+\delta)-\bar{\imath}-\varepsilon)) - v(\beta(i), s(i))\, di.$$

Dividing by 2ε and taking limits as $\varepsilon \to 0$ yields

$$v(\beta(\bar{\imath})+\delta, \sigma(\tilde{\imath})) - v(\beta(\bar{\imath}), s(\bar{\imath})).$$

The third term divided by 2ε is (again after substituting for $\tilde{s}^{\delta,\varepsilon}$)

$$\frac{1}{2\varepsilon}\int_{\check{\imath}}^{\hat{\imath}} v(\beta(i), \sigma(\beta^{-1}(\beta(i)-\delta)+i-\bar{\imath}-\varepsilon)) - v(\beta(i), s(i))\, di.$$

In order to evaluate the limit as $\varepsilon \to 0$, we apply L'Hopital's rule. The derivative of the integral with respect to ε is

$$\int_{\check{\imath}}^{\hat{\imath}} -\frac{\partial v(\beta(i), \sigma(\beta^{-1}(\beta(i)-\delta)+i-\bar{\imath}-\varepsilon))}{\partial s}\, \sigma'(\beta^{-1}(\beta(i)-\delta)+i-\bar{\imath}-\varepsilon)\, di$$

$$+\{v(\beta(\hat{\imath}), \sigma(\beta^{-1}(\beta(\hat{\imath})-\delta)+\hat{\imath}-\bar{\imath}-\varepsilon)) - v(\beta(\hat{\imath}), s(\hat{\imath}))\}\,\frac{d\hat{\imath}}{d\varepsilon}$$

$$-\{v(\beta(\check{\imath}), \sigma(\beta^{-1}(\beta(\check{\imath})-\delta)+\check{\imath}-\bar{\imath}-\varepsilon)) - v(\beta(\check{\imath}), s(\check{\imath}))\}\,\frac{d\check{\imath}}{d\varepsilon}.$$

The above expression equals 0 when $\varepsilon = 0$,[16] and so the third term converges to 0 as $\varepsilon \to 0$.

Thus, $\lim_{\delta\to 0}\{\lim_{\varepsilon\to 0} \Delta_{\delta,\varepsilon}/2\varepsilon\}/\delta =$

$$\lim_{\delta\to 0}\frac{1}{\delta}\left\{\int_{\bar{\imath}}^{\tilde{\imath}} -\frac{\partial v(\beta(i), \sigma(i))}{\partial s}\,\sigma'(i)\, di + v(\beta(\bar{\imath})+\delta, \sigma(\tilde{\imath})) - v(\beta(\bar{\imath}), s(\bar{\imath}))\right\}.$$

Applying L'Hopital's rule to the first term, the derivative of the numerator with respect to δ is (recall that $\tilde{\imath} = \beta^{-1}(\beta(\bar{\imath})+\delta)$)

$$-\frac{\partial v(\beta(\tilde{\imath}), \sigma(\tilde{\imath}))}{\partial s}\,\sigma'(\tilde{\imath})\,\frac{1}{\beta'(\tilde{\imath})},$$

while the derivative of the second term is

$$\frac{\partial v(\beta(\bar{\imath})+\delta, \sigma(\tilde{\imath}))}{\partial b} + \frac{\partial v(\beta(\bar{\imath})+\delta, \sigma(\tilde{\imath}))}{\partial s}\,\sigma'(\tilde{\imath})\,\frac{1}{\beta'(\tilde{\imath})},$$

so that

$$\lim_{\delta\to 0}\frac{1}{\delta}\left\{\lim_{\varepsilon\to 0}\frac{\Delta_{\delta,\varepsilon}}{2\varepsilon}\right\} = \left.\frac{\partial v(\beta(\bar{\imath})+\delta, \sigma(\tilde{\imath}))}{\partial b}\right|_{\delta=0} = \frac{\partial v(\beta(\bar{\imath}), \sigma(\bar{\imath}))}{\partial b}.$$

Since $\sigma(\bar{\imath}) = \tilde{s}(\beta(\bar{\imath}))$, this is the expression on the right hand side of (5).

[16] The first term is zero, since $\hat{\imath}$ and $\check{\imath}$ both equal $\beta^{-1}(\beta(\bar{\imath})+\delta) = \tilde{\imath}$. The second and third terms cancel, since, $d\hat{\imath}/d\varepsilon|_{\varepsilon=0} = d\check{\imath}/d\varepsilon|_{\varepsilon=0}$.

APPENDIX C. PROOF FOR SECTION 5

Proof of Proposition 3. As in the proof of Lemma 1, i_t denotes the tth discontinuity point of β^* and σ^*. Since (β^*, σ^*) is efficient,

$$\begin{aligned} v(\beta^*(i_t-), \sigma^*(i_t-)) - \psi(\beta^*(i_t-), i_t) - c(\sigma^*(i_t-), i_t) \\ = v(\beta^*(i_t+), \sigma^*(i_t+)) - \psi(\beta^*(i_t+), i_t) - c(\sigma^*(i_t+), i_t). \quad \text{(C.1)} \end{aligned}$$

Equilibrium requires

$$x(i_t+) - \psi(\beta^*(i_t+), i_t) = x(i_t-) - \psi(\beta^*(i_t-), i_t) \quad \text{(C.2)}$$

and

$$p(i_t+) - c(\sigma^*(i_t+), i_t) = p(i_t-) - c(\sigma^*(i_t-), i_t), \quad \text{(C.3)}$$

where $x(i_t+)$ $(x(i_t-))$ is the share of a buyer with attribute $\beta^*(i_t+)$ $(\beta^*(i_t-))$ and $p(i_t+)$ $(p(i_t-))$ is the share of a seller with attribute $\sigma^*(i_t+)$ $(\sigma^*(i_t-))$. If the stable payoffs do not satisfy these equalities, then clearly buyers and sellers close to i_t (either just above or just below) have an incentive to deviate. From (C.1), if $x(i_t+)$ satisfies (C.2), then $p(i_t+) = v(\beta^*(i_t+), \sigma^*(i_t+)) - x(i_t+)$ necessarily satisfies (C.3).

We first observe that the payoffs implied by (C.2) and (C.3) are consistent with stability (i.e., with (7)):

$$\begin{aligned} x(i_t+) + p(i_t-) &= x(i_t-) + p(i_t-) + \psi(\beta^*(i_t+), i_t) - \psi(\beta^*(i_t-), i_t) \\ &= v(\beta^*(i_t-), \sigma^*(i_t-)) + \psi(\beta^*(i_t+), i_t) - \psi(\beta^*(i_t-), i_t) \\ &\geqslant v(\beta^*(i_t+), \sigma^*(i_t-)), \end{aligned}$$

since $v(\beta^*(i_t-), \sigma^*(i_t-)) - \psi(\beta^*(i_t-), i_t) - c(\sigma^*(i_t-), i_t) \geqslant v(\beta^*(i_t+), \sigma^*(i_t-)) - \psi(\beta^*(i_t+), i_t) - c(\sigma^*(i_t-), i_t)$.

We need to show that (C.2) and (C.3), together with (5) and (6), are sufficient for equilibrium. Fix $(x^*(0), p^*(0))$ such that $x^*(0) + p^*(0) = v(\beta^*(0), \sigma^*(0))$. The payoffs (x^*, p^*) are now obtained from (A.4), (A.5), (C.2), and (C.3). From Proposition 1, these payoffs are stable. These determine the payoffs to a buyer (seller) choosing any attribute in the range of β^* (σ^*). Attributes outside the range are dealt with according to (8). Let b_t^* solve $v(b, \sigma^*(i_t+)) - p(i_t+) = v(b, \sigma^*(i_t-)) - p(i_t-)$ and set $b_0^* = 0$ and $b_{T+1}^* = \bar{B}(i)$ (and similarly for s_t^*). Then, $\beta^*(i_t-) < b_t^* < \beta^*(i_t+)$, $v(\beta^*(i_t-), \sigma^*(i_t+)) - p(i_t+) < v(\beta^*(i_t-), \sigma^*(i_t-)) - p(i_t-)$, and $v(\beta^*(i_t+), \sigma^*(i_t+)) - p(i_t+) > v(\beta^*(i_t+), \sigma^*(i_t-)) - p(i_t-)$. Then, for $b \in [\beta^*(i_t-), b_t^*]$, $\tilde{x}(b) = v(b, \sigma^*(i_t-)) - p(i_t-)$, and for $b \in [b_t^*, \beta^*(i_t+)]$, $\tilde{x}(b) = v(b, \sigma^*(i_t+)) - p(i_t+)$. Similar statements hold for the seller.

Consider now the buyer's problem (the argument for the seller is symmetric). We first argue that $\beta^*(i)$ is a maximizing attribute choice for buyer $i \in [i_t, i_{t+1}]$ from the attribute set $[b_t^*, b_{t+1}^*]$.[17] The problem for buyer $\hat{\imath}$ is to choose $b \in [b_t^*, b_{t+1}^*]$ to maximize $\tilde{x}(b) - \psi(b, \hat{\imath})$. Consider first choices of $b \in (\beta^*(i_t+), \beta^*(i_{t+1}-))$. Since buyer $\hat{\imath}$'s payoff function is differentiable over that domain (by Proposition 1), any maximizing choice of $b \in (\beta^*(i_t+), \beta^*(i_{t+1}-))$ must satisfy the first order condition

$$\tilde{x}'(b) = \frac{\partial \psi(b, \hat{\imath})}{\partial b}.$$

By construction,

$$\tilde{x}'(\beta^*(i)) = \frac{\partial v(\beta^*(i), \sigma^*(i))}{\partial b} = \frac{\partial \psi(\beta^*(i), i)}{\partial b} \qquad \forall i \in (i_t, i_{t+1}).$$

Suppose that $\tilde{x}'(b) = \partial \psi(b, i)/\partial b$ for some $b \neq \beta^*(i)$, $b \in (\beta^*(i_t+), \beta^*(i_{t+1}-))$. Since $b \in (\beta^*(i_t+), \beta^*(i_{t+1}-))$, there exists $\tilde{\imath}$ with $\beta^*(\tilde{\imath}) = b$ and so

$$\frac{\partial \psi(b, i)}{\partial b} = \tilde{x}'(b) = \frac{\partial \psi(b, \tilde{\imath})}{\partial b},$$

which is impossible, since $\partial\psi/\partial b$ is a strictly decreasing function of i. Thus, the first order condition has a unique solution in $b \in (\beta^*(i_t+), \beta^*(i_{t+1}-))$.

We now argue that $\beta^*(\hat{\imath})$ a local maximizer for $\hat{\imath}$. In what follows, partial derivatives are indicated by subscripts. It is enough to show that the second derivative of buyer $\hat{\imath}$'s payoff function is strictly negative. The second derivative is

$$v_{bb}(\beta^*(\hat{\imath}), \sigma^*(\hat{\imath})) + v_{bs}(\beta^*(\hat{\imath}), \sigma^*(\hat{\imath})) \left.\frac{d\tilde{s}}{db}\right|_{b=\beta^*(\hat{\imath})} - \psi_{bb}(\beta^*(\hat{\imath}), \hat{\imath}). \qquad \text{(C.4)}$$

Now, $\frac{d\tilde{s}}{db}\big|_{b=\beta^*(\hat{\imath})} = (d\sigma^*(\hat{\imath})/di)(d\beta^*(\hat{\imath})/di)^{-1}$ and $d\beta^*/di > 0$, so that (C.4) can be rewritten as

$$(d\beta^*(\hat{\imath})/di)^{-1}\left\{(v_{bb} - \psi_{bb})\left(\frac{d\beta^*}{di}\right) + v_{bs}\left(\frac{d\sigma^*}{di}\right)\right\} = (d\beta^*(\hat{\imath})/di)^{-1}\psi_{bi} < 0.$$

[17] The same argument shows that for buyers in the bottom interval $[0, i_1)$, $\beta^*(i)$ is optimal in the set $[0, b_1^*]$ and that, for buyers in the top interval $(i_T, 1]$, $\beta^*(i)$ is optimal in the set $[b_T^*, \bar{B}(i)]$.

Thus, $\beta^*(i)$ is the unique optimal choice from $(\beta^*(i_t+), \beta^*(i_{t+1}-))$. By continuity, $\beta^*(i)$ is an optimal choice for $i = i_t$ and i_{t+1} from $[\beta^*(i_t+), \beta^*(i_{t+1}-)]$.

We now turn to choices of $b \notin (\beta^*(i_t+), \beta^*(i_{t+1}-))$. Since stable matchings require positive assortative matching in attributes, if buyer $\hat{\imath}$ chooses $b \in [b_t^*, \beta^*(i_t+))$, then he or she is effectively matched with the seller with attribute $\sigma^*(i_t+)$, while a choice of $b \geqslant \beta^*(i_{t+1}-)$ leads to a match with $\sigma^*(i_{t+1}-)$. In the first case, $\tilde{x}(b) = v(b, \sigma^*(i_t+)) - \tilde{p}(\sigma^*(i_t+))$, while in the second, $\tilde{x}(b) = v(b, \sigma^*(i_{t+1}-)) - p(\sigma^*(i_{t+1}-))$.

We first consider $b \leqslant \beta^*(i_t+)$ and argue to a contradiction. Suppose there exists $b \leqslant \beta^*(i_t+)$ such that

$$\tilde{x}(\beta^*(\hat{\imath})) - \psi(\beta^*(\hat{\imath}), \hat{\imath}) < v(b, \sigma^*(i_t+)) - \tilde{p}(\sigma^*(i_t+)) - \psi(b, \hat{\imath}).$$

Let $\epsilon \equiv v(b, \sigma^*(i_t+)) - \tilde{p}(\sigma^*(i_t+)) - \psi(b, \hat{\imath}) - [\tilde{x}(\beta^*(\hat{\imath})) - \psi(\beta^*(\hat{\imath}), \hat{\imath})] > 0$. Since $\tilde{p}$ is continuous, there exists an $i < \hat{\imath}$ (and close to i_t) such that $|\tilde{p}(\sigma^*(i)) - \tilde{p}(\sigma^*(i_t+))| < \epsilon/2$. For this i,

$$v(\beta^*(i), \sigma^*(i)) - v(b, \sigma^*(i)) \geqslant \psi(\beta^*(i), i) - \psi(b, i) > \psi(\beta^*(i), \hat{\imath}) - \psi(b, \hat{\imath}),$$

where the first inequality follows from the optimality of (β^*, σ^*) for i and the second from $\partial\psi/\partial b\partial i < 0$. Then,

$$\begin{aligned}\tilde{x}(\beta^*(\hat{\imath})) - \psi(\beta^*(\hat{\imath}), \hat{\imath}) &\geqslant \tilde{x}(\beta^*(i)) - \psi(\beta^*(i), \hat{\imath}) \\ &= v(\beta^*(i), \sigma^*(i)) - \psi(\beta^*(i), \hat{\imath}) - \tilde{p}(\sigma^*(i)) \\ &> v(b, \sigma^*(i)) - \psi(b, \hat{\imath}) - \tilde{p}(\sigma^*(i)) \\ &> v(b, \sigma^*(i_t+)) - \psi(b, \hat{\imath}) - \tilde{p}(\sigma^*(i_t+)) - \epsilon/2 \\ &= \tilde{x}(\beta^*(\hat{\imath})) - \psi(\beta^*(\hat{\imath}), \hat{\imath}) + \epsilon - \epsilon/2,\end{aligned}$$

which implies $0 \geqslant \epsilon$, a contradiction.

We now consider $b \geqslant \beta^*(i_{t+1}-)$. Note first that it is obviously a best reply for buyer i_{t+1} to choose $\beta^*(i_{t+1}-)$. Consider the difference between buyer i's payoff from following β^* and choosing b:

$$\begin{aligned}\Delta(i; b) \equiv \tilde{x}(\beta^*(i)) - \psi(\beta^*(i), i) - [v(b, \sigma^*(i_{t+1}-)) \\ - \tilde{p}(\sigma^*(i_{t+1}-)) - \psi(b, i)].\end{aligned}$$

Differentiating with respect to i yields

$$\frac{\partial \Delta(i;b)}{\partial i} = (\tilde{x}'(\beta^*(i)) - \psi_b(\beta^*(i), i))\,\frac{d\beta^*}{di} - \psi_i(\beta^*(i), i) + \psi_i(b, i)$$

$$= \left(\frac{\partial v(\beta^*(i), \sigma^*(i))}{\partial b} - \psi_b(\beta^*(i), i)\right)\frac{d\beta^*}{di} - \psi_i(\beta^*(i), i) + \psi_i(b, i)$$

$$= \psi_i(b, i) - \psi_i(\beta^*(i), i) = \int_{\beta^*(i)}^{b} \psi_{bi} < 0,$$

so that if $\Delta(\hat{\imath}; b) < 0$ for some $b > \beta^*(i_{t+1}-)$, then $\Delta(i_{t+1}, b) < 0$, contradicting the optimality of $\beta^*(i_{t+1}-)$ for buyer i_{t+1}.

We now argue that $\beta^*(i)$ is a maximizing attribute choice for buyer $i \in [i_t, i_{t+1}]$ from the full attribute set $[0, \bar{B}(i)]$. Fix $i \in [i_t, i_{t+1}]$, $t \geqslant 1$, and consider an attribute $b \in [b^*_{t-1}, b^*_t)$. Then

$$x(i) - \psi(\beta^*(i), i) \geqslant \tilde{x}(\beta^*(i_t+)) - \psi(\beta^*(i_t+), i)$$

and

$$x(i_t-) - \psi(\beta^*(i_t-), i_t) \geqslant \tilde{x}(b) - \psi(b, i_t).$$

Combining these two inequalities with

$$x(i_t+) - \psi(\beta^*(i_t+), i_t) = x(i_t-) - \psi(\beta^*(i_t-), i_t)$$

gives

$$x(i) - \psi(\beta^*(i), i) \geqslant \tilde{x}(b) - \psi(b, i_t),$$

and so

$$x(i) - \psi(\beta^*(i), i) \geqslant \tilde{x}(b) - \psi(b, i).$$

That is, $\beta^*(i)$ is a maximizing choice for i from $[b^*_{t-1}, b^*_{t+1}]$. An obvious induction completes the argument. ∎

APPENDIX D. DETAILS OF THE CONTINUOUS EXAMPLE FROM SECTION 2

The buyer and seller populations are each the interval $[\underline{i}, \bar{i}] = [.2, .3]$.[18] The cost functions are given by $\psi(b, i) = b^5/(5i)$ and $c(s, j) = s^5/(5j)$, and the surplus function is

$$v^*(b, s) = \begin{cases} bs, & \text{if } bs \leqslant \frac{1}{2}, \\ 2(bs)^2, & \text{if } bs > \frac{1}{2}. \end{cases}$$

Net surplus is maximized by matching buyer i with seller $j = i$. The net surplus maximizing choices, (β^*, σ^*), are

$$\beta^*(i) = \begin{cases} \sqrt[3]{i}, & \text{if } i < i^*, \\ 4i, & \text{if } i \geqslant i^*, \end{cases} \qquad \sigma^*(j) = \begin{cases} \sqrt[3]{j}, & \text{if } j < i^*, \\ 4j, & \text{if } j \geqslant i^*, \end{cases}$$

where $i^* = (3/2^9)^{3/10} \approx 0.21$. Note that $\underline{i} < i^*$. The net surplus function $bs - (b^5 + s^5)/(5i)$ is maximized by setting $b(i) = \sqrt[3]{i}$ and $s(j) = \sqrt[3]{j}$, and the value of net surplus is $3i^{2/3}/5$. The net surplus function $2(bs)^2 - (b^5 + s^5)/(5i)$ is maximized by $b(i) = 4i$ and $s(j) = 4j$, and the value of this net surplus is $2i^4 4^4/5$. The index i^* equates the net surpluses $3i^{2/3}/5$ and $2i^4 4^4/5$. Finally, note that $\sqrt[3]{i^*} \cdot \sqrt[3]{i^*} \approx 0.36 < \frac{1}{2}$ and $4i^* \cdot 4i^* \approx 0.73 > \frac{1}{2}$.

To complete the description of the ex ante contracting equilibrium, we describe the attribute returns $(\tilde{x}, \tilde{p})$. Let $\underline{b} = \underline{s} = \sqrt[3]{\underline{i}}$, $b^*_- = s^*_- = \sqrt[3]{i^*}$, and $b^*_+ = s^*_+ = 4i^*$. Fixing an arbitrary division of the bottom surplus $(\tilde{x}(\underline{b}), \tilde{p}(\underline{s}))$, we use Proposition 1 to set

$$\tilde{x}(b) = \begin{cases} b^2/2 - \tilde{x}(\underline{b}), & \text{if } b \in (\underline{b}, b^*_-) \\ b^4 - \tilde{x}(b^*_+), & \text{if } b \in (b^*_+, 4\bar{i}), \end{cases}$$

where $\tilde{x}(b^*_+) = \tilde{x}(b^*_-) + (b^*_+)^5/(5i^*) - (b^*_-)^5/(5i^*)$, and

$$\tilde{p}(s) = \begin{cases} s^2/2 - \tilde{p}(\underline{s}), & \text{if } s \in (\underline{s}, s^*_-) \\ s^4 - \tilde{p}(s^*_+), & \text{if } s \in (s^*_+, 4\bar{i}), \end{cases}$$

where $\tilde{p}(s^*_+) = \tilde{p}(s^*_-) + (s^*_+)^5/(5i^*) - (s^*_-)^5/(5i^*)$. We extend $\tilde{x}$ and $\tilde{p}$ to $\Re_+$ using (8). It is straightforward to verify that this is an ex post contracting equilibrium.

There are two inefficient equilibria. In the underinvestment equilibrium, buyers choose attributes according to $\hat{\beta}_1(i) = \sqrt[3]{i}$ for all $i \in [\underline{i}, \bar{i}]$ and sellers

[18] This is equivalent to renormalizing the buyer and seller indexes by setting $i' = (i - \underline{i})/(\bar{i} - \underline{i})$.

choose attributes according to $\hat{\sigma}_1(j) = \sqrt[3]{j}$ for all $j \in [\underline{i}, \bar{\imath}]$. In the overinvestment equilibrium, buyers choose attributes according to $\hat{\beta}_2(i) = 4i$ for all $i \in [\underline{i}, \bar{\imath}]$ and sellers choose attributes according to $\hat{\sigma}_2(j) = 4j$ for all $j \in [\underline{i}, \bar{\imath}]$.

Consider first the underinvestment case. We now argue it is not profitable for buyer $\bar{\imath} = .3$ to deviate. The attribute returns are given by $\tilde{x}_1(b) = b^2/2 - \tilde{x}_1(\underline{b})$ for $b \in [\underline{b}, \sqrt[3]{\bar{\imath}}]$ and $\tilde{p}_1(s) = s^2/2 - \tilde{p}_1(\underline{s})$ for all $s \in [\underline{s}, \sqrt[3]{\bar{\imath}}]$. Consider the problem (implied by $\bar{\imath}$ taking the "price" to be paid to seller $j = \bar{\imath} = .3$ as given) of maximizing $v(b, \sqrt[3]{\bar{\imath}}) - p(\bar{\imath}) - \psi(b, \bar{\imath})$. Let $B_1 = \{b \colon b\sqrt[3]{\bar{\imath}} \leqslant \frac{1}{2}\}$ and $B_2 = \{b \colon b\sqrt[3]{\bar{\imath}} \geqslant \frac{1}{2}\}$. Maximizing the above objective function over $b \in B_1$ implies $b = \sqrt[3]{\bar{\imath}}$, with value $(.3)^{2/3} - (.3)^{2/3}/5 - p(\bar{\imath}) \approx .359 - p(\bar{\imath})$. Consider now maximizing the objective function over $b \in B_2$. The solution to the first order condition is $\sqrt[3]{4}(.3)^{5/9} \approx .81 \in B_2$, with value $2(\sqrt[3]{4}(.3)^{5/9})^2(.3)^{2/3} - (\sqrt[3]{4}(.3)^{5/9})^5/(1.5) - p(\bar{\imath}) \approx .355 - p(\bar{\imath}) < .358 - p(\bar{\imath})$. Finally, note that $\sqrt[3]{\bar{\imath}} < 4i^*$, so that a buyer (for example) with index i above, but just near i^*, cannot deviate to $4i$ and match with a seller of attribute $4i$ (see Proposition 4).

Consider now the overinvestment case. It is not profitable for buyer $\underline{i} = .2$, for example, to deviate because the marginal reward of increasing attribute is so high, even matching with the lowest attribute seller. The attribute returns are given by $\tilde{x}_2(b) = b^4 - \tilde{x}_2(4\underline{i})$ for $b \in [4\underline{i}, 4\bar{\imath}]$ and $\tilde{p}_2(s) = s^4 - \tilde{p}_2(4\underline{i})$ for all $s \in [4\underline{i}, 4\bar{\imath}]$. Consider the problem (implied by $\underline{i}$ taking the "price" to be paid to seller $j = \underline{i} = .2$ as given) of maximizing $v(b, 4\underline{i}) - p(\underline{i}) - \psi(b, \underline{i})$. Now, let $B_1 = \{b \colon b(4\underline{i}) \leqslant \frac{1}{2}\}$ and $B_2 = \{b \colon b(4\underline{i}) \geqslant \frac{1}{2}\}$. Maximizing the above objective function over $b \in B_1$ gives the boundary solution $b = .625$, so that the optimal attribute is $4\underline{i} = .8$.

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