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# Rational Expectations and the Measurement of a Stock's Elasticity of Demand

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**ABSTRACT**

Scholes [1] considered the effect of secondary sales of large blocks of stock on the price of the stock. However, he only looked at price changes occurring just before and just after the sale took place. It is argued here, using a simple model, that if traders have rational expectations they may anticipate the sale, and prices could reflect this possibility long before it actually occurs. To determine the full effect, it may therefore be necessary to consider the price path many months, or even years, before the sale.

IN A WELL-KNOWN PAPER, Scholes [1] considered the effect on a stock's price of the sale of a large secondary block of stock. The purpose of this exercise was to test the predictions of the price-pressure hypothesis against those of the substitution hypothesis. Adherents of the former suggest that when the size of a trade of a particular stock is large its price must fall to induce investors to purchase these additional shares. Those who support the substitution hypothesis suggest that the demand curve for assets is essentially flat because investors can always substitute other assets or combinations of assets to obtain a similar income stream.

Scholes considered changes in price around the date of secondary sales. He found that on average prices fell permanently by around 2 percent but that the size of this fall was insensitive to the quantity of stock that came on the market. The fall in price was mainly attributed to new information that the seller possessed on average; when the identity of the seller became known, the price fell further if it was likely that he had inside information and, if not, it rose. Overall, Scholes suggested that the results were supportive of the substitution hypothesis.

The purpose of this paper is to argue that the effect of a sale of stock on its price depends crucially on the degree to which the likelihood of this event has been taken into account by traders. If people think a large block of stock is likely to come on the market at some point, the current price will already reflect this; when the actual block of stock comes on the market the price may not change very much even though the demand curve for the stock is downward sloping. In general, it is therefore not possible to investigate a stock's elasticity of demand by looking at the change in price in the period around the sale. It may be necessary to consider the price path over many preceding months, or even years, to find the overall effect.

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A simple rational expectations model is developed in Section I to demonstrate this. In Section II, the determination of prices is considered and it is shown how current prices already reflect the possibility of future sales of stock. Finally, Section III contains a summary and a number of implications of the model.

## I. The Model

The time structure of the model is taken to be discrete with two periods, 1 and 2. The initial point in time at the beginning of period 1 is time 0. Assets are traded at time 0 and at the end of period 1.

There are three assets, one safe, which is denoted $m$, and two risky, which are denoted $x$ and $y$. The safe asset can be thought of as money and earns no interest. It is taken to be numeraire and has unit price. Assets $x$ and $y$ have normally distributed returns which are received at the end of period 2. Asset $x$ is the one whose price will be the focus of the model. Asset $y$ can be thought of as the rest of the market. There is little loss of generality in having only one risky asset apart from $x$ because the risky asset $y$ and the safe asset $m$ can be combined to form portfolios with a large variety of risk characteristics. The total per capita supplies of $x$ and $y$ are denoted $S_x$ and $S_y$.

Everybody is taken to have the same constant elasticity utility function which depends on their wealth $W_2$ at the end of the second period. Since all traders have the same preferences it is possible, without loss of generality, to set the degree of absolute risk aversion equal to unity by an appropriate choice of units for wealth.

$$u = -e^{-W_2} \tag{1}$$

This assumption is adopted for simplicity; it enables relatively simple explicit expressions for prices to be obtained. For ease of notation it is also assumed everybody has the same initial wealth $W_0$.

The population can be divided into two groups: a proportion $\nu_a$ are of type $a$, and the remainder are of type $b$. The difference between the groups concerns their after-tax payoffs. If the tax structure remains unchanged before the end of period 1, then the mean and variance of after-tax payoffs of assets $x$ and $y$, for both groups are $(\mu_y,\ \sigma_y^2)$ and $(\mu_x,\ \sigma_x^2)$, respectively, and their covariance is $\sigma_{xy}$. However if the tax structure is changed before the end of period 1, then group $a$'s mean after-tax payoff from asset $x$ becomes $\mu_{x\tau}$ which is less than $\mu_x$; all the other after-tax payoffs remain the same. Both groups can observe the change in tax structure and are aware of its implications; they all agree its probability of occurrence is $\pi$.

Scholes looked at the effect of secondary sales of large blocks of stock by single transactors. In the context of this model, these will occur when the type $a$ people group together and hold their stock in some form of mutual fund. Provided they remain price takers, it does not matter whether their behavior is analyzed in terms of the mutual fund or as individuals. For simplicity, the latter approach is adopted here.

## II. The Determination of Prices

The model is solved in two steps. The first is to find the prices at the end of period 1, in the case where the tax change has not occurred and also where it has occurred. The second involves using these to find prices in period 0.

### *A. Prices at t = 1*

(i) *The tax change has not occurred.* Let the price of $x$ and $y$ in this case be denoted $P_{1x}$ and $P_{1y}$, respectively.

Since the tax change has not occurred, everybody has the same after-tax returns to assets. Using the normality of returns and denoting the demand of a person of type $j$ for $x$, $y$, and $m$ at $t = 1$ by $X_{1j}$, $Y_{1j}$, and $M_{1j}$, respectively, it can be shown that their expected utility $EV_{1j}$ is given by

$$EV_{1j} = -\exp\{-[M_{1j} + X_{1j}\mu_x + Y_{1j}\mu_y - \tfrac{1}{2}(X_{1j}^2\sigma_x^2 + Y_{1j}^2\sigma_y^2 + 2X_{1j}Y_{1j}\sigma_{xy})]\} \quad (2)$$

Now the wealth of people of type $j$ in period 1, $W_{1j}$, is given by

$$W_{1j} = P_{1x}X_{1j} + P_{1y}Y_{1j} + M_{1j} \quad (3)$$

Thus,

$$EV_{1j} = -\exp\{-[W_{1j} + X_{1j}(\mu_x - P_{1x}) + Y_{1j}(\mu_y - P_{1y}) - \tfrac{1}{2}(X_{1j}^2\sigma_x^2 + Y_{1j}^2\sigma_y^2 + 2X_{1j}Y_{1j}\sigma_{xy})]\} \quad (4)$$

The first-order conditions for the choice of $X_{1j}$ and $Y_{1j}$ are therefore

$$\mu_x - P_{1x} - X_{1j}\sigma_x^2 - Y_{1j}\sigma_{xy} = 0 \quad (5)$$

$$\mu_y - P_{1y} - Y_{1j}\sigma_y^2 - X_{1j}\sigma_{xy} = 0 \quad (6)$$

Solving these equations simultaneously gives

$$X_{1j} = [\mu_x - P_{1x} - \frac{\sigma_{xy}}{\sigma_y^2}(\mu_y - P_{1y})]/Z\sigma_x^2 \quad (7)$$

$$Y_{1j} = [\mu_y - P_{1y} - \frac{\sigma_{xy}}{\sigma_x^2}(\mu_x - P_{1x})]/Z\sigma_y^2 \quad (8)$$

where

$$Z = 1 - \frac{\sigma_{xy}^2}{\sigma_x^2\sigma_y^2} \quad (9)$$

Equilibrium requires

$$X_{1a} = X_{1b} = S_x \quad (10)$$

$$Y_{1a} = Y_{1b} = S_y \quad (11)$$

This gives

$$P_{1x} = \mu_x - S_x\sigma_x^2 - S_y\sigma_{xy} \quad (12)$$

$$P_{1y} = \mu_y - S_y\sigma_y^2 - S_x\sigma_{xy} \quad (13)$$

(ii) *The tax change has occurred.* Next consider the case where the tax change has occurred and let $P_{1x\tau}$ and $P_{1y\tau}$ denote the prices of $x$ and $y$, respectively.

Similar to the above, it can be shown that the demands of type $a$'s and $b$'s are

$$X_{1a\tau} = \left[\mu_{x\tau} - P_{1x\tau} - \frac{\sigma_{xy}}{\sigma_y^2}(\mu_y - P_{1y\tau})\right] \Big/ Z\sigma_x^2 \tag{14}$$

$$Y_{1a\tau} = \left[\mu_y - P_{1y\tau} - \frac{\sigma_{xy}}{\sigma_x^2}(\mu_{x\tau} - P_{1x\tau})\right] \Big/ Z\sigma_y^2 \tag{15}$$

$$X_{1b\tau} = \left[\mu_x - P_{1x\tau} - \frac{\sigma_{xy}}{\sigma_y^2}(\mu_y - P_{1y\tau})\right] \Big/ Z\sigma_x^2 \tag{16}$$

$$Y_{1b\tau} = \left[\mu_y - P_{1y\tau} - \frac{\sigma_{xy}}{\sigma_x^2}(\mu_x - P_{1x\tau})\right] \Big/ Z\sigma_y^2 \tag{17}$$

Equilibrium requires that

$$\nu_a X_{1a\tau} + (1 - \nu_a) X_{1b\tau} = S_x \tag{18}$$

$$\nu_a Y_{1a\tau} + (1 - \nu_a) Y_{1b\tau} = S_y \tag{19}$$

Substituting for the demands of types $a$ and $b$ and solving simultaneously for prices gives

$$P_{1x\tau} = \nu_a \mu_{x\tau} + (1 - \nu_a)\mu_x - S_x \sigma_x^2 - S_y \sigma_{xy} \tag{20}$$

$$P_{1y\tau} = P_{1y} \tag{21}$$

*B. Prices at $t = 0$*

It can be seen from (21) that the price of asset $y$ is independent of whether or not the tax change has occurred. It follows that during period 1 asset $y$ is riskless. The price at time 0 of asset $y$ will therefore be equal to the price at the end of period 1 since the interest rate on riskless assets is zero:

$$P_{0y} = P_{1y} \tag{22}$$

Also the disposition of assets in individuals' portfolios between the riskless asset $m$ and asset $y$ will be arbitrary; they are perfect substitutes for this period.

Turning next to the determination of $P_{0x}$, it can be shown using (13)–(17), (20), and (21) that for type $j$'s

$$EV_{0j} = -\pi \exp[-(W_{1\tau j} + D + F_j)] - (1 - \pi)\exp[-(W_{1j} + D)] \tag{23}$$

where

$$D = \frac{1}{2}\,(S_x^2\sigma_x^2 + S_y^2\sigma_y^2 + 2S_xS_y\sigma_{xy}) \tag{24}$$

$$F_a = \frac{1}{2}\,(1 - \nu_a)^2(\mu_x - \mu_{x\tau})^2\,\frac{\sigma_y^2}{\sigma_y^2\sigma_x^2 - \sigma_{xy}^2} \tag{25}$$

$$F_b = \frac{1}{2}\,\nu_a^2(\mu_x - \mu_{x\tau})^2\,\frac{\sigma_y^2}{\sigma_y^2\sigma_x^2 - \sigma_{xy}^2} \tag{26}$$

Now

$$W_{1\tau j} = X_{0j}P_{1x\tau} + Y_{0j}P_{1y\tau} + M_{0j} \tag{27}$$

$$W_{1j} = X_{0j}P_{1x} + Y_{0j}P_{1y} + M_{0j} \tag{28}$$

where $X_{0j}$, $Y_{0j}$, and $M_{0j}$ are the amounts of the risky asset and safe asset, respectively, that are chosen by type $j$'s in period 0.

In period 0, people are also restricted by their budget constraints. For type $j$'s, this is of the form

$$W_0 = X_{0j}P_{0x} + Y_{0j}P_{0y} + M_{0j} \tag{29}$$

Substituting from (29) into (27) and (28) and using (22), it follows that (23) can be written in the form

$$\begin{aligned} EV_{0j} = {} & -\pi\,\exp\{-[W_0 + X_{0j}(P_{1x\tau} - P_{0x}) + D + F_j]\} \\ & - (1 - \pi)\exp\{-[W_0 + X_{0j}(P_{1x} - P_{0x}) + D]\} \end{aligned} \tag{30}$$

Choosing $X_{0j}$ to maximize this gives the following demand function

$$X_{0j} = \frac{1}{\nu_a(\mu_x - \mu_{x\tau})}\left\{\log\left[\frac{(1 - \pi)(P_{0x} - P_{1x})}{\pi(P_{1x\tau} - P_{0x})}\right] + F_j\right\} \tag{31}$$

Equilibrium requires

$$\nu_a X_{0a} + (1 - \nu_a)X_{0b} = S_x \tag{32}$$

Using this together with (31), it can be shown that

$$P_{0x} = \frac{(1 - \pi)}{(1 - \pi) + \pi\,\exp G}\,P_{1x} + \frac{\pi\,\exp G}{(1 - \pi) + \pi\,\exp G}\,P_{1x\tau} \tag{33}$$

where

$$G = \nu_a(\mu_x - \mu_{x\tau})S_x - \nu_a F_a - (1 - \nu_a)F_b \tag{34}$$

The price of asset $x$ at time 0 is therefore a weighted average of the prices at time 1 when the tax change has occurred and when it has not occurred. It already reflects the possibility that there may be large sales of asset $x$ in the future. It is thus not possible to measure the effect of the sale by looking at the relationship between the change in price $(P_{0x} - P_{1x\tau})$ and the quantity sold when a large secondary sale occurs.

## III. Summary and Conclusions

It has been shown in the context of a simple illustrative model that, if anticipated, the effect of a secondary sale of stock on its price may occur well before the sale actually takes place. The reason for the sale in the model presented was a tax change; many other similar exogenous justifications for the sale could have been adopted. The other type of reason for a secondary sale is that the seller has private information that the stock is overvalued. In this case, there will be an additional change in price due to the adverse information signalled by the sale. Again, some of this change may occur before the sale, if the sale is anticipated.

Scholes found that the fall in price immediately after the sale was largely independent of the quantity sold, and that subsequently, when the identity of the seller became known, if it turned out he was likely to have inside information the price continued to fall, but if not it rose. Scholes' conclusion was that the price change was mainly due to the signal contained in the sale and that the price-pressure hypothesis could therefore be rejected.

The model presented above shows that this conclusion does not necessarily follow from the results obtained; an alternative explanation is possible. The change in price observed at the date of the secondary sale could have two components: the first being the price-pressure component and the second the result of the adverse information implied by the sale. In terms of explaining Scholes' results, it is quite plausible that the larger the sale of stock, the better anticipated it will be, so that even the price-pressure component may not vary very much with the quantity sold. For example, suppose the costs for forecasting secondary distributions are independent of the scale of distribution. If the magnitude of the price change did increase with the size of the sale, the rewards to predicting large sales would be greater than those obtained from predicting small sales. This would not be consistent with equilibrium since it would result in more resources being devoted to forecasting those conditions which lead to large secondary distributions. However, it is possible an equilibrium could occur where the magnitude of the price change does not vary by the size of distribution, since the profits from forecasting all sizes of distribution would be the same. As far as the difference in price paths when the identity of the seller becomes known is concerned, this can be explained as being due to the adverse information component. It then follows from the results of the model presented above that even though the price-pressure component observed at the time of the sale may be small, it is not possible to reject the price-pressure hypothesis; the price change around the date of sale could be a significant underestimate of the overall price-pressure, because most of this may have occurred well before the sale took place.

In conclusion, it is not generally possible to measure the effect of a secondary sale by looking at the price path just before and just after as Scholes does; it may be necessary to go back many months or even years before. His results cannot therefore be regarded as conclusive in deciding between the price-pressure hypothesis and the perfect substitution hypothesis. There may have been significant adjustments in price as the former hypothesis suggests, but these may have occurred well in advance. More research that takes into account this possibility is required to settle the issue. In addition, his conclusions concerning issues of new stock and antitrust divestitures having no effect on share prices are also not definitely proven. In these cases, there may again be significant adjustments in price, but these may occur well in advance of the events that caused them.

In general, observing the price change at the time of any event does not measure the true impact of that event, if it is anticipated at all. In other contexts, many event studies have found systematic changes in price well before the event occurred. The same could be true for secondary sales.

## REFERENCE

1. M. S. Scholes. "The Market for Securities: Substitution versus Price Pressure and the Effects of Information on Share Prices." *Journal of Business* 45 (April 1972), 179–211.