Games and Economic Behavior **33**, 72–89 (2000)
doi:10.1006/game.1999.0773, available online at http://www.idealibrary.com on IDEAL®

Specialization of Labor and the Distribution of Income[1]

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Received September 15, 1997

We analyze a model in which workers must be allocated to tasks to produce. There are differences among the workers in absolute ability that are independent of the activity they perform. We demonstrate a unique competitive equilibrium that determines both wages and the allocation of workers to tasks. The equilibrium wage has the property that workers assigned to the lowest "value-added" tasks will receive a "premium" above their identifiable contribution to value simply for filling these least valuable positions. We examine how the equilibrium wage function changes as the production process becomes more specialized and the division of labor increases. *Journal of Economic Literature* Classification Numbers: D31, D33, J31.

Key Words: division of labor; specialization of labor; income distribution.

1. INTRODUCTION

The importance of specialization of labor as a source of economic welfare has been recognized at least since Adam Smith's description of pin manufacture. Economists since Smith have focused on two distinct sources of economic gains from specialization. The first was economies of scale: the division of a complex job into many simpler tasks enabled each worker to master individual tasks to a degree that was impossible when he or she

[1]This research was partially supported by the National Science Foundation. We thank Ken Burdett, Masahiro Okuno-Fujiwara, and Jaewoo Ryoo for helpful comments and suggestions.

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0899-8256/00 $35.00

was responsible for the entirety of the more complex job. The second source of gains from specialization relies on different relative abilities of workers at different tasks. When workers differ in this way, the division of the complex job into smaller tasks allows each worker to work exclusively on the task for which he or she has a comparative advantage.

Our aim in this paper is to investigate how specialization may generate gains when workers seem to differ only in absolute advantage. In our model, workers differ in ability, but differences are independent of the activity. We consider a particular production process that consists of a finite number of tasks, each of which contributes a specific amount to the final output and each of which must be done in order for there to be any output. In our model, specialization will be interpreted as dividing a job into tasks with different workers carrying out the tasks. The gains will arise from the possibility of assigning higher ability workers to higher value tasks.

When there are gains from specialization, a natural question arises as to how those gains are distributed among the workers. For the simple structure of the production process we assume, there will be a unique competitive equilibrium outcome that determines both the allocation of workers to tasks and the wages those workers will receive. The wage any given worker will receive will naturally be closely related to the worker's ability and the value of the task to which he or she is assigned. It will not, however, be solely determined by these; because each task must be performed by some worker, the production process has "Leontief-like" characteristics. Those workers assigned to the lowest value jobs will receive something like a premium above their identifiable contribution to value simply for filling the least valuable positions. Besides characterizing the income distribution for a given division of the activities into tasks, we investigate the effects on income distribution when the technology changes so that there is a finer task division, that is, who wins and who loses when a task is subdivided.

We should emphasize that we do *not* think our model describes the only or even the most important source of gains from specialization. Rather, we think of our model as being complementary to models which identify economies of scale and relative advantage as important causes of specialization. We do believe, though, that the gains from specialization that we identify are important. The gains arise from the particular technology that we assume, but the aspects of the technology that are important making gains possible are reflected in many real-world production processes.

The next section describes in detail the nature of the gains from specialization we are interested in, and lays out the formal model and results. Discussion of our results and relationship to other work is left to the last section.

2. MODEL

Before presenting the formal model, it will be helpful to present a simple example that illustrates how gains from specialization arise. In both the example and the formal model, we will make assumptions about both workers' abilities and the production technology that focus attention on the way the gains arise. Many aspects of the model are special. Our choices have been motivated by expository simplicity; most can be relaxed without changing the qualitative nature of the results. We discuss the main assumptions of the model further in the last section.

We think of any job as being a portfolio of many simple tasks. For example, assembling an automobile involves installing a transmission, bolting in seats, wiring in a radio, attaching trim, painting the body, etc. The job of assembling an automobile can obviously be broken down into subtasks; in fact, it can probably be done in an infinite number of different ways. Furthermore, for any way in which the job can be broken down, there is undoubtedly a way to break down further some of the subtasks. Wiring in a radio can be broken down so that first, the radio is set in place, and subsequently the wires attached appropriately. The possibilities of any task being further subdivided seem endless.

We assume that workers differ in ability, but in a very restricted way. Suppose that a high ability worker can carry out one of the tasks involved in the assembly of the automobile in a way that that component will perform forever without needing repair, while if done by a low ability worker, the component will perform for a strictly positive, but shorter period of time. It follows then that an automobile in which the radio was installed by the high ability worker is worth more than an identical automobile except for having a radio installed by the low ability worker. If the radio installed by the low ability worker would last only half the life of the automobile, that radio will add only half the value to a car as a radio installed by the high ability worker (ignoring discounting). (We assume that a worker's ability does not affect the speed with which he can carry out a task, only the value added for that task.) Suppose further that workers differ in ability uniformly across all tasks. That is, we would say that one worker was twice as able as a second if the value of any task done by the first was twice the value of that task when done by the second (where the value of a task done by a worker is value added to the automobile). To summarize, workers differ by ability but there is no comparative advantage, only absolute advantage.

Suppose the assembly of automobiles can be decomposed in the manner described above. If the job is not divided into subtasks, but each of the two workers assembles an automobile alone, we would have two automobiles, with the automobile assembled by the high ability worker being worth twice the automobile assembled by the low ability worker. Now suppose that the entire assembly is broken into two groups of tasks, each of which takes one-half the total assembly time. Each of the two workers can perform one group of tasks, and the pair is able to assemble two automobiles in the same amount of time it would take if they did not specialize. If the two tasks are randomly determined, specialization will add nothing. But suppose that the tasks can be designed so that one consists of "high value" activities for which the ability is relatively more important, say installing the electrical system. If we can identify activities that constitute half the total time for the job but contribute three quarters of the total possible value, we can assign this task to the high ability worker, with the remainder being the task for the low ability worker. Now if v is the value of an automobile assembled solely by the less able worker ($0.75v$ from the "high value" activity and $0.25v$ from the "low value" activity), the value of an automobile assembled by the specialized pair will be $1.75v$ (0.75 m $\times 2 + 0.25$ m $\times 1$). The total value of the two automobiles that can now be assembled by the two workers specializing is then $3.5v$, which is larger than the value of the output without specialization, $v + 2v = 3v$.

This example illustrates the primary issue we wish to investigate: differences in ability, even when uniform across tasks, can lead to gains from specialization. The general model below formalizes the structure of the example. The point of the paper, however, is not that there are gains from such specialization. Rather, it is to investigate how those gains will be distributed between workers of differing abilities.

Formally, an economy consists of a continuum of individuals uniformly distributed on $I \equiv (0,1]$, each of whom has one unit of time.[4] Individuals differ in their ability; there is an ability function $X: I \to \mathscr{R}$, where $X(i)$ $(i \in I)$ is the ability of person i. We assume that $X(i)$ is a decreasing function of i, i.e., i is at least as able as $i' > i$, and that $X(\ )$ is continuous. Some people choose to become entrepreneurs and hire workers from those who choose not to be entrepreneurs in order to produce goods.

There are $N < \infty$ different tasks with θ_n the fraction of total time that must be devoted to the nth task, where the θ_n's are non-negative rational numbers, and $\sum_{n=1}^{N} \theta_n = 1$. Each person owns one unit of time and can divide it among any number of tasks. Agent i_1, an entrepreneur, hires a finite set of people with ability levels of $X(i_2), \ldots, X(i_M)$ where M, the number of employees including himself, is also determined by i_1. Some of them may have the same ability. Agent i then assigns θ_{nm} units of the time of person i_m $(m = 1, \ldots, M)$ to task n. It is assumed that $\sum_{n=1}^{N} \theta_{nm} = 1$ for all $m = 1, \ldots, M$. Let $X = (X(i_1), \ldots, X(i_M))$ and $\Theta = \{\theta_{nm}\} \in \mathscr{R}^{N \times M}$.

[4] The set $(0, 1]$ is endowed with the Lebesgue measure. In the following, any function will be assumed to be measurable.

The output of this firm is

$$F(X,\Theta) = \underset{1\le n\le N}{\mathrm{Min}} \left\{ \frac{\sum_{m=1}^{M} \theta_{nm}}{\theta_n} \right\} \cdot \sum_{n=1}^{N} \theta_n f^n\left(\overline{X}_n\right). \tag{1}$$

where $\overline{X}_n$ is the weighted average of workers' ability who are assigned to task n, i.e., $\overline{X}_n = \sum_{m=1}^{M} \theta_{nm} X(i_m)/\sum_{m=1}^{M}\theta_{nm}$. The value of the product is determined both by quantity and quality of the product. The quantity produced is captured by the Leontief function in (1), while the average quality is a function of the (weighted) average ability of workers in each task. It is assumed that f^n $(n = 1,\dots,N)$ is differentiable, increasing and concave in $\overline{X}_n$, and $f^n(0) = 0$. Moreover, we assume that $f^{n'}(X) > f^{n+1'}(X')$ holds for all X and X' where $f^{n'}(\ )$ is the derivative of f^n. That is, task n is uniformly more sensitive to ability level than task $n+1$.

We now consider a competitive economy in which given a wage schedule $W\colon I \to \mathscr{R}$, each person simultaneously decides whether he becomes an entrepreneur or worker, and if he becomes an entrepreneur, what ability workers to hire and how to apportion their labor among the tasks. We should point out that being an entrepreneur requires no time or resources. An entrepreneur in this model is simply a worker who coordinates the hiring and assignment of workers to tasks. Given a wage schedule $W\colon X(I) \to \mathscr{R}$, person i's problem if he chooses to be an entrepreneur is

$$\begin{aligned} &\underset{X,\Theta}{\mathrm{Max}}\, F(X,\theta) - \sum_{x'\in X} W(x') + W(X(i)) \\ &\text{s.t. } X(i) \in X. \end{aligned} \tag{2}$$

An individual who chooses to be a worker gets $W(X(i))$. Therefore, person i chooses to be an entrepreneur if the solution to (2) is greater than $W(X(i))$, and chooses to be a worker if it is less than $W(X(i))$. Let $E \subset I$ be the set of individuals who choose to be entrepreneurs; $E^c \equiv I \setminus E$ is then the set of workers as a consequence of the maximization. A measurable tuple of individuals' decisions $(E; \{X^i, \Theta^i\}_{i\in E})$ is called a *decision profile* if (X^i, Θ^i) is a solution to (2). Then a pair consisting of a wage schedule and a decision profile under the wage schedule is a *competitive equilibrium* (or simply an *equilibrium*) of this economy if supply equals demand, i.e., for any measurable subset D of $\mathscr{R}$,

$$\int_{i\in E} \left|\left\{ j : X^i_j \in D \right\}\right| \, d\mu(i) = \mu\left(X^{-1}(D)\right). \tag{3}$$

It is straightforward to show that $W(\cdot)$ is an increasing function of ability in a competitive equilibrium. Our first result is the zero-profit condition, i.e., that every agent is indifferent between being an entrepreneur and being a worker.[5]

LEMMA 1. *In an equilibrium, every agent is indifferent between being an entrepreneur and being a worker.*

Proof. Let $(W, (E; \{X^i, \Theta^i\}_{i \in E}))$ be a competitive equilibrium. As a consequence of the optimization, any agent i in E weakly prefers being an entrepreneur to being a worker, and any worker weakly prefers being a worker to being an entrepreneur. For agent $i \in E$ we have

$$F(X^i, \Theta^i) - \sum_{x' \in X^i} W(x') \geq 0. \tag{4}$$

Suppose the above inequality is strict. Then a worker j employed by agent i has an incentive to become an entrepreneur, choose (X^i, Θ^i), and get

$$F(X^i, \Theta^i) - \sum_{x' \in X^i} W(x') + W(X(j)), \tag{5}$$

which is strictly greater than $W(X(j))$ by virtue of strict inequality in (4). This violates the condition for optimality. Thus, any entrepreneur is indifferent between being an entrepreneur and being a worker.

Next, consider agent $j \in E^c$. Let i be an entrepreneur employing j. By choosing to become an entrepreneur and (X^i, Θ^i), agent j could get

$$F(X^i, \Theta^i) - \sum_{x' \in X^i} W(x') + W(X(j)), \tag{6}$$

which is equal to $W(X(j))$ since the first two terms add up to zero as (4) holds with equality for $i \in E$. Thus, a worker is also indifferent between being an entrepreneur and being a worker. Q.E.D.

LEMMA 2. *In a competitive equilibrium, for any i and j with $X(i) > X(j)$ and any k and l with $k < l$, if i's time is allocated to task l, then j's time is never allocated to task k.*

Proof. Let $(W, (E; \{X^i, \Theta^i\}_{i \in E}))$ be a competitive equilibrium. Suppose the contrary, i.e., that there exist agents i and j with $X(i) > X(j)$ and tasks k and l with $k < l$ such that a positive portion $\theta(i, l)$ of agent i's time is allocated to task l, and a positive portion $\theta(j, k)$ of agent j's time

[5] It is straightforward to verify that in this model the set of core allocations is the same as the set of competitive allocations. For the core interpretation of the set, the entrepreneur's role is unnecessary.

is allocated to task k. Let $\theta = \min\{\theta(i, l), \theta(j, k)\}$. Suppose that agent h employs i and agent h' employs j (some of h, h', i, and j may be the same, though i and j are always different). If $h = h'$ holds, then agent $h \in E$ switches θ portion of i's time and j's time between k and l and can increase his profit by at least $\theta[f^{k'}(X) - f^{l'}(X')]\,[X(i) - x(j)]$ for some X and X', which is obtained by using the intermediate value theorem. This expression is positive due to our assumption on $f^{n'}(\ )$. Therefore, the initial state is not an equilibrium. Suppose next that $h \neq h'$. The argument is basically the same as the case of $h = h'$ except that we now have to combine the two production units and apply Lemma 1. In this case, some agent, say h, has an incentive to choose to become an entrepreneur and $X^h \circ X^{h'} \equiv (X(i_1^h), \dots, X(i_{M_h}^h), X(i_1^{h'}), \dots, X(i_{M_{h'}}^{h'}))$ and assign the same tasks as in Θ^h and $\Theta^{h'}$ except for i and j; agent h switches their task assignment as in the case of $h = h'$. By this arrangement, agent h gains by at least $\theta[f^{k'}(X) - f^{l'}(X')]\,[X(i) - X(j)] > 0$ for some X and X' since h gives agent h' the wage equal to the profit of agent h' in the original situation by virtue of Lemma 1. Thus, the original situation is not a competitive equilibrium, a contradiction. Q.E.D.

Let $I_n = \sum_{k=1}^{n} \theta_k$, and $X_n = X(I_n)$ for $n = 1, \dots, N$. Lemma 2 implies that if $X(\ )$ is strictly decreasing, we can partition I into n intervals, $(0, I_1], (I_1, I_2], \dots, (I_{N-1}, 1]$, in which everyone in $(I_{k-1}, I_k]\,(k = 2,3, \dots, N)$ except (possibly) the agent at the boundary specializes in task k. The next lemma determines the relative wages among workers assigned to the same task. Let EX_n be the average ability of people assigned to task n, i.e.,

$$EX_n = \frac{1}{I_n - I_{n-1}} \int_{I_{n-1}}^{I_n} X(i)\, di.$$

LEMMA 3. *Any firm operates at the point where $f^{n'}(\overline{X}_n) = f^{n'}(EX_n)$ for all $n = 1, \dots, N$. Moreover, if agents i and j each have all of their time assigned to the same task n, we have*

$$W(X(i)) - W(X(j)) = f^{n'}(EX_n)[X(i) - X(j)]. \tag{7}$$

Proof. First, suppose that there exists a firm with M workers operating at a point with $f^{n'}(\overline{X}_n) \neq f^{n'}(EX_n)$ for some $n = 1, \dots, N$. Consider the case of $f^{n'}(\overline{X}_n) < f^{n'}(EX_n)$. Then there exists another firm that operates at a point where $f^{n'}(\overline{X}'_n) \geq f^{n'}(EX_n)$. This is due to the previous lemma. Let M' be the number of workers in the second firm. If we combine these two firms, and assign workers to their original tasks, then this merged firm gains positive profit. Indeed, the concavity of f^l's implies

$$f^l\left(\frac{M\overline{X}_l + M'\overline{X}'_l}{M + M'}\right) \geq \frac{Mf^l\left(\overline{X}_l\right) + M'f^l\left(\overline{X}'_l\right)}{M + M'}$$

for all $l = 1, \ldots, N$. Furthermore, since $f^{n'}(\overline{X}_n) \neq f^{n'}(\overline{X}'_n)$, the above relation holds with strict inequality for $l = n$. Thus, the original profile is not an equilibrium. We follow the same argument in the case of $f^{n'}(\overline{X}_n) < f^{n'}(EX_n)$ to establish the first claim.

Second, suppose that (7) does not hold. Assume without loss of generality that the LHS is greater than the RHS. If agent i is an employee, then his employer replicates the firm sufficiently many times, replaces i with j in one of the replicated configuration, and gets a higher profit. Indeed, this employer's profit will be increased by approximately $f^{n'}(EX_n)\{X(j) - X(i)\} - [W(X(j)) - W(X(i))] > 0$. If agent i is an entrepreneur, then a worker in this firm has an incentive to become an entrepreneur, replicate this firm sufficiently many times, replace i with j in one of them, and get a higher profit. Q.E.D.

Lemma 3 implies that the wage of agent i assigned to task n ($n = 1, \ldots, N$) is determined by the following formula

$$W(X(i)) = \alpha_n X(i) + R_n, \tag{8}$$

where $\alpha_n = f^{n'}(EX_n)$, and R_n is the premium (not necessarily positive) associated with task n.

LEMMA 4. *$W(\cdot)$ is continuous on $X((0,1])$.*

Proof. Suppose not. Then from the monotonicity of $W(\cdot)$, there exist i and $j > i$ such that

$$W(X(i)) - W(X(j)) > \alpha_1[X(i) - X(j)] \tag{9}$$

holds. If agent i is an employee, his employer is better off by replacing agent i with another agent with ability $X(j)$ for the same task since agent j's value of productivity exceeds that of agent i. If agent i is an employer, then agent j has an incentive to become an entrepreneur, replicate the company that i belongs to, replace i with j in one of replicated configuration. If agent j makes the same job allocation as agent i, using himself (or a worker with the same ability as j) in place of i, then agent j can get approximately

$$W(X(i)) + \alpha_1[X(j) - X(i)], \tag{10}$$

for sufficient number of replication. Equation (10) is greater than $W(X(j))$. Q.E.D.

Due to the above Lemmata, the shape of an equilibrium wage schedule is completely determined by the distribution of abilities and the production function. It is a continuous piecewise linear function with the number of kinks being equal to the number of tasks minus one. W is convex and decreasing. Indeed, by continuity of $W(\cdot)$, we have

$$\alpha_{n-1} X(I_{n-1}) + R_{n-1} = \alpha_n X(I_{n-1}) + R_n$$

for all $n = 2, \dots, N$. Thus, we obtain

$$R_n = \sum_{k=n}^{N-1} (\alpha_{n+1} - \alpha_n) X(I_n) + R_N, \tag{11}$$

for all $n = 1, \dots, N-1$. Equation (11) and a zero-profit condition imply the uniqueness of the equilibrium wage schedule. Therefore, the last lemma to be proven before the uniqueness result is the following, which comes from the zero-profit condition.

LEMMA 5. *In an equilibrium, we have*

$$\sum_{n=1}^{N} \theta_n R_n = 0. \tag{12}$$

Proof. Suppose that $S = \{i_1, \dots, i_M\}$ with $i_1 \leq \cdots \leq i_M$ participate in a firm. The optimal allocation of agents to N tasks is to assign θ_{nm} fraction of agent m's time to task n where $\{\theta_{nm}\}$ satisfies

$$\theta_{nm} = \left| \left(\sum_{m'=1}^{m-1} \theta_{nm'}, \sum_{m'=1}^{m} \theta_{nm'} \right) \cap (I_{n-1}M, I_n M) \right|. \tag{13}$$

Let $\Theta = \{\theta_{nm}\}$. Then we have

$$W(X(i_m)) \cdot \sum_{n=1}^{N} \theta_{nm} = \sum_{n=1}^{N} \theta_{nm} [\alpha_n X(i_m) + R_n] \tag{14}$$

by virtue of Lemmata 3 and 4. By Lemma 1, which states that the profit of an entrepreneur is equal to his wage in an equilibrium, we get

$$\begin{aligned} \sum_{m=1}^{M} \sum_{n=1}^{N} \alpha_n \theta_{nm} X(i_m) &= \sum_{m=1}^{M} W(X(i_m)) \cdot \sum_{n=1}^{N} \theta_{nm} \\ &= \sum_{m=1}^{M} \sum_{n=1}^{N} \theta_{nm} [\alpha_n X(i_m) + R_n]. \end{aligned}$$

Subtracting the LHS from the RHS, we obtain Eq. (12). Q.E.D.

From eqs. (11) and (12), we obtain

$$R_n = -\sum_{k=n}^{N-1} (\alpha_k - \alpha_{k+1}) X(I_k) + \sum_{n=1}^{N} \left[\theta_n \sum_{k=n}^{N-1} (\alpha_k - \alpha_{k+1}) X(I_k) \right]. \tag{15}$$

We note that if f is strictly concave, then the average ability of workers on any given task must be the same at all firms. If not, an entrepreneur could "merge" the firms (that is, hire all workers employed by firms with differing average ability), assign workers to the same task as before, and output will increase.

We now state the uniqueness result.

THEOREM. *There exists a unique equilibrium wage schedule.*

3. COMPARATIVE STATICS

3.1. *Effects of Finer Division of Labor*

Adam Smith argued that the division of labor increases productivity by increasing technological advance and by saving switching time. We will show in this section that division of labor can result in an increase in output even in the absence of these two effects. Consider two technologies, F and F'. Let α and α' be functions from I to $[0, \infty)$ generated by F and F', respectively, where $\alpha(i) = \alpha_n \equiv f^{n'}(EX_n)$ if $i \in (I_{n-1}, I_n]$, and similarly for α'. Then F' is said to be a *mean-preserving spread* of F if

$$\int_0^1 [\alpha'(i) - \alpha(i)]\, di = 0$$

and

$$\int_0^i [\alpha'(\iota) - \alpha(\iota)]\, d\iota \geq 0,$$

for all $i \in I$ with strict inequality for some $i \in I$.

We want to argue that comparing technologies whose associated α's are such that one is a mean preserving spread of the other is the relevant experiment in our framework. First, preserving the mean holds constant the average productivity, isolating the effect of changes in the composition of tasks. Second, certain changes in technology that seem naturally to be finer divisions of labor are represented by mean preserving spreads. For example, consider two technologies F and F' with $\theta_i' = \theta_i$, $i = 1, \ldots, n$,

$\alpha'_i = \alpha_i$, $i \neq j, j+1$, $\alpha_j = \alpha_{j+1} = [\alpha'_j + \alpha'_{j+1}]/2$. That is, the technologies F and F' are identical except that the j'th task and the $j'+1$st can be thought of as a subdivision of the jth and $j+1$st tasks, which are essentially a single task. It is straightforward to see that F' is a mean preserving spread of F.

Dividing a single task into subtasks is not the only way of generating mean preserving spreads, however. Consider two technologies F and F' with $n = 2$, $\theta'_1 = \theta_i$, $i = 1, 2$, $\alpha'_1 > \alpha_1$, and $\alpha'_2 < \alpha_2$. F' will then be a mean preserving spread of F if the means are the same, that is, if $\alpha'_1 + \alpha'_2 = \alpha_1 + \alpha_2$, despite the fact that it is clear that F' was not generated by subdividing a task associated with F. We can think of F' as having arisen by a redefinition of the tasks associated with F in which some of the more productive activities in the second task of F are moved to the first task, being replaced with some of the less productive activities of the first task. This type of mean preserving spread should be thought of as a redesign of the tasks in which there is an increased concentration of the most productive activities in the task that will be performed by the more productive workers.

We then have the following result.

THEOREM. *Equilibrium total output under technology F' is greater than or equal to that under technology F for any distribution of abilities if F' is a mean-preserving spread of F.*

Proof. Consider two technologies F and F'. Assume that F' is a mean-preserving spread of F. Since α and α' are step functions, it is sufficient to output goes up (weakly) in the case when spread is made only once, i.e., there exists i^* such that $\alpha'(i) \geq \alpha(i)$ holds if $i < i^*$ and $\alpha'(i) \leq \alpha(i)$ holds if $i > i^*$. The total output under technology F is $\int_0^1 \alpha(i)X(i)\,di$, while that under F' is $\int_0^1 \alpha'(i)X(i)\,di$. Subtracting the former from the latter, we obtain

$$\int_0^1 [\alpha'(i) - \alpha(i)]X(i)\,di = \int_0^{i^*} [\alpha'(i) - \alpha(i)]X(i)\,di + \int_{i^*}^1 [\alpha'(i) - \alpha(i)]X(i)\,di.$$

Since X is decreasing in i, and (2) holds, (3) is greater than or equal to

$$\int_0^{i^*} [\alpha'(i) - \alpha(i)]X(i^*)\,di + \int_{i^*}^1 [\alpha'(i) - \alpha(i)]X(i^*)\,di,$$

which is non-negative. Q.E.D.

Given our results, it is a trivial exercise to extend our model to an economy with a continuum of tasks, where $\alpha\colon (0,1] \to \mathscr{R}_+$ is a decreasing function (not necessarily a step function). We could then apply the above theorem directly to get the same result, that is, the total output under α' is greater than that under α for any ability distribution if and only if α' is a mean-preserving spread of α.

Given an ability distribution X and the average productivity of tasks $E\alpha = \int \alpha(i)\,di$, we can determine the supremum of output obtained through all possible finer divisions of labor, i.e., mean-preserving spreads. Let X_{sup} be the supremum of abilities. Then it is easy to verify that $X_{\text{sup}} E\alpha$ is the highest possible output obtained through refining division of labor. Indeed, let $\alpha^l\colon I \to \mathscr{R}_+ (l = 1, 2, \dots)$ satisfy $\alpha^l(i) = E\alpha l$ if $i \le 1/l$ and zero otherwise. Then the total output y^l under α^l is

$$y^l = \int_0^1 X(i)\,\alpha(i)\,di = E\alpha \cdot l \cdot \int_0^{1/l} X(i)\,di.$$

Using the continuity of X, this expression goes to $E\alpha X_{\text{sup}}$. On the other hand, the output cannot exceed it since we have

$$\int_0^1 \alpha(i) X(i)\,di \le \int_0^1 \alpha(i) X_{\text{sup}}\,di = E\alpha \cdot X_{\text{sup}}.$$

In words, this simply means that we can keep average productivity constant by letting the productivity of all tasks except the most productive go to zero while letting the productivity of the most productive task get increasingly large. If such technological innovation is possible, then even if the average ability is finite, there is no upperbound to the benefit of division of labor if the ability of the highest able person tends to infinity, i.e., $X(i)$ tends to infinity as i goes to zero.[6]

3.2. *Income Distribution*

In this subsection, we focus on a special type of mean-preserving spread. Consider a production technology F. We divide the kth task into two to change the technology to F' in such a way that the average productivity of the kth task remains constant, i.e.,

$$\alpha_k'\theta_k' + \alpha_k''\theta_k'' = \alpha_k\theta_k,$$

with $\alpha_{k-1} > \alpha_k' > \alpha_k'' > \alpha_{k+1}$. Observe that an effect of the division of labor argued by Adam Smith is expressed if $\alpha_k'\theta_k' + \alpha_k''\theta_k'' > \alpha_k\theta_k$ holds,

[6] It would be interesting to endogenize the division into tasks. Given our result that for any division a finer division yields an output gain, any endogenization will necessarily be driven by the explicit and implicit costs of effecting the finer division.

i.e., if the weighted average of productivity increases through the division of labor. Since the argument of the previous section holds under the new technology as well, the new wage scheme W' and the premium scheme R' satisfy (15). Since the lemmata in the previous section hold under both technologies F and F', we clearly have

$$R'_n - R'_{n-1} = R_n - R_{n-1},$$

for $n = 1, \ldots, k-1$ and $n = k+2, \ldots, N$. In other words, there exist r' and r'' such that $R'_n = R_n + r'$ holds for all $n < k$, and $R'_n = R_n + r''$ holds for all $n > k$.

LEMMA. *We have $r' < r''$ (resp. $r' > r''$) whenever $X(\ )$ is strictly concave (resp. strictly convex) on $[I_{k-1}, I_k]$.*

Proof. From Lemmata 3 and 4, we have the relations

$$\alpha_{k-1} X_{k-1} + R_{k-1} = \alpha_k X_{k-1} + R_k$$

and

$$\alpha_k X_k + R_k = \alpha_{k+1} X_k + R_{k+1}$$

for the original technology and

$$\alpha_{k-1} X_{k-1} + R'_{k-1} = \alpha'_k X_{k-1} + R'_k,$$
$$\alpha'_k X'_k + R'_k = \alpha''_k X'_k + R''_k,$$

and

$$\alpha''_k X_k + R''_k = \alpha_{k+1} X_k + R'_{k+1}$$

for the new technology. Eliminating R_k for the original technology and R'_k and R''_k for the new technology, we get

$$\Delta R \equiv R_{k+1} - R_{k-1} = \alpha_{k-1} X_{k-1} - \alpha_k X_{k-1} + \alpha_k X_k - \alpha_{k+1} X_k,$$

and

$$\begin{aligned}\Delta R' &\equiv R'_{k+1} - R'_{k-1} \\ &= \alpha_{k-1} X_{k-1} - \alpha'_k X_{k-1} + \alpha'_k X'_k - \alpha''_k X'_k + \alpha''_k X_k - \alpha_{k+1} X_k,\end{aligned}$$

respectively. Therefore, $\Delta R' > \Delta R$ if and only if

$$\alpha'_k (X_{k-1} - X'_k) + \alpha''_k (X'_k - X_k) < \alpha_k (X_{k-1} - X_k).$$

Since $\alpha'_k > \alpha''_k$ and $\alpha'_k \theta'_k + \alpha''_k \theta''_k = \alpha_k \theta_k$, $\Delta R' > \Delta R$ if and only if

$$\frac{X_{k-1} - X'_k}{X_{k-1} - X_k} < \frac{\theta'_k}{\theta_k},$$

which holds whenever $X(\ )$ is strictly concave. The same logic is applied to conclude that $r' > r''$ if $X(\ )$ is strictly convex.

There is an important implication of this lemma in the case that the density function of ability is single-peaked, for example, a normal distribution. An ability function X has the *single-peaked density property* if there exists x^* such that for any $\epsilon > 0$, $X^{-1}(x - \epsilon) - X^{-1}(x) > X^{-1}(x) - X^{-1}(x + \epsilon)$ holds if $x - \epsilon > x^*$, and the reverse (strict) inequality holds if $x^* > x + \epsilon$. The ability level x^* is the peak of this (pseudo) density. If $X(\ )$ satisfies this property, then it is shown that the ability function is strictly convex for $i < X^{-1}(x^*)$ and strictly concave for $i > X^{-1}(x^*)$. A direct consequence of this argument is stated below.

PROPOSITION. *Assume that an ability function has a single-peaked density property with the single peak of* x^*. *Then* $r' > r''$ *holds if* $I_k > x^*$, *while* $r' < r''$ *holds if* $I_{k-1} < x^*$.

Intuitively, this proposition says that the division of a hard task is more likely to be preferred by the rich than by the poor, and vice versa.

3.3. *Effects of Education*

A striking statement, which is a direct consequence of Eq. (15), is the following.

PROPOSITION. *If* $X(I_k)$ *increases* $(k = 1, \ldots, N)$, *then* $W(X)$ *increases* (*resp. decreases*) *if* $X < X(I_k)$ (*resp.* $X > X(I_k)$) *holds.*

Suppose that education increases the ability of individuals. Then this proposition implies that the rich are worse off if the level of education for the poor goes up.

4. PRODUCTION FUNCTION WITHOUT STRICT COMPLEMENTARITY

We have been able to derive quite sharp results about the wage function and how it changes for changes in the distribution of abilities for our model. That model is fairly special and it is important to understand what characteristics drive our results. In particular, for our production function the quantity was determined by the minimum amount of time devoted to the individual tasks. We want to emphasize that this is not a knife edge case; for production functions that are close to that analyzed above, but without the strict complementarity, the wage function will be close to that in the limit case.

Consider a two task case with a modified production function of the following form

$$F(X,\Theta) = G(\Theta)^{1/\gamma} H(X,\Theta),$$

where

$$G(\Theta) = \left[\left(2\sum_{m=1}^{M}\theta_{1m}\right)^{\gamma} + \left(2\sum_{m=1}^{M}\theta_{2m}\right)^{\gamma}\right]$$

and

$$H(X,\Theta) = \tfrac{1}{2}\alpha_1\overline{X}_1 + \tfrac{1}{2}\alpha_2\overline{X}_2.$$

Here, $G^{1/\gamma}$ corresponds to the quantity of the goods, while H is the value of each unit of the goods. We recover the original production function (1) for which the two tasks are perfect complements if we let γ go to negative infinity. Conversely, we get a linear production function, i.e., the two tasks are perfect substitutes, if γ equals one. In other words, γ can be interpreted as parametrizing the degree of substitutability of the two tasks. Differentiating this production function with respect to θ_{nm}, we obtain

$$\frac{\partial F}{\partial \theta_{nm}} = \frac{F}{\gamma G} 2\left(2\sum_{m'}\theta_{nm'}\right)^{\gamma-1} + G^{1/\gamma}\frac{\alpha_n}{2}\left[-\frac{\overline{X}_n}{\sum_{m'}\theta_{nm'}} + \frac{x_m}{\sum_{m'}\theta_{nm'}}\right].$$

If we consider the second term, $G^{1/\gamma}$, X_n, and $\Sigma_{m'}\theta_{nm'}$ all depend on x_m, the ability of the mth agent, but as the number of agents goes to infinity, the affect of a change in x_m on those terms goes to zero. Hence, as the number of agents gets large, first term is asymptotically unaffected by changes in x_m. Similarly, the second term of the expression converges to an expression that is linear in x_m. Moreover, if the size of the firm is sufficiently large, the coefficient of x_m is arbitrarily close to $G^{1/\gamma}\alpha_n/2\Sigma_{m'}\theta_{nm'}$, which in turn converges to α_n as γ goes to negative infinity when the firm optimally allocates workers. This follows because as γ goes to negative infinity, $G^{1/\gamma}$ converges to $2\Sigma_{m'}\theta_{nm'}$ since $2\Sigma_{m'}\theta_{1m'}$ and $2\Sigma_{m'}\theta_{2m'}$ are arbitrarily close when there are a large number of agents optimally allocated. Thus, if there is perfect competition and agents are paid their marginal product, as γ goes to infinity, these production functions converge to that analyzed in the previous sections, and the wage schedules converge to the wage function for that limit case. It is clear then, that it is not the Leontief production function that drives our results. Rather, it is the separability between G, the quantity component independent of ability of workers, and H, the quality component affected by ability of workers, that gives rise to the wage schedule we observe in the original model.

5. DISCUSSION

5.1. *Robustness of Our Results*

We are able to derive sharp results on the effect of specialization of labor on income distribution because we assume a specific technology. It is clear that the particular technology cannot be applicable to many industries. It does seem to be roughly appropriate for some industries however. It is important to note that while the particular form of the production function is important for tractability, the qualitative nature of our results does not seem to hinge on the exact form of the production function. In the previous section, we relaxed the assumption of strict complementarity of tasks. It is difficult to find closed form solutions to other perturbations of the basic model, but there is nothing to suggest that there is any radical change in the equilibria of a perturbed model.

We also assumed that among workers there was no comparative advantage to make clear that gains from specialization were not the result of comparative advantage, but rather arose because of the production technology. One could imagine a more realistic case in which workers exhibited both absolute and comparative advantage. While it might be difficult to characterize the equilibria of our model in this case, there is nothing to suggest that the gains from specialization we have identified would disappear; one would expect that these gains would be incremented by the gains from specialization that comparative advantage generates.

5.2. *Related Literature*

There is a long literature on the analysis of the gains from specialization. One strand of that literature takes as given economies of scale arising from increased specialization. If we observe anything less than complete specialization, there must be something that offsets the benefits from increased specialization. Stigler (1951) identified the extent of the market as a limit to specialization: increased specialization must entail increased scale, and at some point the extent of the market places limits on the possible scale. Kim (1989) develops a model in which the cost of mismatch between workers and jobs increases with specialization, giving workers a reason to limit the specialized skills they obtain. In Murphy (1986), there is uncertainty about what sector will be good; less specialization exposes workers to less risk. In Becker and Murphy (1992), increasing coordination costs stemming from the larger scale associated with greater specialization limit the equilibrium degree of specialization.

The second strand of literature focuses on comparative advantage across workers as the source of the gains from specialization. The principle of comparative advantage dates back at least to Ricardo; Rosen (1978) formally introduced the concept into a model to determine the optimal allocation of workers to jobs. MacDonald and Markusen (1985) analyze a model in which absolute as well as relative advantage influence the assignment of workers to jobs. In their model, there is a public input (such as administration), the level of which affects the marginal productivity of other workers. This external effect makes it sometimes optimal to allocate a person of very high ability in the public input to that job even if he has a relative disadvantage in this job.

Assignment models that emphasize the assignment of workers to jobs are closest to the model presented here. [See Sattinger (1993) for an excellent survey of assignment models in labor markets.] An early example of a model of this type that is commonly used in labor economics is Roy's model [Roy (1951)]. Roy analyzed the distribution of earnings when workers of differing abilities chose between different sectors of an economy in which to work. In his model, there is no constraint on the number of workers who choose a particular sector, whereas in our model, the proportion of workers that are assigned to each task must be the same. Our model is closer to differential rents assignment models [see Sattinger (1993), p. 845]. These models allow constraints on the number of workers that can be assigned to any job, and, hence, are closer to ours. There is a significant difference, however, in the form of the production function we consider. Differential rents models typically assume that aggregate output is the sum of the individual workers' outputs, while we have assumed a technology that yields no output unless all tasks are completed. This constraint that the less important tasks *must* be done has a real effect on the distribution of earnings. If the value of output were merely the sum of the values of each of the workers, we would assign all workers to the high value task.

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