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COMMON REASONING ABOUT ADMISSIBILITY

ABSTRACT. We analyze common reasoning about admissibility in the strategic and extensive form of a game. We define a notion of sequential proper admissibility in the extensive form, and show that, in finite extensive games with perfect recall, the strategies that are consistent with common reasoning about sequential proper admissibility in the extensive form are exactly those that are consistent with common reasoning about admissibility in the strategic form representation of the game. Thus in such games the solution given by common reasoning about admissibility does not depend on how the strategic situation is represented. We further explore the links between iterated admissibility and backward and forward induction.

1. INTRODUCTION

A well known problem with non-cooperative game theory is that Nash equilibria are seldom relevant for predicting how the players will play. The equilibria of a game do not represent all the possible outcomes. Rather, they represent the set of self-enforcing agreements: had the players known their respective choices before playing the game, then they must have constituted an equilibrium. Some game theorists have argued that predictability must involve what Binmore (1987/88) has called an “eductive” procedure. When asking how the players’ deductive processes might unfold, one must usually specify some basic principles of rationality, and then examine what choices are consistent with common knowledge of the specified principles. The advantage of this approach is that it is possible to refine our predictions about how players might choose without assuming that they will coordinate on a particular equilibrium. Principles such as iterated strict dominance or rationalizability (Pearce 1984), (Bernheim 1984) are examples of how it is possible to restrict the set of predictions using rationality arguments alone. In this paper we embrace the eductive viewpoint, and examine the game-theoretic implications of adopting the classic admissibility postulate of decision theory as a candidate for a rationality principle. An admissible choice is a choice that is not weakly dominated, and we take rationality to coincide with admissibility. We assume admissibility to be common knowledge, and describe players’ common reasoning about admissibility in the strategic and extensive forms of a game. Common reasoning about
admissibility in the extensive form leads to iterated elimination of weakly dominated strategies (IWD).

However, a player might be indifferent between two strategies, one of which is weakly dominated by the other, if she treats as null the state on which the weakly dominant act is strictly preferred. To guarantee that a player will always eliminate a weakly dominated strategy, we have to assume that no state of the world is treated as null by the players. This means that a player’s full belief that a strategy will be played is not to be interpreted as treating the event that it won’t be played as null. As some philosophers have argued (Harper 1976), (McGee 1994), a state that is not considered an epistemically serious possibility need not be treated as null. This means that a player’s conditional preference for a strategy, given such an “impossible” state, can be nontrivially defined. A related approach using lexicographic probabilities to reconcile IWD and Bayesian decision theory is taken in Blume et al. (1991), Stahl (1995) and Asheim and Dufwenberg (1996).

We do not take a stance on this issue; instead, we investigate the consequences of applying iterated admissibility, as an independent choice principle, to finite games of perfect and imperfect information. In the last part of the paper we explore the relationship between IWD in the extensive and strategic forms of a game.

One commonly held disadvantage of IWD is that—unlike iterated strict dominance—different orders of deletion can result in different solutions. A standard solution to this problem is to delete at each round all weakly dominated strategies of all players (Rochet 1980), (Moulin 1986), (Harper 1991). We support this view by arguing that order-free elimination of weakly dominated strategies captures common reasoning about admissibility in the strategic form. In the extensive form of a game, a strategy may prescribe choices in parts of the tree that will never be reached if that strategy is played. If we evaluate strategies only with respect to information sets that are consistent with them (i.e., information sets that can be reached if the strategy is played), we are led to the concept of sequential proper admissibility: A strategy is sequentially properly admissible in a game tree just in case the strategy is admissible at each information set that is consistent with the strategy. A striking result of our paper is that, for finite extensive form games with perfect recall, the strategies that are consistent with common reasoning about sequential proper admissibility in the extensive form are exactly those that are consistent with common reasoning about admissibility in the strategic form representation of the game. Thus in these games, the solution given by common reasoning about admissibility does not depend on how the strategic situation is represented.

Like iterated strict dominance and rationalizability, application of iterated weak dominance (IWD) has the advantage that it does not require advanced computation of equilibria. It is therefore a more global condition than backward and forward induction principles, some of whose features IWD is held to capture. Though backward and forward induction principles are understood to be local conditions, in that they provide a test which can only be applied after the equilibria of a game have been computed, we think that our characterization of IWD captures some crucial features of both principles. For example, we show that, in generic finite games of perfect information, common reasoning about weak admissibility yields exactly the backward induction solution. And in finite games of imperfect information, common reasoning about admissibility yields typical forward induction solutions. Thus backward and forward induction seem to follow from one principle, namely that players’ choices should be consistent with common reasoning about admissibility. This result may seem questionable, as it is also commonly held that backward and forward induction principles are mutually inconsistent. That is, if we take backward and forward induction principles to be restrictions imposed on equilibria, then they lead to contradictory conclusions about how to play. We show that the problem with the examples one finds in the literature is that no constraints are set on players’ forward induction “signals”. We define a credible forward induction signal in an extensive game as a signal consistent with common reasoning about sequential admissibility. Thus the examples in the literature which purport to show the conflict between backward and forward induction principles involve forward induction signals that are not credible.

2. EXTENSIVE FORM GAMES

We introduce the basic notions for describing games in extensive form. Note that our formalization is limited to finite games, and that we restrict players to only play pure strategies. A finite extensive form game for players $N = 1, 2, \ldots, n$ is given by a game tree $T$ with finitely many nodes $v$, root $r$, payoff functions $u_i$ which assign a payoff to each player $i$ at each terminal node in $T$, and information sets $I_i$ for each player $i$. For each node $x$ in $T$, $I(x)$ is the information set containing $x$. A pure strategy $s_i$ for player $i$ in a game tree $T$ assigns a unique action, called a move, to each information set $I_i$ of player $i$ in $T$. We denote the set of $i$'s pure strategies in $T$ by $S_i(T)$ (in what follows, the term “strategy” always refers to pure strategies). A strategy profile in $T$ is a vector $(s_1, s_2, \ldots, s_n)$ consisting of one strategy for each player $i$. We denote the set of pure strategy profiles in $T$ by $S(T)$; i.e. $S(T) = \times_{i \in N} S_i(T)$. We use $s$ to denote a generic
strategy profile. It is useful to denote a vector of length \( n - 1 \) consisting of strategy choices by player \( i \)'s opponents by \( s_{-i} \). We write \( S_{-i}(T) \) for the set of strategy profiles of \( i \)'s opponents, i.e. \( S_{-i}(T) = \times_{j \in N - \{i\}} S_j(T) \).

Given a strategy profile \( s \), we use \( s[i] \) to denote the strategy of player \( i \) in \( s \), and \( s[-i] \) to denote the strategy profile of \( i \)'s opponents in \( s \).

In the games we consider, the root is the only member of its information set (i.e. \( I(r) = \{r\} \)), so that a strategy profile \( s \) in \( T \) determines a unique maximal path \( < r, x_1, x_2, \ldots, x_n > \) from the root \( r \) to a terminal node \( x_n \); we refer to this path as the play sequence resulting from \( s \), and denote it by \( play(s) \). When a strategy profile \( s \) in \( T \) is played, each player receives as a payoff the payoff from the terminal node reached in the play sequence resulting from \( s \). With some abuse of notation, we use \( u_i \) to denote both a function from strategy profiles to payoffs for player \( i \), as well as a function from terminal nodes to a payoff for player \( i \), and define \( u_i(s) = u_i(x) \), where \( x \) is the terminal node in the play sequence \( play(s) \). For a finite game tree \( T \), the height of a node \( x \) in \( T \) is denoted by \( h(x) \), and defined recursively by \( h(x) = 0 \) if \( x \) is a terminal node in \( T \), and \( h(x) = 1 + \max \{h(y) : y \text{ is a successor of } x \text{ in } T\} \) otherwise.

An important part of players' deliberation about which strategy to choose in a given game consists of ruling out possibilities about how the game might be played. Though players may use different principles to exclude some plays of the game, any such reasoning will result in a game tree restricted to those possibilities consistent with the application of a given principle. The following definitions allow us to describe this notion precisely.

**Definition 1** Restricted Game Trees

- Let \( T \) be a finite game tree for \( N = 1, 2, \ldots, n \) players.
- \( T[V] \) is the restriction of \( T \) to \( V \), where \( V \) is a subset of the nodes in \( T \). All information sets in \( T[V] \) are subsets of information sets in \( T \).
- \( T_x \) is the game tree starting at node \( x \) (i.e. \( T_x \) is the restriction of \( T \) to \( x \) and its successors.) If \( I(x) = \{x\} \), then \( T_x \) is called a subgame.
- If \( s_i \) is a strategy for \( T \) and \( T' \) is a restriction of \( T \), \( s_i[T'] \) is the strategy that assigns to all information sets in \( T' \) the same choice as in \( T \). Formally, \( s_i[T'](I'_i) = s_i(I_i) \), where \( I_i \) is the (unique) information set in \( T \) that contains all the nodes in \( I'_i \). Note that \( s_i[T'] \) is not necessarily a strategy in \( T' \); for the move assigned by \( s_i \) at an information set \( I_i \) in \( T \) may not be possible in \( T' \).
- If \( s \) is a strategy profile in \( T \) and \( T' \) is a restriction of \( T \), \( s[T'] \) is the strategy vector consisting of \( s[i][T'] \) for each player \( i \).

3. Common Reasoning about Rationality

We may assume that in deliberating players use some principle to rule out plays of the game that are inconsistent with that principle. One such principle is rationality. In the next sections we explore the consequences of adopting two candidates for a rationality principle: weak admissibility and admissibility. In the first case, a player never plays a strictly dominated strategy, whereas in the second case also weakly dominated strategies are eliminated.

A player who is reasoning, say, with the help of admissibility would not go very far in eliminating plays of the game inconsistent with it, unless he assumes that the other players are also applying the same principle. In the game of Figure 1, for example, player 1 could not eliminate a priori any play of the game unless he assumed player 2 never plays a dominated strategy.

In general, even assuming that other players are rational might not be enough to rule out possibilities about how a given game might be played. Players must reason about other players' reasoning, and such mutual reasoning must be common knowledge. Unless otherwise specified, we shall assume that players have common knowledge of the structure of the game and of rationality, and examine how common reasoning about rationality unfolds.

3.1. Strict Dominance and Subgame Perfection

This section explores in detail the implications of common reasoning about weak admissibility, the requirement that players should avoid strictly dominated actions. We show that in finite games of perfect information, common reasoning about weak admissibility gives exactly the same results as Zermelo's backward induction algorithm, which in finite games of perfect information corresponds to Selten's notion of subgame perfection (cf.
Our procedure for capturing common reasoning about sequential weak admissibility in $T$ is the following. First, eliminate at each information set in $T$ all moves that are inconsistent with weak admissibility, i.e. strictly dominated choices. The result is a restricted game tree $T'$.

Repeat the pruning procedure with $T'$ to obtain another restricted game tree, and continue until no moves in the resulting game tree are strictly dominated. Note that the recursive pruning procedure does not start at the final information sets. Our procedure allows players to consider the game tree as a whole and start eliminating branches anywhere in the tree by applying weak admissibility. To illustrate the procedure, look at the game of Figure 1. $R$ is eliminated at 2's information set in the first iteration, and then $c$ is eliminated for player 1 because, after $R$ is eliminated, either $a$ or $b$ yield player 1 a payoff of 1 for sure, while $c$ yields 0. The pruning procedure is formally defined as follows. For a given game tree $T$, let $\text{Weak} - Ad_1(T) = \{ s_i \in S_i(T) : s_i \text{ is sequentially weakly admissible in } T \}$, and let $\text{Weak} - Ad(T) = \times_{i \in N} \text{Weak} - Ad_i(T)$.

**DEFINITION 3 Common Reasoning about Sequential Weak Admissibility**

- Let $T$ be a finite game tree for $N = 1, 2, \ldots, n$ players.
- The strategies in $T$ consistent with common reasoning about sequential weak admissibility are denoted by $CR_{WA}(T)$, and are defined as follows:
  1. $WA^0(T) = S(T)$.
  2. $WA^{j+1}(T) = \text{Weak} - Ad(T|WA^j(T))$.
  3. $s \in CR_{WA}(T) \iff \forall j : s|[T|WA^j(T)] \in WA^{j+1}(T)$.

If $T$ is a finite game tree, the set of strategies for player $i$, $S_i(T)$ is finite, and our procedure will go through only finitely many iterations. To be precise, let $\max = \sum_{i \in N} |S_i| - 1$; then the procedure will terminate after max iterations, i.e. for all $j \geq \max$, $WA^j(T) = WA^{j+1}(T)$.

We introduce the concept of Nash equilibrium and one of its refinements, subgame perfection, for generic finite games in extensive form. A strategy $s_i$ in a game tree $T$ is a best reply to a strategy profile $s_{-i}$ of $i$'s opponents if there is no strategy $s'_i$ for player $i$ such that $u_i(s'_i, s_{-i}) > u_i(s_i, s_{-i})$. A strategy profile $s$ is a Nash equilibrium if each strategy $s[i]$ in $s$ is a best reply against $s[-i]$. A strategy profile $s$ is a subgame perfect equilibrium if for each subgame $T_0$ of $T$, $s[T_0]$ is a Nash equilibrium of $T_0$. We say that a strategy $s_i$ in $T$ is consistent with subgame perfection if there is a subgame perfect strategy profile $s$ of which $s_i$ is a component strategy, i.e. $s_i = s[i]$. We denote the set of player $i$'s strategies in $T$ that are consistent with subgame perfection by $SPE_i(T)$.
and define the set of strategy profiles consistent with subgame perfection by $\text{SPE}_i(T) = \times_{i \in N} \text{SPE}_i(T)$). Note that not all strategy profiles that are consistent with subgame perfection are subgame perfect equilibria. In Figure 2, all strategy profiles are consistent with subgame perfection, but $L$, $ba'$ and $R$, $ab'$ are not equilibria, since in equilibrium 1 must be playing a best reply to 2's strategy.

Finally, $T$ is a game of perfect information if each information set $I$ of $T$ is a singleton. The game in Figure 2 is a game of perfect information.

A standard approach to finite games of perfect information is to apply Zermelo's backwards induction algorithm which yields the set of strategy profiles that are consistent with subgame perfection, i.e. $\text{SPE}(T)$ (cf. Osborne and Rubinstein 1994, Ch.6.2)). Common reasoning about weak admissibility, as defined by the procedure $WA$, does not follow Zermelo's backwards induction algorithm. For example, suppose that in a game tree a move $m$ at the root is strictly dominated by another move $m'$ at the root for the first player. Common reasoning about weak admissibility rules out $m$ immediately, but the backwards induction algorithm eliminates moves at the root only at its last iteration. Nonetheless, our first result is that in games of perfect information, the final outcome of the two procedures is the same: In these games, the strategies that are consistent with common reasoning about sequential weak admissibility are exactly those consistent with subgame perfection.

**PROPOSITION 1.** Let $T$ be a finite game tree of perfect information. Then a strategy $s_i$ is consistent with common reasoning about sequential weak admissibility in $T$ if and only if $s_i$ is consistent with subgame perfection. That is, $\text{CRWA}(T) = \text{SPE}(T)$.

In games of imperfect information, the equivalence between strategies consistent with subgame perfection and those consistent with common reasoning about sequential weak admissibility fails in both directions. Figure 1 shows that a strategy profile $s$ may be a subgame perfect equilibrium although $s$ is not consistent with common reasoning about sequential weak admissibility: The strategy profile $(c, R)$ is a subgame perfect equilibrium, but $R$ and (hence) $c$ are not consistent with common reasoning about sequential weak admissibility. And in Figure 3, $a$ is not strictly dominated for player 2, but $a$ is neither a best reply to $L$ nor to $R$.

Although $a$ is not strictly dominated, $a$ seems like a bad choice because it never gives player 2 a better payoff than the alternatives and sometimes gives her less. In other words, $a$ is weakly dominated. In the remainder of this paper, we investigate how players might reason about a game on the assumption that no player will choose a weakly dominated strategy.

4. **SEQUENTIAL WEAK DOMINANCE AND FORWARD INDUCTION**

4.1. **Weak Dominance**

Informally, a strategy $s_i$ is weakly dominated by another strategy $s_i'$ at an information set $I_i$ in a game tree $T$ if $s_i'$ never yields less to $i$ at $I_i$ than $s_i$, does, and sometimes yields more. For example, in the game of Figure 3, $a$
We define a procedure to capture common reasoning about sequential admissibility analogous to common reasoning about sequential weak admissibility. To illustrate the procedure, consider Figure 4. Common reasoning about admissibility rules out \( b \) as a choice for player 2 because \( b \) is weakly dominated. Then given that only \( a \) remains at 2's decision node, \( R_1 \) (strictly) dominates \( L_1 \) for player 1. So the only play consistent with common reasoning about sequential admissibility is for player 1 to play \( R_1 \) and end the game. Note however that common reasoning about sequential weak admissibility, i.e. the standard backwards induction procedure, is consistent with both \( R_1 \) and the play sequence \( L_1, b, L_2 \). So even in games of perfect information, common reasoning about sequential admissibility may lead to stronger results than common reasoning about sequential weak admissibility.

For a given game tree \( T \), let \( \text{Seq} - \text{Ad}_i(T) = \{ s_i \in S(T) : s_i \text{ is sequentially admissible in } T \} \), and let \( \text{Seq} - \text{Ad}(T) = \times_{i \in \mathbb{N}} \text{Seq} - \text{Ad}_i(T) \).

**DEFINITION 5** Common Reasoning about Sequential Admissibility

- Let \( T \) be a finite game tree with players \( N = 1, 2, \ldots, n \).
- The strategies in \( T \) consistent with common reasoning about sequential admissibility are denoted by \( \text{CR}_\text{Seq}(T) \), and are defined as follows:
  1. \( \text{Seq}^0(T) = S(T) \).
  2. \( \text{Seq}^{i+1}(T) = \text{Seq} - \text{Ad}(T|\text{Seq}^i(T)) \).
  3. \( s \in \text{CR}_\text{Seq}(T) \iff \forall j : s[T|\text{Seq}^j(T)] \in \text{Seq}^{j+1}(T) \).

We have seen that common reasoning about sequential admissibility can lead to stronger results than common reasoning about sequential weak admissibility; we next show that the former never leads to weaker results than the latter. The key is to observe that if a strategy \( s_i \) is strictly dominated in a game tree \( T \), \( s_i \) will be strictly dominated in a restriction of \( T \). The next lemma asserts the contrapositive of this observation: If a strategy \( s_i \) is admissible in a restriction of \( T \), \( s_i \) is not strictly dominated in \( T \).

**LEMMA 2.** If \( T \) is a restriction of \( T' \) and \( s_i \) is sequentially admissible in \( T \), then there is an extension \( s'_i \) of \( s_i \) to \( T' \) such that \( s'_i \) is sequentially weakly admissible in \( T' \).

This means that our procedure \( \text{Seq} \) yields, at each stage \( j \), a result that is at least as strong as that of common reasoning about weak admissibility, the procedure \( \text{WA} \). Hence we have the following proposition.
PROPOSITION 3 Let $T$ be a finite game tree. If a play sequence is consistent with common reasoning about sequential admissibility in $T$, then that play sequence is consistent with common reasoning about sequential weak admissibility. That is, $\{\text{play}(s) : s \in CR_{Seq}(T)\} \subseteq \{\text{play}(s) : s \in CR_{WA}(T)\}$.

4.2. Forward Induction

It is commonly held that iterated weak dominance (i.e. iterated sequential admissibility) captures some of the features of backward and forward induction. Fudenberg and Tirole (1993, p. 461) thus state that: “Iterated weak dominance incorporates backward induction in games of perfect information: The suboptimal choices at the last information sets are weakly dominated; once these are removed, all subgame-imperfect choices at the next-to-last information sets are removed at the next round of iteration; and so on. Iterated weak dominance also captures part of the forward induction notions implicit in stability, as a stable component contains a stable component of the game obtained by deleting a weakly dominated strategy”.

Indeed, we have previously shown that, in finite games of perfect information, common reasoning about weak admissibility yields exactly the backward induction solution. In this section we show how, in finite games of imperfect information, common reasoning about admissibility yields typical forward induction solutions. Thus backward and forward induction seem to follow from one principle, namely that players’ choices should be consistent with common knowledge of (and common reasoning about) admissibility. This result may seem questionable, as it is also commonly held that backward and forward induction principles are mutually inconsistent (Kohlberg and Mertens 1986), (Myerson 1991). That is, if we take backward and forward induction principles to be restrictions imposed on equilibria, then they may lead to contradictory conclusions about how to play.

A backward induction principle states that each player’s strategy must be a best reply to the other players’ strategies, not only when the play begins at the initial node of the tree, but also when the play begins at any other information set. A forward induction principle says that players’ beliefs should be consistent with sensible interpretations of the opponents’ play. Thus a forward induction principle restricts the range of possible interpretations of players’ deviations from equilibrium play. Deviations should be constructed as ‘signals’ (as opposed to mistakes), since players should privilege interpretations of the opponents’ play that are consistent with common knowledge of rationality. The typical example of a contradiction between backward and forward induction principles would be a game of imperfect information, where one may apply forward induction in one part of the tree, and then use the conclusion for a backward induction argument in a different part of the tree (Kohlberg 1990).

The game of Figure 5 is taken from (Kohlberg 1990, p.10). Since player I, by choosing $y$, could have received 2, then by forward induction if he plays $n$ he intends to follow with $T$; but for the same reason $II$, by choosing $D$, shows that she intends to play $R$, and hence – by backward induction – $I$ must play $B$. What seems to be at stake here is a conflict between different but equally powerful intuitions. By playing $D$, player $II$ is committing herself to follow up with $R$, and thus player $I$ would be safe to play $y$. On the other hand, once player $I$’s node has been reached, what happened before might be thought of as strategically irrelevant, as $I$ now has a chance—by choosing $n$—of signaling his commitment to follow with $T$. Which commitment is firmer? Which signal is most credible?
We must remember that players make their choices about which strategy to adopt after a process of deliberation that takes place before the game is actually played. During deliberation, we have argued, players will employ some shared principle that allows them to rule out some plays of the game as inconsistent with it. A plausible candidate is admissibility. Let us now see how the ex ante deliberation of the players might unfold in this game by applying the procedure \( \text{Seq}(T) \) to the strategies \( \text{UL}, \text{UR}, \text{DL}, \text{DR} \) and \( yT, yB, nT, nB \). Note that if we recursively apply this game to the concept of sequential admissibility presented in the previous section, we must conclude that the only strategies consistent with common reasoning about sequential admissibility are \( \text{UR} \) and \( yT \). Indeed, common reasoning about sequential weak admissibility alone yields this result. For during the first round of iteration, the strategy \( nB \) of player I is eliminated because this strategy is strictly dominated by any strategy that chooses \( y \) at I's first choice node. Similarly, the strategy \( DL \) of player II is immediately eliminated because this strategy is strictly dominated by any strategy that chooses \( U \) at the root. So after the first round of elimination, I's second information set is restricted to the node reached with \( nT \), and her choices at this information set are restricted to \( R \) only. This means in turn that \( y \) now strictly dominates \( nT \) at I's first information set, and \( U \) strictly dominates \( DR \) at the root. Finally, the strategies \( yB \) and \( UL \) are not strategies in the restricted tree obtained after the first round of elimination, and therefore they are eliminated. After the second round of elimination, only \( UR \) and \( yT \) survive. Thus we predict that players who deliberate according to a shared admissibility principle will expect \( U \) to be chosen at the beginning of the game.

A brief comment about the intuitive plausibility of our procedure is now in order. Note that the procedure we propose does not allow the players to discount whatever happens before a given information set as strategically irrelevant. For example, if player II were to choose \( D \), player I should not keep playing as if he were in a new game starting at his decision node. We rather suggest that I should expect II to follow with \( R \), if given a chance. In which case he should play \( y \) and player II, who can replicate I's reasoning, will in fact never play \( D \). On the other hand, playing \( D \) to signal that one wants to continue—if given a chance—with \( R \) would make little sense, since II must know that \( nB \) is never going to be chosen, and \( R \) makes sense only if it follows \( nB \). In other words, \( D \) is not a rational move for player II. Similar reasoning excludes \( nB \) as a rational strategy for player I.

The problem with Kohlberg's and similar examples is that no constraints are set on players' forward induction "signals". We define the notion of a credible signal in an extensive form game, and show that the credible signals are the signals consistent with common reasoning about sequential admissibility (much as Selten's subgame-perfect equilibria characterize "credible threats"). Thus the examples in the literature which purport to show the conflict between backward and forward induction principles involve forward induction signals that are not credible.

The following definition formulates the notion of a forward induction signal in general, and a credible forward induction signal in particular. The idea is this: Let us consider a move \( m \) at a given information set \( I_i \), and ask what future moves of player i at lower information sets \( I'_i \) are consistent with sequential admissibility and the fact that \( m \) was chosen at \( I_i \). If there are future moves that are consistent with sequential admissibility and the fact that \( m \) was chosen at \( I_i \), then we take the move \( m \) at \( I_i \) to be a signal that player i intends to follow with one of those moves at \( I'_i \). But we argue that in order for this signal to be credible to i's opponents, at least one of the future admissible moves must be consistent with common reasoning about sequential admissibility in \( T \).

We say that an information set \( I'_i \) in a game tree \( T \) is reachable from another information set \( I_i \) with a strategy \( s_i \) if there are nodes \( x \in I_i \), \( y \in I'_i \) such that some play sequence that is consistent with \( s_i[T_x] \) contains \( y \).

**DEFINITION 6** Let \( T \) be a game tree with information set \( I_i \). Let \( T[I_i] \) denote the restriction of \( T \) to nodes in \( I_i \) and successors of nodes in \( I_i \).

- A strategy \( s_i \) is consistent with forward induction at \( I_i \) if \( s_i \) is sequentially admissible at \( I_i \).

- A move \( m \) at an information set \( I_i \) is a forward induction signal for \( S_i^* \) at a lower information set \( I'_i \) (written \( \text{< } I_i : m, I'_i : S_i^* \text{ >} \)), where
  \[ \text{< } I_i : m, I'_i : S_i^* \text{ >} \iff \]
  1. \( s_i(I_i) = m; \)
  2. \( I'_i \) is reachable from \( I_i \) with \( s_i; \)
  3. \( s_i \) is consistent with forward induction at \( I_i \).

- A forward induction signal \( \text{< } I_i : m, I'_i : S_i^* \text{ >} \) is credible if some strategy \( s_i \) in \( S_i^* \) is consistent with common reasoning about sequential admissibility in \( T \), i.e. \( s_i \in CR_{\text{Seq}(T)} \).

Let us illustrate these concepts in the game of Figure 5. According to our definitions, the only strategy that chooses \( n \) at I's first information set and is consistent with forward induction is \( nT \). So \( \text{< } I_i : n, I'_i : \{nT\} \text{ >} \) is a forward induction signal, where \( I'_i \) denotes I's first information set and
$I^2_1$ denotes $I$’s second information set. However, $< I^1_1 : n, I^2_2 : \{nT\} >$ is not a credible signal. For $nT$ is inconsistent with common reasoning about sequential admissibility, since such reasoning rules out $L$ at $I$’s second information set. Similarly for player II, $< I^1_2 : D, I^2_2 : \{DR\} >$ is a forward induction signal. But it is not a credible signal, since $DR$ is inconsistent with common reasoning about sequential admissibility. Hence neither forward induction signal is credible, as “sending” either signal is inconsistent with common reasoning about sequential admissibility as defined by $CR_{Seq}$.

In terms of reasoning about admissibility, the difference between Kohlberg’s and our analysis is this. Kohlberg applies admissibility once to argue that $D$ is a forward induction signal for $R$ and $n$ is a forward induction signal for $T$. But if we assume that admissibility is common knowledge among the players, then neither $D$ nor $n$ are credible signals. Indeed, common knowledge is not even needed to get to this conclusion: it is sufficient to apply admissibility twice to get the same result.

5. COMMON REASONING ABOUT ADMISSIBILITY IN THE EXTENSIVE AND STRATEGIC FORMS

A game $G$ in strategic form is a triple $(N, S_{i \in N}, u_{i \in N})$, where $N$ is the number of players and, for each player $i \in N$, $S_i$ is the set of pure strategies available to $i$, and $u_i$ is player $i$’s utility function. Given a strategy profile $s = (s_1, \ldots, s_n)$, we let $u_i(s)$ denote the payoff to player $i$ when players follow the strategies $(s_1, \ldots, s_n)$. Consider the set of strategy profiles $S = S_1 \times S_2 \times \cdots \times S_n$, and two strategies $s_i, s'_i \in S_i$ of player $i$. Player $i$’s strategy $s_i$ is weakly dominated by her strategy $s'_i$ in $S_i$ just in case:

1. for all $n - 1$-tuples $s_{-i}$ chosen by $i$’s opponents that are consistent with $S$, $u_i(s_i, s_{-i}) \leq u_i(s'_i, s_{-i})$ and
2. for at least one $n - 1$-tuple $s_{-i}$ consistent with $S$, $u_i(s_i, s_{-i}) < u_i(s'_i, s_{-i})$.

A strategy $s_i$ is weakly dominated given $S$ just in case there is a strategy $s'_i$ consistent with $S$ such that $s'_i$ weakly dominates $s_i$ in $S$. A strategy $s_i$ is admissible in $S$ just in case $s_i$ is not weakly dominated given $S$. We denote the strategic form of an extensive form game $T$ by the collection $S(T)$ of strategies in $T$, with payoffs defined as in $T$.

Our goal in this section is to determine what reasoning in the strategic form of a game corresponds to common reasoning about sequential admissibility. To this end we characterize what properties a strategy $s_i$ must satisfy in the extensive form $T$ of a game in order to be admissible in the strategic form $S(T)$. The key idea is to evaluate a strategy only with respect to information sets that can be reached by the given strategy. For example, in the game of Figure 4, the strategy $(R_1, R_2)$ for player 1 yields the same payoff as $(R_1, L_2)$. Hence neither strategy weakly dominates the other in the normal form, although $(R_1, L_2)$ is sequentially admissible and $(R_1, R_2)$ is not. Evaluating strategies only with respect to information sets that are consistent with them leads to what we call proper weak dominance, and proper admissibility. So in the game of Figure 4, $(R_1, R_2)$ is properly admissible.

We say that an information set $I$ in a game tree $T$ is reachable with a strategy $s_i$ if some node in $I$ is consistent with $s_i$.

**DEFINITION 7** Sequential Proper Admissibility

- Let $T$ be a finite game tree.
- A strategy $s_i$ is properly weakly dominated at an information set $I_i$ belonging to $i$ in $T$ just in case $I_i$ is reachable with $s_i$ and $s_i$ is weakly dominated at $I_i$.
- A strategy $s_i$ is properly admissible at an information set $I_i$ just in case $s_i$ is not properly weakly dominated at $I_i$.
- A strategy $s_i$ is sequentially properly admissible in $T$ if and only if $s_i$ is properly admissible at each information set $I_i$ in $T$ that belongs to player $i$.

We define the result of common reasoning about sequential proper admissibility in the by now familiar way. For a given game tree $T$, let $Seq - PA_i(T) = \{s_i \in S_i(T) : s_i$ is sequentially properly admissible in $T\}$, and let $Seq - PA(T) = \times_{i \in N}Seq - PA_i(T)$.

**DEFINITION 8** Common Reasoning About Sequential Proper Admissibility

- Let $T$ be a game tree, with players $N = 1, 2, \ldots, n$.
- The strategies in $T$ consistent with common reasoning about sequential proper admissibility are denoted by $CR_{PSeq}(T)$, and are defined as follows:
  1. $PSeq^0(T) = S(T)$.
  2. $PSeq^{i+1}(T) = Seq - PA(T)[PSeq^i(T)]$.
  3. $s \in CR_{PSeq}(T) \iff \forall j : s[T]PSeq^i(T) \in PSeq^{i+1}(T)$.
The two notions of sequential admissibility are equivalent in terms of their predictions about how the game will be played. That is, exactly the same play sequences are consistent with both restrictions.

Lemma 4 Let $T$ be a finite game tree. Then the play sequences consistent with sequential admissibility are exactly those consistent with sequential proper admissibility. That is, \{play($s$) : $s$ is sequentially admissible in $T$\} = \{play($s$) : $s$ is sequentially properly admissible in $T$\}.

From this fact it follows immediately that common reasoning about sequential admissibility yields the same predictions as common reasoning about proper sequential admissibility.

Proposition 5 Let $T$ be a finite game tree. Then the play sequences consistent with common reasoning about sequential admissibility are exactly those consistent with common reasoning about sequential proper admissibility. That is, \{play($s$) : $s$ $\in$ $CR_{Seq}(T)$\} = \{play($s$) : $s$ $\in$ $CR_{Pseq}(T)$\}.

However, it is not always the case that a strategy that is admissible in the strategic form of a game is properly admissible in an extensive form of the game. For example, in the game of Figure 6, the strategy $L$ is properly weakly dominated for player 2 at her information set: at node $y$, $R$ yields a higher payoff than $L$, and starting at node $x$, both choices yield the same. On the other hand, node $y$ cannot be reached when 2 plays $L$, so that $L$ is admissible in the strategic form of the game, yielding 2's maximal payoff of 1. The game in Figure 6 has the strange feature that if 2 plays $R$ at $x$ to arrive at $y$, she has 'forgotten' this fact and cannot distinguish between $x$ and $y$. Indeed, this is a game without perfect recall. Perfect recall is defined as follows.

Definition 9 (Kuhn) Let $T$ be a finite game tree. Then $T$ is an extensive form game with perfect recall if and only if for each information set $I_i$ belonging to player $i$, and each strategy $s_i$ in $T$, all nodes in $I_i$ are consistent with $s_i$ if any node in $I_i$ is.

We note that if $T$ is a game with perfect recall, then all restrictions of $T$ satisfy perfect recall. The next proposition shows that in extensive form games with perfect recall, the notion of proper weak dominance coincides exactly with admissibility in the strategic form.

Proposition 6 Let $T$ be a finite game tree with perfect recall. Then a strategy $s_i$ for player $i$ is admissible in the strategic form $S(T)$ if and only if $s_i$ is sequentially properly admissible in $T$.

Consider a game $G$ in strategic form. We define an order-free iterative procedure for eliminating weakly dominated strategies. If $S$ is a set of strategy profiles, let $Admiss_i(S)$ be the set of all strategies $s_i$ for player $i$ that are consistent with $S$ and admissible given $S$, and let $Admiss(S) = \times_{i \in N} Admiss_i(S)$.

Definition 10 Common Reasoning About Admissibility in the Strategic Form

- Let the strategic form of a finite game $G$ be given by $(N, S_i \in N, u_i \in N)$, and let $S = S_1 \times S_2 \times \cdots \times S_n$ be the set of strategy profiles in $G$.
- The strategies in $S$ consistent with common reasoning about admissibility are denoted by $CR_{Ad}(S)$, and are defined as follows.
  1. $Ad^0(S) = S$.
  2. $Ad^{j+1}(S) = Admiss(Ad^j(S))$.
  3. $CR_{Ad}(S) = \cap_{j=0}^{\infty} Ad^j(S)$.

The procedure goes through at most $\sum_{i \in N} |S_i - 1|$ iterations; that is, for all $j \geq \sum_{i \in N} |S_i - 1|$, $Ad^j(S) = Ad^{j+1}(S)$.
For example, consider the game in Figure 7. In the first iteration, player 1 will eliminate c, which is weakly dominated by b, and player 2 will eliminate R, which is dominated by L and M. Since admissibility is common knowledge, both players know that the reduced matrix only contains the strategies a, b and L, M. Common reasoning about admissibility means that both players will apply admissibility to the new matrix (and know that they both do it), and since now L dominates M, both will know that M is being eliminated. Finally, common reasoning about admissibility will leave b, L as the unique outcome of the game.

Our main result is that in games with perfect recall, iterated sequential proper admissibility and order-free elimination of inadmissible strategies in the strategic form yield exactly the same result.

**THEOREM 7** Let T be a finite game tree with perfect recall. A strategy profile s is consistent with common reasoning about sequential proper admissibility in T if and only if s is consistent with common reasoning about admissibility in the strategic form of T. That is, \( CR_{FS} (T) = CR_{Ad} (S(T)) \).

It is noteworthy that if the order-free elimination of inadmissible strategies in the normal form yields a unique solution, then that solution is a Nash equilibrium (Bicchieri 1993).

General existence is now easy to establish.

**PROPOSITION 8** For all finite games G with pure strategy profiles S, \( CR_{Ad} (S) \neq \emptyset \).

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**6. PROOF OF RESULTS**

For the proof of proposition 1, we rely on the well-known one-deviation property of subgame perfect equilibrium: If it is possible for one player to profitably deviate from his subgame perfect equilibrium strategy \( s_i \), he can do so with a strategy \( s'_i \) that deviates from \( s_i \) only once.

**LEMMA 0** Let T be a finite game tree of perfect information. Then s is a subgame perfect equilibrium in T if and only if for each node \( x \), for each player \( i \), \( u_i(s_i, s_{[-i]}, x) \geq u_i(s'_i, s_{[-i]}, x) \), whenever \( s_i \) and \( s'_i \) differ only at \( x \).

**Proof.** See (Osborne and Rubinstein 1994, Lemma 98.2).

For the next proposition, we note that if T is finite, then our iterative procedure goes through only finitely many iterations. In particular, this means that if a strategy \( s_i \) is strictly dominated given \( CR_{WA}(T) \), then \( s_i \) is not in \( CR_{WA}(T) \).

**PROPOSITION 1** Let T be a finite game tree of perfect information. Then a strategy \( s_i \) is consistent with common reasoning about sequential weak admissibility in T if and only if \( s_i \) is consistent with subgame perfection. That is, \( CR_{WA}(T) = SPE(T) \).

**Proof.** We prove by induction on the height x of each node that \( CR_{WA}(T_x) = SPE(T_x) \). The proposition follows when we take x to be the root of T.

Base Case, \( h(x) = 1 \). Then all successors of x are terminal nodes. Let player \( i \) be the player to move at x. Let \( max(x) \) be the maximum payoff player \( i \) can achieve at x (i.e., \( max(x) = max\{u_i(y) : y \) is a successor of x\}). Then \( s_i | T_x \) is consistent with subgame perfection at x if and only if \( s_i | T_x \) yields i the maximum payoff \( max(x) \), which is exactly when \( s_i | T_x \) is not strictly dominated at x.

Inductive Case: Assume the hypothesis in the case when \( h(y) < h(x) \) and consider x.

\( (\Rightarrow) \): Let s be a strategy profile consistent with common reasoning about sequential weak admissibility (i.e., \( s \in CR_{WA}(T_x) \)). Suppose that it is player \( i \)'s turn at x. For each player \( j \), \( s[j] | T_x \) is consistent with subgame perfection in each proper subgame \( T_j \) of \( T_x \), by the inductive hypothesis and the fact that \( s[j] \) is consistent with common reasoning about sequential weak admissibility in \( T_x \). So the implication \( (\Rightarrow) \) is established if we show that \( s[i] \) is consistent with subgame perfection in \( T_x \). Let \( y \) be the successor of x that is reached when \( i \) plays \( s[i] \) at x. Let \( max(y) \) be the maximum that \( i \) can achieve given common reasoning about sequential weak admissibility.
when he follows $s[i]$ (i.e., $\max(y) = \max\{u(i, s[i], s_{-i}, x) : s_{-i}$ is consistent with $CRW_A(T_2)\}$). For each $y'$ that is a successor of $x$, let $\min(y')$ be the minimum that $i$ can achieve given common reasoning about sequential weak admissibility when he follows $s[i]$ in $T_y$. Then we have (*) that $\max(y) \geq \min(y')$ for each successor $y'$ of $x$. For otherwise player $i$ can ensure himself a higher payoff than $s[i]$ can possibly yield, by moving to some successor $y'$ of $x$ and continuing with $s[i]$. That is, the strategy $s'[i]$ which moves to $y'$ at $x$ and follows $s[i]$ below $y'$ strictly dominates $s[i]$ in $T_y CRW_A(T_2)$. But since $T$ and hence $T_2$ is finite, this contradicts the assumption that $s[i]$ is consistent with $CRW_A(T_2)$. Now by inductive hypothesis, $CRW_A(T_y') = SPE(T_y')$ for each successor $y'$ of $x$. So there is a subgame perfect equilibrium $s_{\text{max}}$ in $T_y$ which yields $i$ the payoff $\max(y) = \max(y')$ in $T_y$ and in which player $i$ follows $s[i]$ (i.e., $s[i] = s_{\text{max}}[i]$). Again by inductive hypothesis, for each successor node $y'$ of $x$ there is a subgame perfect equilibrium $s'_{\text{min}}$ in $T_y'$ which gives player $i$ the payoff $\min(y')$ and in which player $i$ follows $s[i]$ in $T_y'$. Now we define a subgame perfect equilibrium $s^*$ in $T_2$ in which player $i$ follows $s[i]$: 

1. $s^*[i](\{x\}) = s[i](\{x\})$, 
2. in $T_y$, $s^*$ follows $s_{\text{max}}$, 
3. in $T_y'$, $s^*$ follows $s'_{\text{min}}$, where $y'$ is a successor of $x$ other than $y$. By our observation (*), there is no profitable deviation from $s^*$ for player $i$ at $x$, and hence by lemma 0, $s^*$ is a subgame perfect equilibrium in $T_2$.

$(\Leftarrow)$ Let $s$ be consistent with subgame perfection in $T_2$. Let $i$ be the player moving at $x$. Consider any strategy $s[j]$ in $s$, where $j \neq i$. Since $j$ is not moving at $x$, $s[j]$ is consistent with common reasoning about sequential weak admissibility in $T_2$ if and only if $s[j]|T_y$ is consistent with common reasoning about sequential weak admissibility in each subgame $T_y$ of $T_2$. Since $s$ is consistent with subgame perfection in $T_2$, there is a subgame perfect equilibrium $s^*$ in $T_2$ in which $j$ follows $s[j]$. Since $s^*$ is subgame perfect, $s^*[T_y]$ is subgame perfect in $T_y$. Hence $s[j]|T_y = s^*[j]|T_y$ is consistent with subgame perfection in $T_y$. By inductive hypothesis, this entails that $s[j]|T_y$ is consistent with common reasoning about sequential weak admissibility in $T_y$. Since this is true for any subgame $T_y$ of $T_2$, the strategy followed by the player who is moving at $x$. We just established that for each iteration $W A^j(T)$ is consistent with common reasoning about weak sequential admissibility, $s^*[-i]$ is consistent with $W A^j(T)$. Since $s^*$ is a subgame perfect equilibrium in $T_2$, $s^*[i]$ is a best reply against $s^*[-i]$ in $T_2$ and each subgame of $T_2$. So in each subgame $T_y$ of $T_2$ (including $T_2$) and at each iteration $W A^j(T)$, $s^*[i]$ is a best reply against some strategy profile of $i$'s opponent consistent with $W A^j(T)$, namely $s^*[-i]|T_y$, and hence $s^*[i]$ is sequentially weakly admissible given $W A^j(T)$. Since $CRW_A(T) = W A^k(T)$ for some $k$, because $T$ is finite, $s^*[i]$ is consistent with common reasoning about sequential weak admissibility. This shows that all strategies in the strategy profile $s$ are consistent with common reasoning about sequential weak admissibility in $T_2$, and completes the proof by induction.

LEMMA 2 If $T$ is a restriction of $T'$ and $s_i$ is sequentially admissible in $T$, then there is an extension $s'_i$ of $s_i$ to $T'$ such that $s'_i$ is sequentially weakly admissible in $T'$.

Proof. We construct $s'_i$ as follows. At each information set $I_i$ in $T'$ such that $I_i$ contains a node in $T$, $s'_i = s_i$. At all other information sets $I_i$, $s'_i$ follows a strategy that is weakly admissible at $I_i$. We claim that $s'_i$ is sequentially weakly admissible in $T'$; let $I_i$ be any information set in $T'$ belonging to $i$.

Case 1: $I_i$ contains a node $x$ in $T$. Since $T$ is a restriction of $T'$, $I_i$ contains all nodes in $I_T(x)$, where $I_T(x)$ is the information set in $T$ containing $x$. So if $s_i$ is strictly dominated in $T'$ at $I_i$, then $s_i$ is strictly dominated in $T$ at $I_T(x)$, contrary to the supposition that $s_i$ is admissible at $I_T(x)$.

Case 2: $I_i$ contains no node $x$ in $T$. By construction, $s_i$ is weakly admissible at $I_i$.

PROPOSITION 3 Let $T$ be a finite game tree. If a play sequence is consistent with common reasoning about sequential admissibility in $T$, then the play sequence is consistent with common reasoning about sequential weak admissibility. That is, $\{\text{play}(s) : s \in CRSeq(T)\} \subseteq \{\text{play}(s) : s \in CRW_A(T)\}$.

Proof. We prove by induction on $j \geq 0$ that for each $j$, $W A^j(T)$ is a restriction of $T|W A^j(T)$.

Base Case, $j = 0$. Then $Seq^0(T) = W A^0(T)$, so the claim is immediate.

Inductive Step: Assume that $T|Seq^j(T)$ is a restriction of $T|W A^j(T)$, and consider $j + 1$. Choose any strategy profile $s$ in $Seq^{j+1}(T)$. By lemma 2, extend each $s[i]$ in $s$ to a strategy $s'[i]$ that agrees with $s[i]$ on information sets that have members both in $T|Seq^j(T)$ and $T|W A^j(T)$, and is sequentially weakly admissible in $T|W A^j(T)$. Call the resulting strategy profile $s'$; $s'$ is in $W A^{j+1}(T)$. Clearly $s$ and $s'$ result in the same play.
sequence, i.e. \( play(s') = play(s) \), because the same actions are taken at each information set. So all nodes that are consistent with \( Seq^{j+1}(T) \) are consistent with \( W A^{j+1}(T) \), which means that \( T \mid Seq^{j+1}(T) \) is a restriction of \( T \mid W A^{j+1}(T) \). This completes the proof by induction.

**Lemma 4** Let \( T \) be a finite game tree. Then the play sequences consistent with sequential admissibility are exactly those consistent with sequential proper admissibility. That is, \( \{play(s): s \text{ is sequentially admissible in } T\} = \{play(s): s \text{ is sequentially properly admissible in } T\} \).

**Proof.** (\( \square \)) Let \( s \) be a sequentially properly admissible strategy profile in \( T \), and let \( x \) be any node reached in \( play(s) \) such that \( I(x) \) belongs to player \( i \). Then \( s[i] \) is admissible at \( I(x) \) since \( I(x) \) is consistent with \( s[i] \). Now we may modify \( s \) to obtain a strategy profile \( s^* \), in which each player \( i \) follows \( s[i] \) at any information set containing a node in \( play(s) \), and follows an admissible strategy at every other information set. Then \( s^* \) is sequentially admissible, and \( play(s^*) = play(s) \).

(\( \square \)) This is immediate because all sequentially admissible strategies are sequentially properly admissible.

**Proposition 5** Let \( T \) be a finite game tree. Then the play sequences consistent with common reasoning about sequential admissibility are exactly those consistent with common reasoning about sequential proper admissibility. That is, \( \{play(s): s \in CRS_{seq}(T)\} = \{play(s): s \in CRS_{seq}(T) \mid PSeq(T) = T \mid Seq(T) \} \).

**Proof.** We prove by induction on \( j \) that for each \( j \geq 0, T \mid Seq(T) = T \mid PSeq(T) \).

**Base Case, \( j = 0 \).** The claim is immediate since \( Seq^0(T) = PSeq^0(T) = S(T) \).

**Inductive Case:** Assume that \( T \mid Seq^j(T) = T \mid PSeq^j(T) \), and consider \( j + 1 \). The claim follows immediately from Lemma 4.

**Proposition 6** Let \( T \) be a finite game tree with perfect recall. Then a strategy \( s_i \) for player \( i \) is admissible in \( S(T) \) if and only if \( s_i \) is sequentially properly admissible in \( T \).

**Proof.** Suppose that a strategy \( s_i \) in \( S(T) \) for player \( i \) is weakly dominated in \( S(T) \). Then there is a strategy \( s_i' \) consistent with \( S(T) \) such that

1. for all strategy profiles \( s_{-i} \) consistent with \( S(T) \), \( u_i(s_i, s_{-i}) \leq u_i(s_i', s_{-i}) \), and

2. for some strategy profile \( s_{-i}^* \) consistent with \( S(T) \), \( u_i(s_i, s_{-i}^*) < u_i(s_i', s_{-i}^*) \).

Let \( x \) be the first node that appears along both the plays of \( s_i \) against \( s_{-i} \) and \( s_i' \) against \( s_{-i} \) at which \( s_i \) deviates from \( s_i' \), so that \( x \in range(\{play(s_i, s_{-i})\} \cap range(\{play(s_i', s_{-i}')\}) \neq I(x) \). Then \( x \) is consistent with \( s_i \) and \( s_i' \) in \( T \). Let \( y \) be any node at \( I(x) \) consistent with \( s_i \) and \( s_i' \), and let \( s_{-i} \) be any strategy profile of \( i \)'s opponents. Then \( u_i(s_i, s_{-i}, y) \leq u_i(s_i', s_{-i}, y) \); for otherwise, by perfect recall, let \( s_{-i} \) be a strategy profile of \( i \)'s opponents such that both \( play(s_i, s_{-i}) \) and \( play(s_i', s_i) \) reach \( y \), and such that \( s_{-i}[T_y] = s_{-i}[T_y] \). Then \( u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i}) \), contrary to the hypothesis that \( s_i \) weakly dominates \( s_i' \). Since we also have that \( u_i(s_i, s_{-i}^*, x) < u_i(s_i', s_{-i}^*, x) \), it follows that \( s_i' \) weakly dominates \( s_i \) at \( I(x) \) so that \( s_i \) is not sequentially admissible.

Suppose that a strategy \( s_i \) is properly weakly dominated at an information set \( I_i \) in \( T \) by strategy \( s_i' \). Then there must be a node \( x \) in \( I_i \) consistent with \( s_i \) and a strategy profile \( s_{-i}^* \) in \( T \) such that \( s_i' \) yields a higher payoff at \( x \) against \( s_{-i} \) than \( s_i \) does, i.e. \( u_i(s_i, s_{-i}^*, x) > u_i(s_i', s_{-i}^*, x) \). Assume without loss of generality that \( x \) is reached by the play sequence of \( i \) against \( s_{-i} \), i.e. \( x \in range(\{play(s_i, s_{-i}^*)\}) \). Now we define a strategy \( s_i^{**} \) that weakly dominates \( s_i \) in \( T \) as follows.

1. At an information set \( I_i \) that does not contain \( x \) or any successor of \( x \), \( s_i^{*}(I_i') = s_i(I_i') \).

2. At an information set \( I_i \) that contains \( x \) or a successor of \( x \), \( s_i^{*}(I_i') = s_i(I_i') \).

We show that \( s_i^{*} \) weakly dominates \( s_i \) in \( S(T) \). Since \( play(s_i, s_{-i}) \) reaches \( x \), \( play(s_i, s_{-i}) \) also reaches \( x \), and so \( u_i(s_i, s_{-i}) = u_i(s_i', s_{-i}^*, x) = u_i(s_i, s_{-i}, x) \geq u_i(s_i', s_{-i}, x) \). Thus \( s_i^{*} \) weakly dominates \( s_i \) in \( S(T) \) if for no \( s_{-i} \) in \( T \), \( u_i(s_i^{*}, s_{-i}) > u_i(s_i^{*}, s_{-i}) \), which we establish now. Let a strategy profile \( s_{-i} \) in \( T \) be given.

**Case 1:** the play sequence of \( (s_i^{*}, s_{-i}) \) does not reach \( I_i(x) \). Then \( play(s_i^{*}, s_{-i}) = play(s_i, s_{-i}) \), and the claim follows immediately.

**Case 2:** the play sequence of \( (s_i^{*}, s_{-i}) \) goes through some node \( y \) in \( I_i(x) \). Since \( x \) is consistent with \( s_i \) and \( T \) is a game with perfect recall, \( y \) is consistent with \( s_i \), and so \( play(s_i, s_{-i}) \) reaches \( y \). As before, we have that (a) \( u_i(s_i, s_{-i}, y) = u_i(s_i, s_{-i}, y) \). Also, \( s_i^{*} \) coincides with \( s_i \) after node \( y \), and so (b) \( u_i(s_i^{*}, s_{-i}) = u_i(s_i^{*}, s_{-i}) \). Since \( s_i^{*} \) weakly dominates \( s_i \) at \( I_i(x) \), and \( y \) is in \( I_i(x) \), it follows that (c) \( u_i(s_i, s_{-i}, y) \geq u_i(s_i, s_{-i}, y) \). Combining (a), (b), and (c) it follows that \( u_i(s_i^{*}, s_{-i}) \geq u_i(s_i, s_{-i}) \). This establishes that \( s_i \) is weakly dominated given \( S(T) \).
THEOREM 7 Let $T$ be a finite game tree with perfect recall. A strategy profile $s$ is consistent with common reasoning about sequential proper admissibility if and only if $s$ is consistent with common reasoning about admissibility in the strategic form of $T$. That is, $\text{CR}_{\text{Seq}}(T) = \text{CR}_{\text{Ad}}(S(T))$.

Proof. We prove by induction on $j$ for which $0 \leq j < \infty$. For $j = 0$, $\text{PSeq}^0(T) = S(T) = \text{Ad}^0(S(T))$.

Base Case, $j = 0$. Then by definition, $\text{PSeq}^j(T) = S(T) = \text{Ad}^0(S(T))$.

Inductive Step: Assume that $\text{PSeq}^j(T) = \text{Ad}^j(S(T))$ and consider $j + 1$. By inductive hypothesis, $T | \text{PSeq}^j(T) = T | \text{Ad}^j(S(T))$. Now a strategy $s_i$ is in $\text{PSeq}^{j+1}(T) \iff s_i$ is in $\text{PSeq}^j(T)$ and $s_i$ is sequentially properly admissible in $T | \text{PSeq}^j(T)$. By inductive hypothesis, the first condition implies that $s_i$ is in $\text{Ad}^j(S(T))$. By Proposition 6 and the facts that $T | \text{PSeq}^j(T) = T | \text{Ad}^j(S(T))$ and that all restrictions of $T$ are games with perfect recall, the second condition implies that $s_i$ is admissible in $S(T | \text{Ad}^j(S(T))) = \text{Ad}^j(S(T))$. So $s_i$ is in $\text{Ad}^{j+1}(S(T))$.

Conversely, a strategy $s_i$ is in $\text{Ad}^{j+1}(S(T)) \iff s_i$ is in $\text{Ad}^j(S(T))$ and $s_i$ is admissible in $\text{Ad}^j(S(T))$. By inductive hypothesis, the first condition implies that $s_i$ is in $\text{PSeq}^j(T)$, and the second condition may be restated to say that $s_i$ is admissible in $S(T | \text{Ad}^j(S(T)))$. By Proposition 6, the second condition then implies that $s_i$ is sequentially properly admissible in $T | \text{Ad}^j(S(T)) = T | \text{PSeq}^j(T)$. Hence $s_i$ is in $\text{PSeq}^{j+1}(T)$. This shows that $\text{PSeq}^{j+1}(T) = \text{Ad}^{j+1}(S(T))$, and completes the proof by induction. □

PROPOSITION 8 For all finite games $G$ with pure strategy profiles $S$, $\text{CR}_{\text{Ad}}(S) \neq \emptyset$.

Proof. The admissible elements in $S^i$ survive at each iteration, $i$, for each player $i$, and there always is an admissible element in each $S^i_j$ since each $S^i_j$ is finite. Hence $S^i \neq \emptyset$ for any $i$, and so $S^{\text{Seq}}_{\text{Ad}}(S) = \text{CR}_{\text{Ad}}(S) \neq \emptyset$. □

NOTES

1 Here and elsewhere, the payoff at a terminal node is given as a pair $(x, y)$, where $x$ is the payoff for player $1$ and $y$ is the payoff for player $2$.

2 This principle corresponds to subgame perfection.

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