The Dotted Circle Method of Drawing 4-Manifold 1-Handles

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1 Introduction and Notation

Kirby calculus techniques are important in the study of 4-manifolds in that they give 3 dimensional mathematicians a way to manipulate their constructions of 4-dimensional manifolds (on a 2-dimensional piece of paper). We assume the reader is familiar with the basic Kirby moves and pictures for k-handles on a 4-manifold. The primary source for this exposition was the text by Gompf and Stipsicz (hereafter referred to as GS)[1].

Some notation:

- D^k: k-dimensional disk
- S^k: k-dimensional sphere
- ν: tubular neighborhood
- ∂X: boundary of X

A 1-handle is $D^1 \times D^3$ attached to a 0-handle D^4 by embedding its attaching sphere $S^0 \times D^3$ in $\partial D^4 = S^3$. The usual notation for 1-handles is $D^3 \coprod D^3$, which is consistent with what we imagine we'd see if we were sitting in the boundary of the 0-handle. We can attach 2-handles to a 1-handle by drawing strands of the attaching knots onto the surfaces of the two balls. These strands are connected by reflecting through the plane perpendicularly bisecting the segment connecting the centers of the balls (Figure (1)).

In this article we discuss the limitation of this notation to consistently depict 2-handle framings, and introduce Akbulut's *dotted circle* notation which solves this problem.

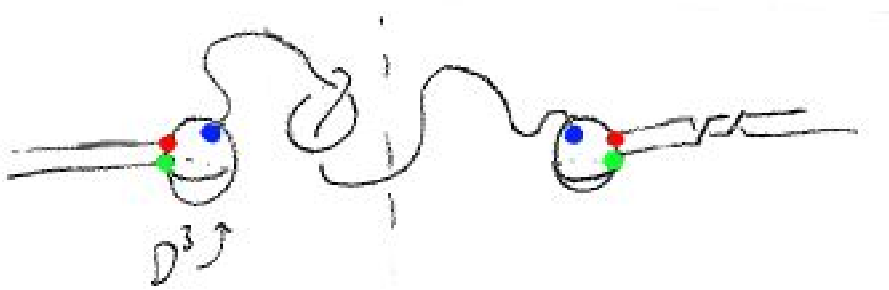


Figure 1: Attaching a 2-handle to a 1-handle.

2 Framing

We review the framing of a k-handle and then discuss methods of visualizing the framing coefficient for 2-handles on 4-manifolds which are attached to $\partial D^4 = S^3$ disjoint of any 1-handles.

In order to specify how to attach a k-handle h to an n-manifold X, we need to know not only the embedding $\phi_0 : S^{k-1} \to \partial X$ which places the attaching sphere of h along a knot in X, but also the *framing*:

Definition 2.1. The *framing* f associated to the attaching sphere of a k-handle on an n-manifold is an identification of the normal bundle $\nu\phi_0(S^{k-1})$ with $S^{k-1} \times \mathbb{R}^{n-k}$. Informally, the framing shows us how to thicken the core $D^k \times 0$ to $D^k \times D^{n-k}$ (fig. (2)).

The set of framings of a k-handle in a n-manifold can be mapped bijectively to the elements of $\pi_{k-1}(O(n-k))$ - this allows us to show that there are unique orientable 1-handle framings for $n \geq 3$, and that there are unique framings for $(n-1)$- (for $n \neq 2$) and n-handles (in general) [see GS 4.1.4]. In our case of interest, $n = 4$, this implies that we need only be concerned with framings of 2-handles.

In a Kirby diagram, we can illustrate a framing f on an attaching knot K in a 3-manifold M^3 by drawing a transverse vector field on K, or (more commonly), drawing a knot K' parallel to K by connecting arrowheads of the vector field.[1] This is called the *double-strand* notation. Sometimes the *blackboard framing* is used, which constrains the transverse vector field on

[1] If $\phi : S^1 \times D^2 \to M^3$ is the attaching map, then $K = \phi|S^1 \times \{0\}$ and $K' = S^1 \times \{p\}$ for some $p \neq 0$ contained in D^2.

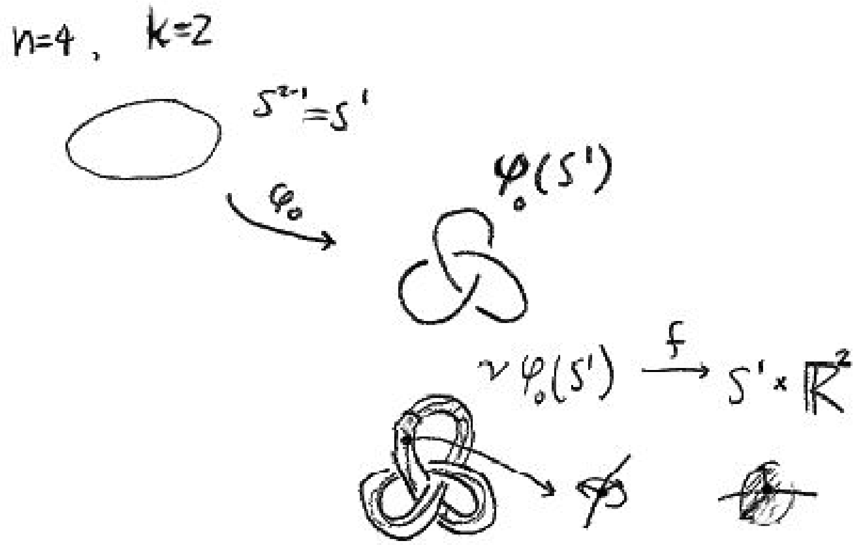


Figure 2: Attaching sphere and framing concepts.

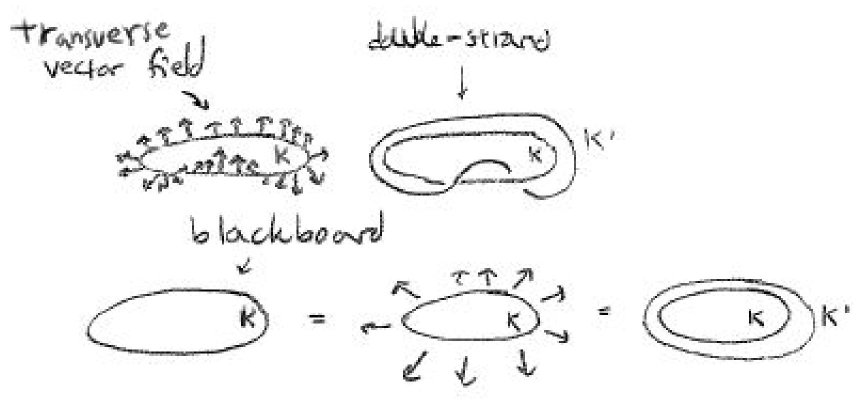


Figure 3: Methods of denoting framing.

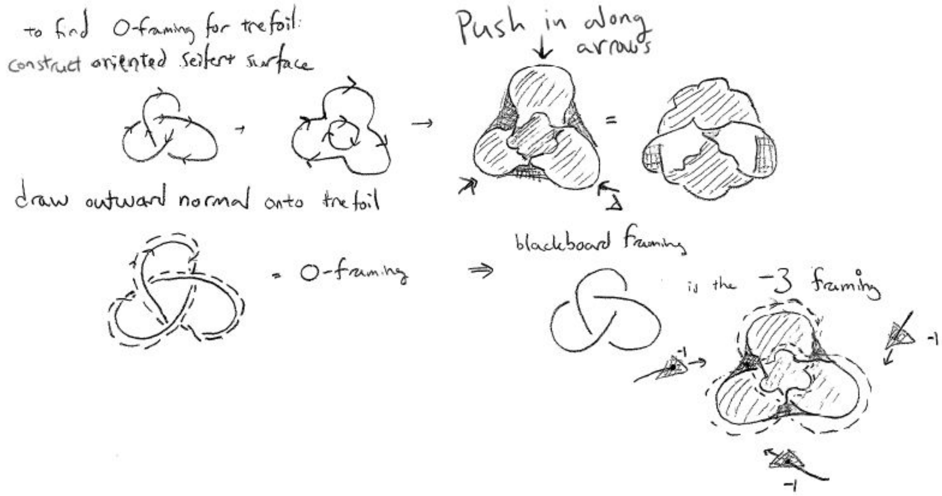


Figure 4: The orientable Seifert surface of a trefoil gives us the canonical 0-framing. Based on GS Exercise 4.5.9.

K to lie in $\mathbb{R}^2$ (the blackboard). For pictures of these, see fig. (3). In any case, once we've specified a framing, we have all the information we need to construct a unique (up to isotopy) oriented basis for each normal fiber if we use the orientation on K induced by S^1 and M.

Since $\pi_1(O(2)) = \mathbb{Z}$, the framings of 2-handles correspond bijectively to the integers. We can use this fact to define a *framing coefficient*, which is an integer specifying the framing of a knot. Here we describe two equivalent ways of finding the canonical framing coefficient of a knot K attached in S^3 (not on any 1-handles); once we have specified the coefficient m of one framing, we can get to the $m \pm n$ framing by adding or subtracting twists in K':

1. If we draw an orientable Seifert surface F bounded by K, drawing K' along an outward normal to F gives us the canonical 0-framing. We can thus find the framing coefficient of a knot drawn with double-strand notation by counting the signed number of times that K' intersects F.

2. A simpler way of finding the framing coefficient of a knot drawn with the blackboard framing is to find the *writhe* $w(K)$, equal to the signed number of self-crossings of the knot. To generalize this to finding the

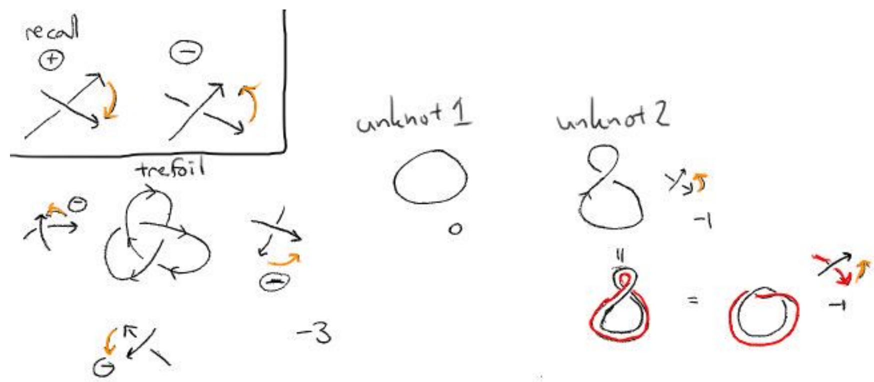


Figure 5: The writhe gives us the canonical framing coefficient of the blackboard framing. Based on GS Fig. 4.23.

> framing coefficient of a knot drawn with double-strand notation, we can compute the *linking number* $lk(K, K')$ of K and K', by counting the signed number of undercrossings of K' under K.

Figure (4) and Figure (5) show how to apply methods 1 and 2 respectively to some simple examples (based on Exercise 4.5.9 and Figure 4.23 in GS, respectively).

The orientability of the Seifert surface for method 1 is essential - Figure (6), also based on Exercise 4.5.9 in GS shows a non-orientable Seifert surface on the trefoil whose outward normal defines the 6-framing rather than the 0-framing.

3 Troublesome Examples

We now work through a few problematic examples of determining framing coefficients of 2-handles running over 1-handles. First, we consider D^2 bundles over surfaces X.

We discussed in tutorial the diagrams for D^2-bundles over orientable surfaces σ_g of genus g. We illustrate one of genus 1 in figure (7). For these bundles, the 0-framing of the 2-handle can still be determined by the 0-section Seifert surface because it is orientable, in this case, a punctured torus.

When we consider framings for D^2 bundles over $\mathbb{RP}^2$, we find that the 0-section Seifert surface is now a non-orientable Möbius band. We saw in

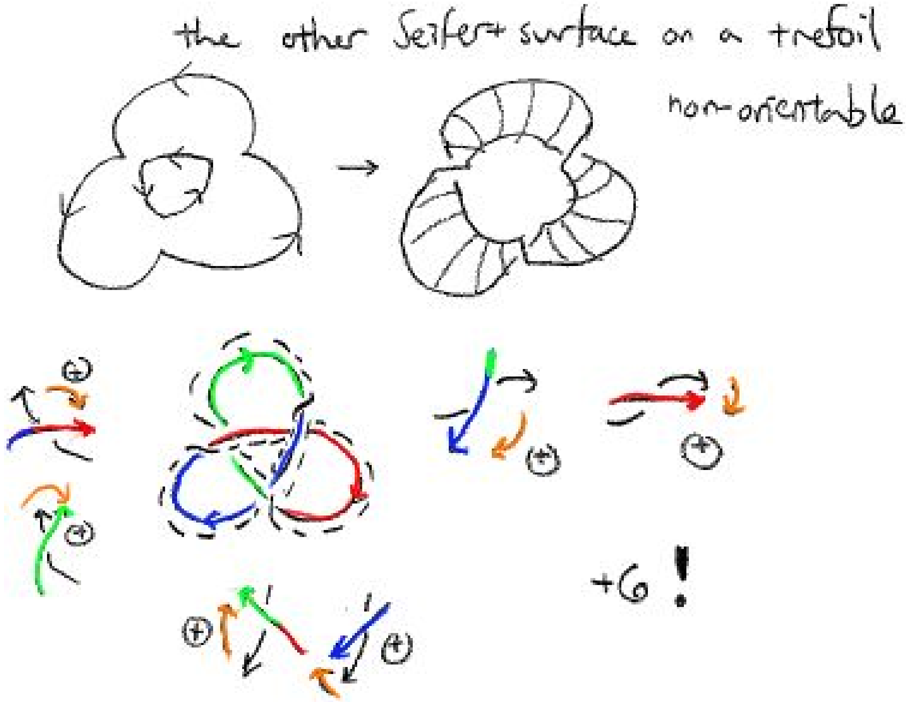


Figure 6: The non-orientable Seifert surface does not define the 0-framing. Based on GS Exercise 4.5.9.

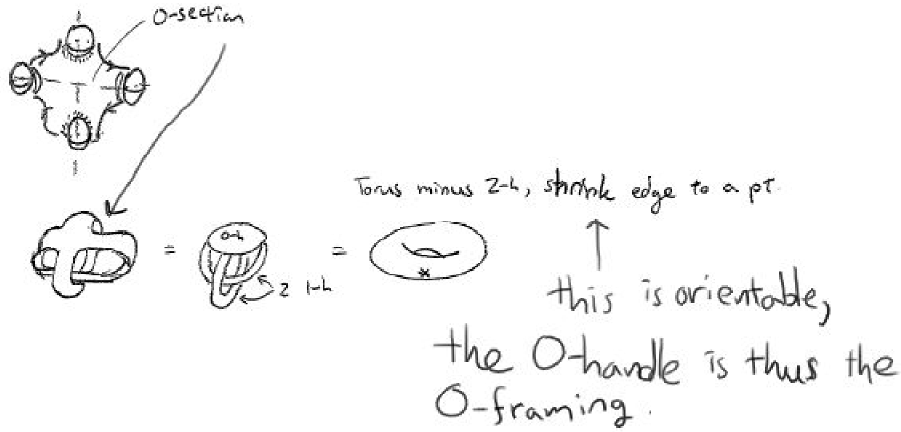


Figure 7: The 0-section Seifert surface of a D^2-bundle over T^2 gives us the 0-framing. Based on GS fig. 4.36.

figure (6) that the outward normal to a non-orientable Seifert surface does not in general determine the canonical 0-framing, so we must try something else.

We can try another method, which is to generalize the writhe method for determining framings. This technique works, and does give us the proper framing coefficient. The writhe of the first Kirby diagram in fig. (8) is +1, and the writhe of the second diagram is -1. Therefore, the coefficients of the blackboard framings are ± 1. Unfortunately, neither of our diagrams are the blackboard framing, so to calculate the framing coefficients, we must add on the number of undercrossings, +1 in the first diagram, and -1 in the second. This gives us framing coefficients of ± 2. However, these two diagrams are isotopic! The isotopy consists of looping one of the strands around the 1-handles.

Another example is Gompf and Stipsicz Exercise 4.4.4 (solution in figure 12.2). In figure (9), we exhibit an isotopy of a 2-handle running over a 1-handle which changes the framing coefficient by ± 2 without changing the embedding! [N.B. It is helpful for visualization to simulate this isotopy with a belt or a ribbon, letting one edge be K, and another edge be K'.]

These two examples show that there's something we're missing regarding the framing of 2-handles running over 1-handles. In particular, we will see that our "isotopies" of the 2-handles in these two examples actually hide

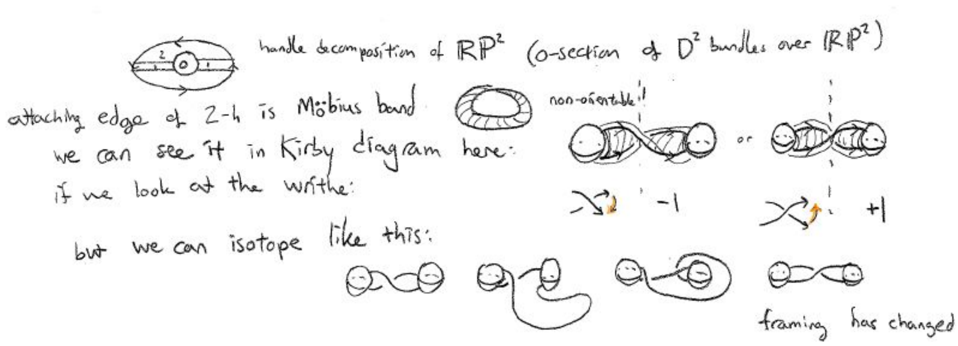


Figure 8: How to determine the 0-framing of a D^2-bundle over $\mathbb{RP}^2$, and the trouble with a certain "twist". Based on GS fig. 4.37, 4.38.

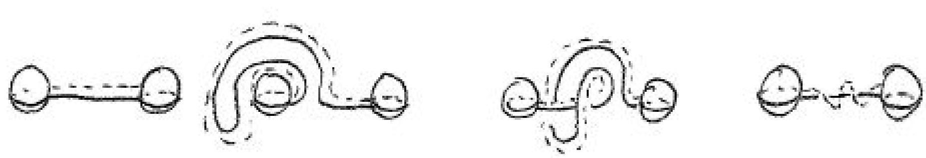


Figure 9: An isotopy which changes the framing? Based on GS Exercise 4.4.4 and figure 12.2.

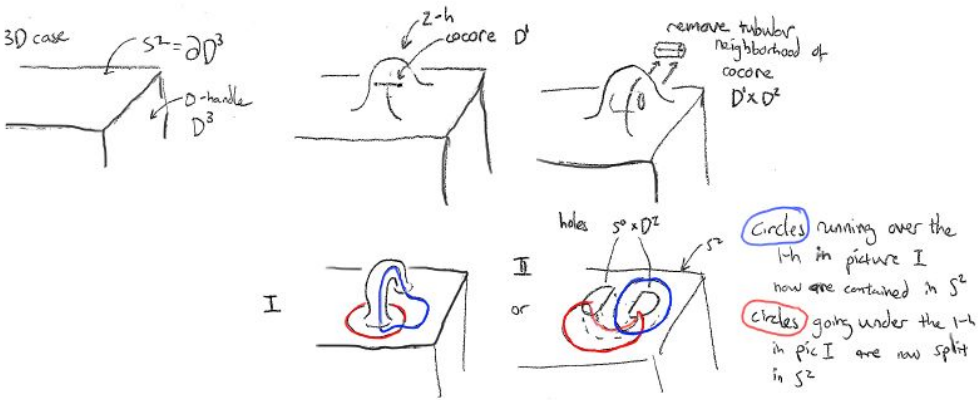


Figure 10: The 3-dimensional analogue of the notions for introducing the dotted-circle notation. Based on GS Fig. 5.34, with suggestions from Andrew Lobb.

handle slides.

4 Dotted circle diagrams

The *dotted circle* notation was introduced by Selman Akbulut in the late 1970s. This notation will help us resolve the difficulties we had with framings above, and also makes it easier to slide 1- and 2-handles. It was originally motivated by a surgery technique, however, we shall discuss here a more elementary way of seeing it.

Recall that we can cancel a 1-handle and 2-handle when the 2-handle runs once over the 1-handle. Thus, attaching a 1-handle to ∂D^4 is equivalent to removing some 2-handle in ∂D^4. To see exactly what this means, consider figure (10) (roughly GS Figure 5.34), which shows the operation in 3 dimensions. In the first image, we identify a 2-handle in the interior of the manifold. We have also labeled the cocore D^1. To remove the 2-handle, it suffices to remove a tubular neighborhood of the cocore,[2] in our figure, this is a cylinder $D^1 \times D^2$.

[2]If we consider the Morse function corresponding to this handle decomposition of our manifold, removing a tubular neighborhood of the cocore corresponds to removing a neighborhood of the critical point with index 2 corresponding to this 2-handle. What remains is homotopy equivalent to the manifold minus the 2-handle.

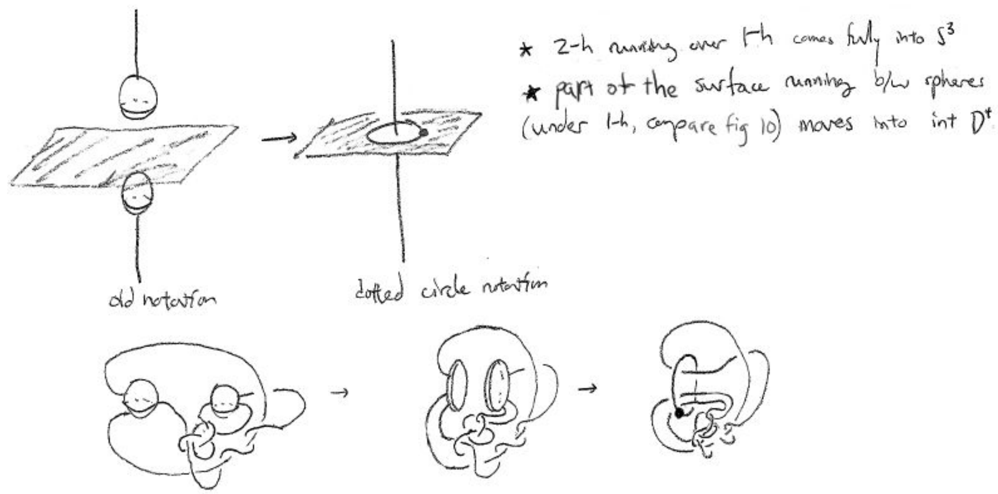


Figure 11: Circles and surfaces in the two notations and a way of visualizing conversion between the usual notation and dotted-circle notation. Based on GS fig. 5.36 and 5.35.

If we push the boundary of the cylinder $S^0 \times D^2$ to the boundary of the manifold S^2, the 1-handle appears as a bridge over a tunnel. In the plane of S^2, the 1-handle looks either like an identification of two disks (corresponding to the usual notation in Heegaard diagrams) or a "missing" $S^0 \times D^2$ (a dotted circle type notation). We see the former if we pull the bridge up and fill in the tunnel, and we see the latter if we bring the tunnel lower and flatten the bridge. We have drawn in circles running over/under the 1-handle in the final two components of figure (10) - we will compare this to 2-handles running over 1-handles and surfaces under 1-handles in 4 dimensions.

In 4 dimensions, the cocore of the 2-handle is an unknotted D^2 in the interior of the manifold. After removing a tubular neighborhood of this disk, we can push the boundary of this disk to S^3, and we draw it by an unlinked unknot decorated with a single dot. This dotted circle thus represents our 1-handle. Now attaching 2-handles to this 1-handle is equivalent to threading the knots through the unknot. Figure (11) shows how circles and surfaces correspond in the two different notations - compare the circles in figure (10). It is useful to think of the dotted circle as a 0-framed unknot in many instances. We will see later that the handle moves in this notation are quite similar in form.

Figure (11) also shows that we can convert between the two notations by pushing the spheres towards another (avoiding other obstacles) while flattening them into pancakes, and then disks. As they get closer, we cut out the interior of the disks, and we end up with the dotted circles. Obviously, if we have any 2-handles attached to the 1-handle, we must keep track of how they move as well, however, the framing does not change as we push the spheres into a dotted circle!

5 Comparison of Kirby moves in each notation

It turns out that handle slides involving dotted circle representations of 1-handles are almost the same as slides for 0-framed unknots. The reason for this is that performing surgery on a 0-framed unknot will give us a 1-handle with the properties we expect from before. So sliding a 0-framed unknot first and then performing the proper surgery is the same as first surgering it into a 1-handle and then sliding.

5.1 2-handle slides over 1-handles

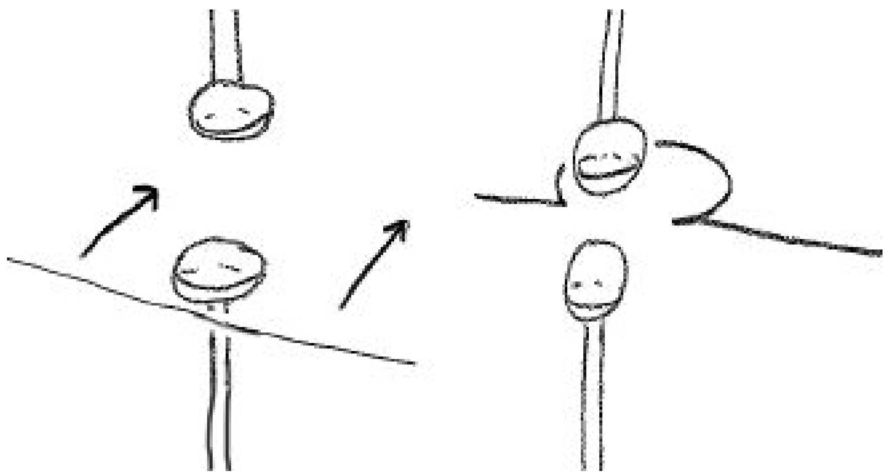
Figure (12) shows this operation in both notations. This move allows us to change the sign of a crossing between the strand that we're sliding and any strand running through the 1-handle. Recall that when we slide a 2-handle h_i with framing n_i embedded on knot K_i over a 2-handle h_j with framing n_j embedded on knot K_j, n_i changes according to the following formula:

$$n_i' = n_i + n_j \pm 2lk(K_i, K_j) \tag{1}$$

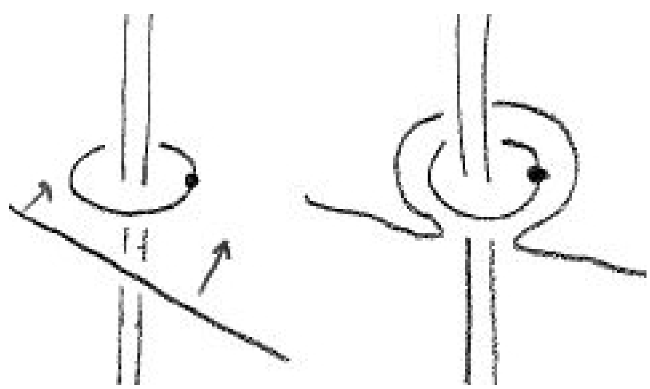
When we slide over a 0-framed unknot, $n_j = 0$, so n_i simply changes by twice the linking number of K_i and the unknot. This is exactly how the framing coefficient changes when sliding over a 1-handle.

5.2 1-handle slides over other 1-handles

Figure (13) shows this operation. As we might expect, when we use the dotted circle notation, 1-handle slides over other 1-handles look like 0-framed unknots sliding over other 0-framed unknots. The one caveat is that the

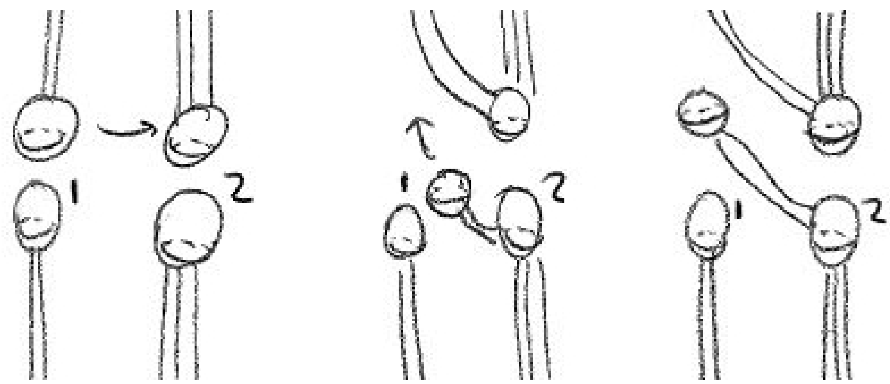


(a)

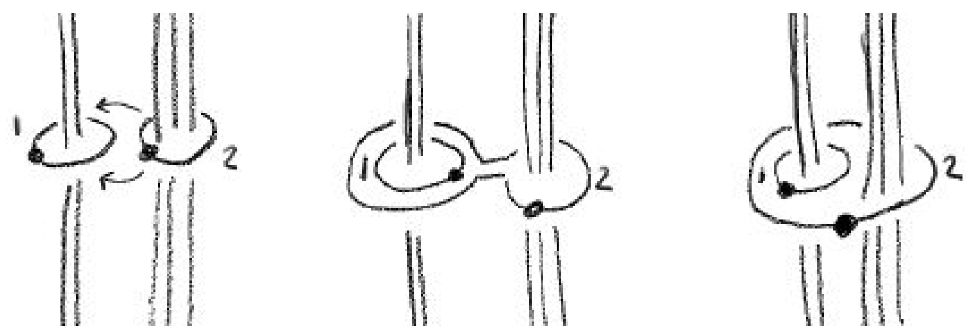


(b)

Figure 12: 2-handle slides over 1-handles. Based on GS Fig. 5.36a, b.

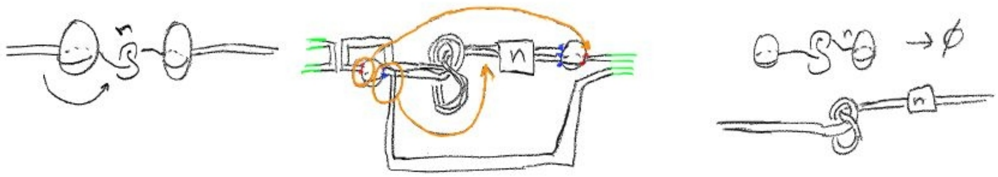


(a)

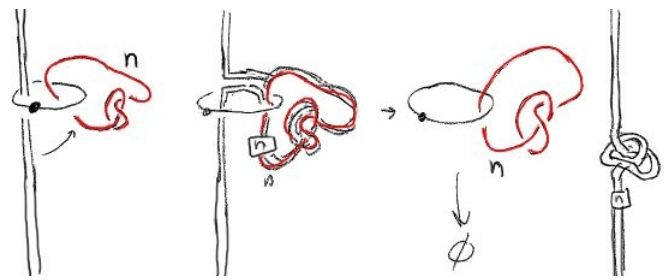


(b)

Figure 13: 1-handle slides over 1-handles. Based on GS Fig. 5.39.



(a)



(b)

Figure 14: Cancelling a 2-handle and 1-handle. Based on GS Fig. 5.38.

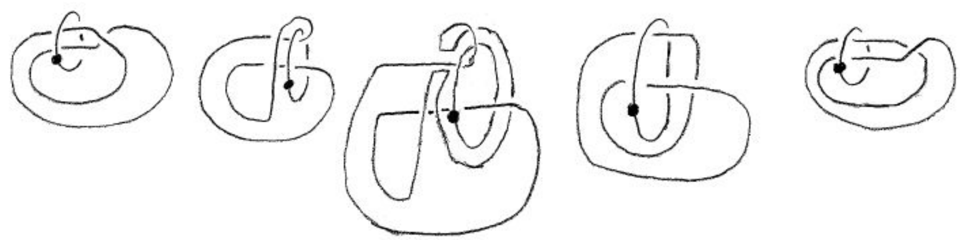
unknot which slides in the picture actually represents the 1-handle which is the target of the sliding, as you can verify from the figure.

5.3 1- and 2- handle creation/cancellation

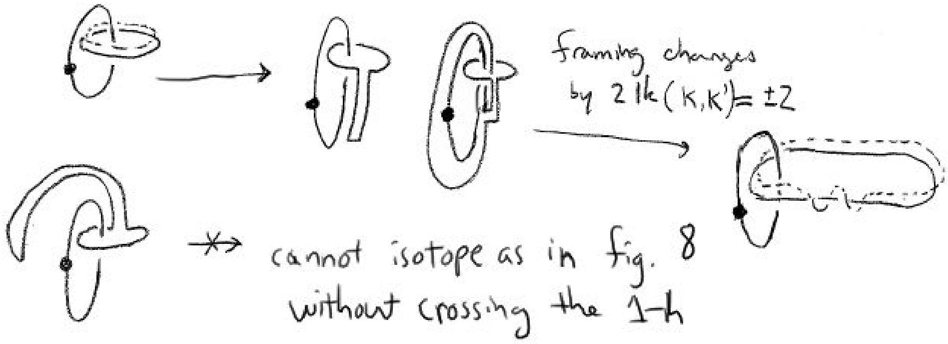
Figure (14) shows a case of handle cancellation where we must first slide non-cancelling 2-handles off the 1-handle. We do this by sliding them over the cancelling 2-handle. This operation brings us back to our first picture for this notation. Since the disk D which spans the unknot appears as a 0-framed meridian, we can identify D with the cocore of the canceled 2-handle - thus removing D is the same as removing the 2-handle.

6 Troublesome Examples Redux

If we change figures (8) and (9) to the new notation (fig. (15(a)), (15(b))), we see that indeed, the isotopies of the 2-handles before corresponded to moves



(a)



(b)

Figure 15: Revisiting the troublesome examples. (a) is based on GS fig. 6.4.

which moved the 2-handles through the region between the 1-handles - in the dotted circle notation, we see that these moves would cause the 2-handles to intersect the unknots. Therefore, to perform these moves, we must actually slide a 2-handle under a 1-handle.

We can now explain the changes in the framing coefficients. In figures (8/15(a)) - the linking number of the 2-handle and the 1-handle is 2, thus equation (1) tells us that the framing coefficient changes by $2 * 2 = 4$. Similarly, in figures (9/15(b)), the linking number is 1, thus the framing coefficient changes by 2.

The dotted circle notation is therefore helpful in that it allows us to more fully understand these nuances of handle moves.

7 Acknowledgements

Thanks to other participants in the morse theory tutorial 2006 and the leader Andrew Lobb.

References

[1] Robert E. Gompf and András I. Stipsicz, *4-Manifolds and Kirby Calculus.* AMS, 1999.