Mathematical Analysis of Households' Saving Problem in a Monetary Economy with Transactions costs

Households' constrained optimization problem

$$\underset{C_1,C_2,m_1,m_2}{\text{Max}} \; U = \ln(C_1) + \frac{\ln(C_2)}{1+\delta}$$

subject to:

$$C_1 + b m_1^{-\gamma} C_1^{1+\gamma} + \frac{C_2 + b m_2^{-\gamma} C_2^{1+\gamma}}{1+r} = W_1^{NF} - \frac{i_1}{1+i_1} m_1 - \frac{m_2}{1+r}$$

Lagrangean

$$\mathcal{L} = \ln(C_1) + \frac{\ln(C_2)}{1+\delta} + \lambda\left(W_1^{NF} - \frac{i_1}{1+i_1}m_1 - \frac{m_2}{1+r} - C_1 + bm_1^{-\gamma}C_1^{1+\gamma} + \frac{C_2 + bm_2^{-\gamma}C_2^{1+\gamma}}{1+r}\right)$$

- Derive first-order conditions (take first derivative of the Lagrangean w.r.t. each choice variable and set equal to zero).
- Also needs second-order conditions to prove it is a maximum, but you can assume they hold (which they do).

First-order conditions:

1) For C_1: $MUC_1 - \lambda MTC_1 = 0$

2) For C_2: $MUC_2 - \frac{\lambda MTC_2}{1+r} = 0$

3) For m_1: $\lambda\left(-\frac{i}{1+i} + MTm_1\right) = 0$

4) For m_2: $\lambda\left(-\frac{1}{1+r} + \frac{MTm_2}{1+r}\right) = 0$

5) For λ: $W_1^{NF} - \frac{i_1}{1+i_1}m_1 - \frac{m_2}{1+r} - C_1 + bm_1^{-\gamma}C_1^{1+\gamma} + \frac{C_2+bm_2^{-\gamma}C_2^{1+\gamma}}{1+r} = 0$

Optimality conditions:

1) For C_1: $MUC_1 = \lambda MTC_1$

2) For C_2: $MUC_2 = \frac{\lambda MTC_2}{1+r}$

3) For m_1: $\frac{i}{1+i} = MTm_1$

4) For m_2: $\frac{1}{1+r} = \frac{MTm_2}{1+r}$

Optimality conditions:

1) For C_1 and C_2:

$$\frac{MUC_1}{MTC_1} = \frac{MUC_2}{MTC_2}(1 + r)$$

2) For m_1: $\frac{i}{1+i} = MTm_1$

3) For m_2: $1 = MTm_2$

Explicit optimality conditions

- Replace MUCs and MTms with derivatives obtained earlier for log utility and IBC with exponential transactions costs

$$\frac{C_2}{C_1} = \frac{(1+r)\left[1+(1+\gamma)bV_1^{\gamma}\right]}{(1+\delta)\left[1+(1+\gamma)bV_2^{\gamma}\right]}$$

$$\frac{i}{1+i} = MTm_1 \quad \Rightarrow \quad m_1 = C_1\left(\frac{i}{1+i}\frac{1}{\gamma b}\right)^{-\frac{1}{1+\gamma}}$$

$$\Rightarrow \quad V_1 = \left(\frac{i}{1+i}\frac{1}{\gamma b}\right)^{\frac{1}{1+\gamma}}$$

$$1 = \gamma b\left(\frac{C_2}{m_2}\right)^{1+\gamma} \quad \Rightarrow \quad m_2 = C_2\left(\frac{1}{\gamma b}\right)^{-\frac{1}{1+\gamma}}, \qquad V_2 = \left(\frac{1}{\gamma b}\right)^{\frac{1}{1+\gamma}}$$