



Solution algorithm for Boz-Mendoza JME 2014

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Two-stage solution method

- At each date t of a sequence of T periods of “observed” realizations of the credit regime, equilibrium allocations and prices are solved in two stages
 1. Use Bayesian learning to find date- t posterior means of regime transition probabilities
 2. Given posterior means, eq. allocations and prices are obtained by solving Kreps’s date- t “anticipated utility optimization problem” (AUOP)
- Caveat: Step 2 *is not* equivalent to Bayesian optimization, which is ruled out by the curse of dimensionality

First stage: Bayesian learning

1. Agents learn as they observe financial regimes, applying Bayes rule to the Beta-Binomial distributions of posteriors of F_{ll} and F_{hh}
2. Regime transition counters

$$n_{t+1}^{ij} = \begin{cases} n_t^{ij} + 1 & \text{if } \kappa_{t+1} = \kappa^j \text{ and } \kappa_t = \kappa^i, \\ n_t^{ij} & \text{otherwise.} \end{cases}$$

3. Initial priors n_0^{ij} (the lower, the “newer” the regime)
4. Mean posteriors (“beliefs”):

$$E_t[F_{hh}^s] = \frac{n_t^{hh}}{n_t^{hh} + n_t^{hl}} \quad E_t[F_{ll}^s] = \frac{n_t^{ll}}{n_t^{ll} + n_t^{lh}}$$



Recall key properties of learning process

1. Convergence to true probabilities in the long run
2. Beliefs about each regime are updated only when it is observed
3. Initial priors drive speed at which optimism builds with financial innovation
 - With low (uninformative) priors, short initial spell of good credit regime leads to highly optimistic beliefs
4. Optimistic beliefs induce optimistic asset pricing, leading collateral constraint to bind in upswing of credit boom

Second stage: Date-t AUOP

- Given date-t posterior means, solve full recursive dynamic optimization problem (“as if they were the true transition probabilities”)
- This works because learning process preserves law of iterated expectations
- Simplify notation for state vector: $\eta \equiv (b, z, \kappa)$
- Solution of each AUOP, has three main steps:
 1. Agents take as given a pricing function $q_t(\eta)$ and solve for $b'_t(\eta)$ and associated consumption plans
 2. A new pricing function is generated using the pricing kernel of the new consumption plans
 3. Iterate to convergence in the pricing function imposed in step 1. and solved for in step 2.

Algorithm steps

1. Define a history of credit regime realizations of length T and initial priors, and compute associated sequence of posteriors using Bayes rule

$$\kappa^T \text{ where } \kappa^t = (\kappa_t, \kappa_{t-1}, \dots, \kappa_1) \\ \{f(F^s | \kappa^t)\}_{t=1}^T$$

2. Take the observed credit regime and posterior means for $t=1$ and construct beliefs about credit regime transition matrix:

$$E_1^s[\kappa' | \kappa] \equiv \begin{bmatrix} E_1[F_{hh}^s] & 1 - E_1[F_{hh}^s] \\ 1 - E_1[F_{ll}^s] & E_1[F_{ll}^s] \end{bmatrix}$$

3. Construct a guess (“conjecture”) of eq. pricing function for $t=1$

$$q_1(b_t, z_t, \kappa_t)$$

Algorithm steps cont'd.

4. Use time-elimination algorithm to find recursive functions that solve AUOP for $t=1$ at given beliefs and conjectured pricing function

$$u'(c_t(\eta)) = \beta R \left[\sum_{z' \in Z} \sum_{\kappa' \in \{\kappa^h, \kappa^l\}} E_t^s[\kappa' | \kappa] \pi(z' | z) u'(c_t(\eta')) \right] + \mu_t(\eta)$$

$$q_t(\eta)(u'(c_t(\eta)) - \mu_t(\eta)\kappa) =$$

$$\beta \left[\sum_{z' \in Z} \sum_{\kappa' \in \{\kappa^h, \kappa^l\}} E_t^s[\kappa' | \kappa] \pi(z' | z) u'(c_t(\eta')) (z' g'(1) + q_t(\eta')) \right]$$

$$c_t(\eta) = z g(1) - \frac{b'_t(\eta)}{R} + b$$

$$\frac{b'_t(\eta)}{R} \geq -\kappa q_t(\eta) 1$$

Algorithm steps cont'd.

5. Step 4 yields these recursive decision rules

$$[b'_1(b, z, \kappa), c_1(b, z, \kappa), \mu_1(b, z, \kappa)]$$

which are applied to the asset pricing condition

$$q_t = E_t^s \left[\sum_{j=0}^{\infty} \left(\prod_{i=0}^j \left(\frac{1}{E_t^s [R_{t+1+i}^q]} \right) \right) z_{t+1+j} g'(l_{t+1+j}) \right]$$

to compute a new (“actual”) eq. pricing function that clears the asset market with the decision rules solved from Step 4

$$\hat{q}_1(b_t, z_t, \kappa_t)$$

Algorithm steps cont'd.

6. Evaluate a convergence criterion for the conjectured and actual pricing functions.
 - If it fails, construct new conjectured pricing function with a Gauss-Siedel rule and return to Step 4
 - If it passes, then the solutions to the AUOP for $t=1$ are the associated functions:

$$\begin{aligned} & [b'_1(b, z, \kappa), c_1(b, z, \kappa), \mu_1(b, z, \kappa)] \\ & q_1(b_t, z_t, \kappa_t) \end{aligned}$$

7. Move to $t=2$. Take the second observation of the credit regime and update history of realizations and mean posteriors matrix.
8. Repeat Steps 3-6 to solve AUOP for $t=2$.
9. Return to Step 7 for $t=3, \dots, T$

Comments

- The final result is a set of T belief matrices and T solutions to AUOPs:

$$E_t^s[\kappa'|\kappa] \quad \text{for } t = 1, \dots, T$$

$$[b'_t(b, z, \kappa), c_t(b, z, \kappa), \mu_t(b, z, \kappa)]_{t=1}^T \quad [q_t(b, z, \kappa)]_{t=1}^T$$

- The above satisfy the definition of the recursive AU competitive equilibrium with learning:

Definition Given a history of realizations κ^T , a recursive AU competitive equilibrium for the economy is given by a sequence of policy functions $[b'_t(b, z, \kappa), c_t(b, z, \kappa), \mu_t(b, z, \kappa)]_{t=1}^T$ and pricing functions $[q_t(b, z, \kappa)]_{t=1}^T$ such that: (a) $b'_t(b, z, \kappa), c_t(b, z, \kappa), \mu_t(b, z, \kappa)$ and $q_t(b, z, \kappa)$ solve the date- t AUOP conditional on $E_t^s[\kappa'|\kappa]$; and (b) $E_t^s[\kappa'|\kappa]$ is the transition probability matrix of κ produced by the date- t posterior density of F^s

Comments cont'd

- Solutions of each AUOP are not functionally related (solving date-t AUOP does not require any element of the solution for other date)
 - The algorithm can be run simultaneously for all T periods
- But sequence of $E_t^s[\kappa' | \kappa]$ for $t=1, \dots, T$ does depend on history (posterior distributions depend on the particular history κ^t)
 - Can solve for different history sets $[\kappa_i^T]_{i=1}^I$, each with a different sequence of posteriors and belief matrices
 - Boz-Mendoza focused on one history for 1997-2007
- When t is large enough for beliefs to converge to “true” trans. matrix, all AUOPs after that date are equal to the standard recursive rat. exp. eq. for the true Markov process of κ

Summary of output objects

1. Sequence of posterior distributions and belief transition matrix for credit regimes for a learning history of T realizations
2. Decision rules and pricing function for each date t of the recursive AU equilibrium
3. Decision rules and pricing function under the true process of credit (i.e. rational expectations without learning)
 - Needed only to contrast with AU eq. with learning (nothing from the “true” rat. ex. eq. is needed to solve the AU equilibrium with learning)

What are the output objects used for?

1. Compute conjectured ergodic distributions for each date t of AU eq. and compare with true ergodic distribution
2. Compute forecast functions of macro aggregates and asset prices to trace dynamics
3. Compare solutions: AU with learning v. full rational expectations v. AU with learning but no Fisherian feedback
4. Compare asset pricing dynamics