Solution Methods for Optimal Time-Consistent Macroprudential Policy

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Solution Method for Decentralized Eq.

Solve for $\{\mathcal{B}(b,s), \mathcal{Q}(b,s), \mathcal{C}(b,s), \nu(b,s), \mathcal{H}(b,s), \mu(b,s)\}$ such that:

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$$zF_n(1, \mathcal{H}(b,s), \nu(b,s)) = G'(\mathcal{H}(b,s)) \tag{4}$$

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$$\begin{aligned} zF_n(1, \mathcal{H}(b,s), \nu(b,s)) &= G'(\mathcal{H}(b,s)) & (4) \\ zF_\nu(1, \mathcal{H}(b,s), \nu(b,s)) &= p_\nu(1 + \theta\mu(b,s)/u'(\mathcal{C}(b,s))) & (5) \end{aligned}$$

$$\begin{aligned} \mathcal{Q}(b,s)u'(c - G(h)) = \; & \beta \mathbb{E}_{s'|s}\Big\{u'(\mathcal{C}(b',s') - G'(\mathcal{H}(b,s)))(\mathcal{Q}(b',s') + \\ & z'F_k(1, \mathcal{H}(b',s'), \nu(b',s'))) + \kappa'\mu(b',s')\mathcal{Q}(b',s')\Big\} \end{aligned} \tag{6}$$

The algorithm follows these steps:

1. Set grids for bonds $G_b = \{b_1, b_2, ...b_M\}$ and shocks $G_s = \{s_1, s_2, ...s_N\}$. We set M=300 and interpolate the functions using piecewise linear interpolation.

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2. Conjecture $\mathcal{B}_k(b, s), \mathcal{Q}_k(b, s), \mathcal{C}_k(b, s), \mathcal{H}_k(b, s), \nu_k(b, s), \mu_k(b, s)$ at time $K, \forall b \in G_b$ and $\forall s \in G_s$ and set $j = 1$

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3. Solve for $\mathcal{B}_{k-j}(b,s), \mathcal{Q}_{k-j}(b,s), \mathcal{C}_{k-j}(b,s), (b,s), \mu_{k-j}(b,s)$ at time $k-j$ using (1)-(6) and $\mathcal{B}_{k-j+1}(b,s), \mathcal{Q}_{k-j+1}(b,s), \mathcal{C}_{k-j+1}(b,s)\mathcal{H}_{k-j+1}(b,s), \mu_{k-j+1}(b,s)$:

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    1. Assume (2) is not binding. Set $\mu_{k-j}(b,s) = 0$ and solve for $\mathcal{H}_{k-j}(b,s)$ using (4). Solve for $\mathcal{B}_{k-j}(b,s)$ and $\mathcal{C}_{k-j}(b,s)$ using (1) and(3) and a root finding algorithm.
    2. Check whether $-\frac{\mathcal{B}_{k-j}(b,s)}{R} + \theta p_\nu \nu_{k-j}(b,s) \leq \kappa \mathcal{Q}_{k-j+1}(b,s)$ holds.
    3. If constraint is satisfied, move to next grid point otherwise, solve for $\mu(b,s), \nu_{k-j}(b,s), \mathcal{H}_{k-j}(b,s), \mathcal{B}_{k-j}(b,s)$ using (2, (3) and(4) with equality.
    4. Solve for $\mathcal{Q}_{k-j}(b,s)$ using (6)

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4. If $sup_{B,s} \| x_{k-j}(b,s) - x_{k-j+1}(b,s) \| < \epsilon$ for $x = \mathcal{B}, \mathcal{C}, \mathcal{Q}, \mu, \mathcal{H}$ we have solved the equilibrium. Otherwise, set $x_{k-j}(b,s) = x_{k-j+1}(b,s)$ and $j \rightsquigarrow j+1$ and go to 3.

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2. Guess future planner's policy rules: $\mathcal{B}, \mathcal{Q}, \mathcal{C}, \mathcal{H}, \nu, \mu \ \forall \ b \in G_b$ and $\forall \ s \in G_s$. The initial guesses are the DE decision rules (same equilibrium is attained starting from alternative policies).

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3. Set the Bellman equation that characterizes the current planner's problem:

$$V(b', s') = \max_{c, b', \mu, h, \nu} u(c - G(h)) + \beta \mathbb{E}_{s'|s} V(b', s') \tag{8}$$

$$c + \frac{b'}{R} = b + zF(k, h, \nu) - p_\nu \nu \tag{9}$$

$$zF_h(k, h, \nu) = G'(h) \tag{10}$$

$$zF_\nu(k, h, \nu) = p_\nu \left(1 + \frac{\theta \mu}{u'(c - G(h))}\right) \tag{11}$$

$$ \tag{12} $$

$$\mu\left(\frac{b'}{R}-\theta p_\nu \nu+\kappa q\right) = 0 \tag{13}$$

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$$\begin{aligned} qu'(c-G(h)) &= \beta \mathbb{E}_{s'|s}\left\{u'(\mathcal{C}(b',s')-G'(\mathcal{H}(b,s)))(\mathcal{Q}(b',s')\right. \\ &\quad \left.+z'F_k(1,\mathcal{H}(b',s'),\nu(b',s'))+\kappa'\mu(b',s')\mathcal{Q}(b',s')\right\} \end{aligned} \tag{15}$$

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4. Solve Bellman eq. using VFI iteration. Assume first that collateral constraint is not binding. For each initial pair (b, s) search over grid of bonds for b' that yields highest value. If constraint does not bind, retain that choice. If it binds, solve for every b', the values of c, h, ν, q, μ that satisfy (9)-(15), with (14) holding with equality.

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5. Denote by $\sigma^i, i = c, q, h, \nu, \mu$ the policy functions that solve Bellman eq. Compute the sup distance between $\mathcal{B}, \mathcal{Q}, \mathcal{C}, \mathcal{H}, \nu, \mu$and $\sigma^i, i = c, q, h, \nu, \mu$. If this distance exceeds 1.0e-6, update $\mathcal{B}, \mathcal{Q}, \mathcal{C}, \nu, \mu$ and return to Step 3.