Workhorse Models of the Small Open Economy

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Objectives and plan of the lecture

- Introduce key issues for analyzing open economy models: stationarity & debt/wealth dynamics; complete v. incomplete markets; prec. savings

- Model 1: Deterministic, 1-sector endowment SOE

- Model 2: Stochastic variant of Model 1 but with incomplete markets

- Quantitative example using a variant of Model 2

- Tradeoffs in the use of global v. local methods, also based on Model 2
WORKHORSE MODEL 1: DETERMINISTIC SMALL OPEN ECONOMY MODEL
Key Assumptions

1. SOE with perfect access to world credit market
2. One-period bonds, fixed world real interest rate
3. Perfect foresight OR Complete Markets
4. Credible commitment to repay
5. Frictionless economy, no distortions
   - CA supports perfect consumption smoothing
   - Long-run NFA is simply annuity value of steady-state trade balance
Intertemporal optimization problem

- **Sequential social planner’s problem:**
  \[
  (I) \sum_{t=0}^{\infty} \beta^t u(c_t)
  \]

  \[
  (II) c_t = y_t - b_{t+1} + b_t R, \quad b_0 \text{ given}, \quad \{y_t\}_{t=0}^{\infty}
  \]

- **Recursive planner’s problem:**
  \[
  (III) V_t(b_t, y_t) = \max_{b_{t+1}} \{u(c_t) + \beta V_{t+1}(b_{t+1}, y_{t+1})\}
  \]

  subject to (II)
Equilibrium conditions

• First-order condition of the recursive problem:

\[ u'(c_t) = \beta V_{1t+1}(b_{t+1}, y_{t+1}) \]

– From envelope theorem (Benveniste-Sheikman eq.)

\[ V_{1t+1}(b_{t+1}, y_{t+1}) = Ru'(c_{t+1}) \]

– So we obtain standard Euler equation:

\[ u'(c_t) = \beta Ru'(c_{t+1}) \]

• Stationarity assumption: \( \beta R = 1 \) \( \Rightarrow \) \( c_t = \bar{c} \) \( \forall t \)

• Closed-form solution (using (II)):

\[ \frac{\bar{c}}{(1 - \beta)} = \left[ \sum_{t=0}^{\infty} \beta^t y_t \right] + b_0 R \quad \Rightarrow \quad \bar{c} = (1 - \beta)W \]
Current account, trade balance and NFA dynamics

• The equilibrium current account is:

\[ b_{t+1} - b_t = y_t - \bar{c} + b_tr \]

• Assume output converges:

\[ y_t \to \bar{y} \quad \text{as} \quad t \to \infty \]

• Stationary equilibrium of CA is zero, and steady states of NFA and NX are given by:

\[
\bar{b} = -\frac{[\bar{y} - \bar{c}]}{r} = -\frac{nx}{r} = -\frac{[\bar{y} - (1-\beta)W]}{r} = \beta W - \frac{\bar{y}}{r}
\]
Stationarity and initial conditions

- Stationary equilibrium is unique, but since wealth depends on initial NFA, $\bar{b}$ and $\bar{c}$ depend on $b_0$ (i.e. steady state depends on initial conditions)

- Borrow when $y_t < \bar{c}$ and save when $y_t > \bar{c}$
  - CA deficit with low $y_t$
  - CA surplus with high $y_t$
  - CA is procyclical!

- Is this a good model of actual CA dynamics?
General equilibrium extension

• Standard production function \((f(K))\) and investment with adjustment costs (Tobin’s Q) can be added easily.

• Consumption, NFA and CA dynamics are analogous to endowment case, but evaluated at eq. sequence of net income (output minus adj. costs) implied by arbitrage cond.

• Optimal decision rule for \(k\) is independent of \(b\) but decision rule for \(b\) depends on \(k\).
Recursive social planner’s problem

\[ V(K, A) = \max_{\{K', A', c\}} \{u(c) + \beta V(K', A')\} \]

s.t. \[ c = f(K) - (K' - K) \left[ 1 + \frac{\phi}{2} (K' - K) \right] \]

\[ - A' + A(1 + r^*) \]

- With a solution characterized by decision rules:

\[ \hat{K}'(K, A), \hat{A}'(K, A) \]
Four key properties

1. \( K_{ss} \) is unique and independent of initial conditions, but \( \bar{c} \), NFA dynamics, and \( \bar{b} \) still depend on \( b_0 \)

2. Fisherian separation: Investment and production dynamics determined by this arbitrage condition:

\[
\frac{d' + q'}{q} \equiv \frac{f'(K') + 1 + \phi(K'' - K')}{1 + \phi(K' - K)} = 1 + r^*
\]

3. Well-defined dynamics, unique steady-state
   – But steady-state Euler eq. does not yield a solution for \( \bar{b} \). Instead, we solve jointly with model’s dynamics

4. Linear approx. methods around det. steady states are not useful for solving these models
   – Even temporary shocks have permanent effects
   – But shooting methods do work
Time-series dynamics
(and a gains from trade argument)
Effects of Shocks

1. Additive (e.g. government expenditures)
   - Permanent: No effect on debt or capital dynamics, equal effects on income profile and consumption.
   - Transitory: No effect on investment dynamics but affects debt dynamics through the effect on permanent income and steady state of $b$.

2. Multiplicative (e.g. productivity, terms of trade)
   - Permanent or transitory: Affect both investment and debt dynamics and steady state of $b$, but only permanent shocks affect $K_{ss}$.

• CA can turn countercyclical (e.g. persistent TFP shocks induce borrowing for investment)
WORKHORSE MODEL 2: STOCHASTIC MODEL WITH INCOMPLETE MARKETS
Uncertainty and Incomplete Markets

• NFA are non-state-contingent, one-period “real” bonds chosen from a finite state space defined by a discrete grid:

\[ B = [b_1 < b_2 < \ldots < b_z] \]

• Income and world interest rate are exogenous.

• Income follows exogenous Markov process with “m” states and known transition prob. matrix:

\[ \bar{y} = [y_1 < y_2 < \ldots < y_m] \quad P(y_i, y_j) \]

• Asset markets are incomplete: \( B \) cannot provide full insurance against income fluctuations.
Sequential planner’s Problem

• Choose $\{b_{t+1}\}_{t=0}^{\infty}$ so as to

$$
\max \ E_0 \left[ \sum_{t=0}^{\infty} \beta^t u(c_t) \right]
$$

s.t.

$$
c_t = y_t - b_{t+1} + b_t R
$$

$$
b_{t+1} \in B \quad P(y_t, y_{t+1}) \text{ known}
$$

$$(b_0, y_0) \text{ given},
$$

…looks very similar to Model 1, but it has very different implications!
Aiyagari’s natural debt limit

- \( u(.) \) is twice differentiable, concave and satisfies the Inada condition:
  \[
  \lim_{c \downarrow 0} u'(c) = \infty
  \]

- Implies that consumption must be positive at all times, and hence the budget constraint implies:
  \[
  b' \geq - \left[ \frac{y_{\text{min}}}{R - 1} \right]
  \]
  - Debt cannot exceed annuity value of lowest income realization, otherwise the agent is exposed to the risk of zero consumption with positive probability
  - Already highlights “global” nature of decision-making in incomplete markets models
Recursive planner’s problem

\[
V(b_n, y_i) = \max_{b' \in B} \left\{ u(y_i - b' + b_n R) + \beta \sum_{j=1} P(y_i, y_j) V(b', y_j) \right\}
\]

for each of the \( mxz \) pairs \((b_n, y_i)\).

• The solution is characterized by:
  1. Decision rule \( b' = g(b, y) \)
  2. Value function \( V(b_n, y_i) \)
  3. Unconditional stationary distribution of \((b, y)\)

\[
\lambda(b, y) = \text{Prob}(b_t = b, y_t = y)
\]
Law of motion of conditional probabilities

- $P(y_t, y_{t+1})$ and $b' = g(b, y)$ induce a law of motion for conditional transition probabilities from date-$t$ states $(b, y)$ to date-$t+1$ states $(b', y')$:

$$\lambda_{t+1}(b', y') = \text{Prob}(b_{t+1} = b', y_{t+1} = y')$$

$$= \sum_{b_t \in B} \sum_{y_t \in \bar{y}} \text{Prob}(b_{t+1} = b'| b_t = b, y_t = y) \times \text{Prob}(y_{t+1} = y'| y_t = y) \times \text{Prob}(b_t = b, y_t = y)$$
Equilibrium Transition Probabilities

- But since \( b' = g(b, y) \) is a unique recursive function of \((b, y)\), the law of motion becomes:

\[
\lambda_{t+1}(b', y') = \sum_b \sum_y \lambda_t(b, y) \text{Prob}(y_{t+1} = y'|y_t = y) \gamma(b', b, y)
\]

\[
\gamma(b', b, y) = \begin{cases} 1 & \iff b' = g(b, y) \\ 0 & \text{otherwise} \end{cases}
\]

- Which can be rewritten as:

\[
\lambda_{t+1}(b', y') = \sum_y \sum_{\{b:b'=g(b,y)\}} \lambda_t(b, y) P(y, y')
\]
Stationary Distribution of Net Foreign Assets

• The stochastic steady state is a joint stationary distribution of NFA and income, which is the fixed point \( \lambda(b, y) \) of the law of motion

\[
\lambda_{t+1}(b', y') = \sum_{y} \sum_{\{b: b' = g(b, y)\}} \lambda_t(b, y) P(y, y')
\]

• Methods to solve for \( \lambda(b, y) \):
  – Iterating to convergence in the law of motion
  – Computing Eigen values of the \((mxz) \times (mxz)\) state transition probability matrix
  – Powering to convergence state transition prob. matrix
Precautionary savings (and the failure of the standard stationarity condition)

• Standard stationarity assumption $\beta R = 1$ fails
  – Euler eq. implies “constant consumption,” but income is always stochastic and NFA is non-state-contingent.
  – Formally: marginal benefit of saving $\beta^t R^t u'(t)$ follows a Supermartingale process, and since Supermartingales converge, it follows that $b' \to \infty$

• Agents self insure, build precautionary savings
  – If $\beta R < 1$, force pushing to borrow and force pushing for prec. savings support stationary distribution
  – Natural Debt Limit imposes lower bound on NFA
  – But the deterministic st. state is always the debt limit!
Graphical illustration

\[ \rho = \frac{1}{\beta} - 1 \]

NDL | R-1
---|---
\( r^* \) | \( \rho \)
\( E(b^*) \) | \( E(b) \)
\( r^A \) | \( r^* \)
Remarks about incomplete markets

• Solving these models generally requires global methods that can tackle state-cont. wealth dist.

• Certainty equivalence of DSGE models solved w. 1st order approx. fails (e.g. higher variance or persistence of shocks increases average NFA)

• Higher order approxs. deviate from certainty equivalence but still differ from global solution (more on this later)

• Prec. savings also affects portfolio structure (wealthier agents tolerate more risk, hold larger shares of risky assets at lower premia)
Example from Durdu, Mendoza & Terrones (2008)

• SOE with exogenous Markov endowment:

\[ V(b, \varepsilon) = \max_{b'} \left\{ \frac{c^{1-\gamma}}{1-\gamma} + \exp(-v(c))E[V(b', \varepsilon')] \right\} \]

subject to

\[ c = \varepsilon y - b' + bR + A \]

\[ b_{t+1} \geq \phi \geq -\min(\varepsilon ty + A)/r \]

• Allows for 2 formulations of rate of time pref.:
  1. Uzawa-Epstein endogenous rate of time preference
  2. Bewley-Aiyagari-Hugget setup with \( \beta R < 1 \)

\[ v(c) = \rho^{UE} \ln(1 + c) \text{ or } \ln(1 + \rho^{BAH}) \]
Calibration

• Discrete state space:

\((b, b') \in B = \{b_1 < b_2 < ... < b_n\} \quad n=1000\)

\(\varepsilon \in E = \{\varepsilon_1 < \varepsilon_2 < ... < \varepsilon_j\} \quad \pi(\varepsilon_{t+1} | \varepsilon_t)\)

• Income process (set to Mexico’s detrended GDP)

\[y_t = \rho_y y_{t-1} + c_t \quad \sigma_y = 3.301\% \quad \rho_y = 0.597\]

\[\sigma_e = \sqrt{\sigma_y^2 (1 - \rho_y^2)} = 2.648\text{ percent}\]

– Discretized using Tauchen-Hussey quadrature method with j=5 (yields process with 3.28% s.d. and AR=0.55)

– Can also use canonical Markov chains (e.g. “simple persistence” rule) to discretize time-series processes
• $E[y] = 1$ for simplicity (variables are GDP ratios)
• $E[b] = -0.44$ Mexico’s average NFA/GDP 1985-2004 in Lane & Milesi Ferretti (06)
• $E[c] = 69.2$ Mexico’s average C/GDP 1965-2005
• $R = 1.059$ Mexico’s country real interest rate from Uribe and Yue (06)
• It follows that $A = y + b(R-1) - c = 0.282$.

• Discount factors and rates of time preference:
  – UE: $\rho^{UE} = \ln(R) / \ln(1 + c) = 0.109 \quad (1 + c)^{-0.109} = 0.944$
  – BAH: $\rho^{BAH} = 0.064$ set by searching for values of ad-hoc debt limit & discount factor that match $E[b]=-0.44$ and sd(c)=3.28% ($\phi = -0.51 \quad \beta = 0.94$)
Calibrated state space

- Vector of income realizations

\[
\begin{array}{c}
1 & -0.075642 \\
2 & -0.035892 \\
3 & 0.0 \\
4 & 0.035892 \\
5 & 0.075642 \\
\end{array}
\]

- Transition prob. matrix of income shocks

\[
\begin{array}{cccccc}
& COL 1 & COL 2 & COL 3 & COL 4 & COL 5 \\
ROW 1 & 0.34500 & 0.52508 & 0.12475 & 0.00513915 & 2.0099D-05 \\
ROW 2 & 0.081986 & 0.47956 & 0.38426 & 0.053385 & 0.00080242 \\
ROW 3 & 0.011257 & 0.22208 & 0.53333 & 0.22208 & 0.011257 \\
ROW 4 & 0.00080242 & 0.053385 & 0.38426 & 0.47956 & 0.081986 \\
ROW 5 & 2.0099D-05 & 0.00513915 & 0.12475 & 0.52508 & 0.34500 \\
\end{array}
\]

- Grid of bonds: spacing=0.001514, nodes=1000, lower bound=-0.5123
### Calibrated parameter values

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Description</th>
<th>Value</th>
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<tbody>
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<td>$\rho^{BAH}$</td>
<td>Rate of time preference in the BAH setup</td>
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<tr>
<td>$\rho^{UE}$</td>
<td>Rate of time preference elasticity in the UE setup</td>
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<td>$\gamma$</td>
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<td>$\phi$</td>
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<td>$b$</td>
<td>Net foreign assets-output ratio</td>
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<td>$A$</td>
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</table>
Transitional and stationary distributions

A. Bewley-Aiyagari-Hugget Preferences

Note: Initial conditions are lowest (b,y) with positive long-run probability
Transitional and stationary distributions

Note: Initial conditions are lowest (b,y) with positive long-run probability
Transitional dynamics of NFA

Note: Dynamics show forecasting function starting from lowest positive prob. B and neutral income shock and plotted as differences relative to long-run averages.
Effects of income variability on precautionary NFA demand
### Unconditional moments

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<th>Baseline</th>
<th>Auto Corr 0.7</th>
<th>Std Dev. 5%</th>
<th>Std Dev. 2.5%</th>
<th>Risk Aver. 5.0</th>
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TRADEOFFS IN THE USE OF GLOBAL V. LOCAL METHODS

(THE IMPORTANCE OF INCOMPLETE MARKETS IN “CLOSING OPEN ECONOMY MODELS”)

Be careful how we “close” op. ec. models

- Schmitt-Grohe & Uribe (JIE 03) proposed three ad-hoc ways to induce stationarity so that local methods can be used:
  1. Debt-elastic interest rate (DEIR) function $r(b - \bar{b})$
  2. Resource cost of holding assets $h(b - \bar{b})$
  3. Rate of time pref. depends on “aggregate” $C(b - \bar{b})$

- They showed these are about equivalent in an RBC moment-matching exercise

- Using this approach, Garcia-Cicco, Pancrazi & Uribe (AER 10) found that RBC-SOE model cannot explain AR behavior of net exports
Autocorrelation functions of $TB/Y$

Autocorr. of Net Exports: Data v. Models

• Garcia-Cicco et al.: In the data, NX is AR(1) but in RBC-SOE model solved with DEIR function it is a near-unit-root process.

• Durdu et al. (2019): this is not a property of the “exact” solution, but a limitation of using the DEIR function

• To show it use this:
  1. Definition of net exports: \( tb_t = b_{t+1} - b_t R^* \)
  2. Assume AR(1) process for NFA: \( b_{t+1} = \rho b_t + \epsilon_{t+1} \)
     and notice DEIR implicitly sets \( \rho \) when specifying \( r(b - \bar{b}) \). Garcia-Cicco et al. set it so that \( \rho \approx 1 \), so that DEIR is “inessential”
Autocorrelations of net exports and NFA

• Combine 1 & 2, solve for AR(1) of net exports:

\[ \rho(nx) = \frac{q^2 \rho + \rho - q - q \rho^2}{1 + q^2 - 2q \rho} \]

where \( q = 1/R^* \)

• \( \rho(nx) \) is a nonlinear function of \( \rho \), so we need “exact” solution for \( \rho \) in order to derive correct results about \( \rho(nx) \)
  – Changing \( \rho \) from 0.95 to 0.999 changes \( \rho(nx) \) from near zero to 0.999!!
  – Knowing true solution of NFA dynamics is critical
Autocorrelations of NFA and NX
Autocorrelations of NFA and NX

Empirically relevant range of AR(1) of NX
Limitations of ad-hoc approach to induce stationarity of NFA positions

• Generally: ad-hoc approach imposes long-run and AR(1) of NFA instead of solving for it

• “Exact” global, non-linear solution is not critical for some business cycle moments, but it is critical for those directly related to NFA and for other key issues:
  1. Global imbalances (accumulation of reserves)
  2. Financial crises & macro-prudential regulation
  3. Sovereign risk
  4. Financial development
...but still ad-hoc approach is widely used

- Allows using local methods that solve quickly and can be applied to large models
- DEIR is by far more common than cost of holding bonds or endogenous discounting
- Majority sets the debt elasticity of DEIR function to "inessential value" of 0.001 following SGU (2003), others calibrate it or estimate it (0.00014-2.8 range)
- Most applications use first-order approximation (1OA), some have used second- and third-order (2OA, 3OA) or risky steady state (RSS)
- Recent methods for occ. binding constraints: OccBin (Iacoviello-Guerrieri), DynareOBC (Holden)
Global v. local methods for op. ec. models
(Durdu, de Groot & Mendoza (2019))

• Compared global solution (FiPIt or VFI) v. 1OA, 2OA, RSS, OccBin & DynareOBC for endowment economy, RBC, and Sudden Stops (RBC w. occ. binding collateral constraint)

• Local methods approximate poorly prec. savings

• Business cycle moments, IRFs, SDFs, and financial crises dynamics & frequency differ

• Best performance requires targeting NFA moments from global sol. (e.g. autocorr. of NFA)

• Various local methods differ mainly on 1st moments
Model 2 again

• Optimization problem:

\[ E_0 \left\{ \sum_{t=0}^{\infty} \beta^t u(c_t) \right\} \]

\[ u(c_t) = \frac{c_t^{1-\sigma}}{1-\sigma}. \]

\[ c_t = e^{z_t \bar{y}} + b_t - qb_{t+1} \]

\[ b_{t+1} \geq -\varphi. \]

• Optimality conditions in recursive form:

\[ c(b, z)^{-\sigma} \geq \beta R \sum_{z'} \pi(z', z) \left[ \left( c(b'(b, z), z') \right)^{-\sigma} \right] \]

\[ c(b, z) = e^{z \bar{y}} + b - qb'(b, z) \]
**FiPIT, a Simple & Fast Global Method**  
Mendoza-Villalvazo (2019)

1. Start iteration $j$ with a conjectured decision rule $\hat{b}_j'(b, z)$

2. Generate the consumption dec. rule implied by that conjecture using the resource constraint
   \[ c_j(b, z) = e^{\tilde{y}} + b - q\hat{b}_j'(b, z) \]

3. Solve for a new consumption dec. rule “directly” using the Euler eq. (assuming $\varphi$ is not binding)
   \[ c_{j+1}(b, z) = \left\{ \beta R \sum_{z'} \pi(z', z) \left[ \left( c_j(\hat{b}_j'(b, z), z') \right)^{-\sigma} \right] \right\}^{-\frac{1}{\sigma}} \]
   - In RHS, we evaluate the j-th iteration cons. dec. rule using the values of the state variables at $t+1$
   - Requires interpolation, because consumption dec. rule is only known at grid nodes
   - No need for a non-linear solver as with endogenous grids method
FiPIT Method Contn’ed

4. Generate new bond’s decision rule $b'_{j+1}(b, z)$ using the resource constraint. If $b'_{j+1}(b, z) \leq -\varphi$, the debt limit binds and we set $b'_{j+1}(b, z) = -\varphi$.

5. Update the initial conjecture for iteration $j+1$:

$$\hat{b'}_{j+1}(b, z) = (1 - \rho)\hat{b'}_j(b, z) + \rho b'_{j+1}(b, z)$$

- Use $0 < \rho < 1$ for unstable iterations, or $\rho > 1$ for slow convergence.

6. Iterate to convergence (until $b'_{j+1}(b, z) = \hat{b'}_j(b, z)$ up to a convergence criterion):

- Analogous to Parameterized Expectations (fixed-point iteration using simulation & regression in Step 3).
- Finite state space better than colocation (occ. bind. constraints).
- Extends easily to 2 endogenous states w. bilinear interpolation.
Local methods

• **1OA, 2OA**: standard approximations of NFA dec. rule applied to approximations of same order to opt. conditions around $b^{dss}$ (assumes $\beta(1+r) = 1$)

• **RSS**: $b^{rss}$ obtained from 2OA of cond. expectation of steady-state Euler eq., solved jointly with 1OA of decision rule around $b^{rss}$ (assumes $\beta(1+r) < 1$)

• Use DEIR to support $b^{dss}$ (necessary in 1OA & 2OA)

\[
\frac{1}{q_t} \equiv 1 + r_t = 1 + r + \psi \left[ e^{b^{dss}} - b_{t+1} - 1 \right]
\]

  – For small perturbations, debt elasticity is $\eta^r \equiv -\psi b^{dss}$

  – $\psi$ can be SGU inessential value (0.001) or calibrated to a target moment (e.g. autocorr. of nfa)
Local methods contn’d

- 2OA to NFA decision rule in dev. form:

\[
\tilde{b}_{t+1} = h_b \tilde{b}_t + h_y \tilde{y}_t + \frac{1}{2} \left( h_{bb} \tilde{b}_t^2 + h_{yy} \tilde{y}_t^2 + h_{\sigma z} \sigma_z^2 \right) + h_{by} \tilde{b}_t \tilde{y}_t
\]

  a) 1OA and RSS have only the first two terms in RHS
  b) RSS uses risky ss. instead of det. ss to define devs.
  c) \( h_b \) has same value regardless of approx. order
  d) \( h_{\sigma z} \sigma_z^2 \) captures effect of income variability on NFA (prec. savings). In RSS it also matters for risky ss.
  e) Quantitatively, all other 2\(^{nd}\) order terms are negligible

- Assuming log utility and i.i.d. income process:

\[
h_b(\psi, b^*) = \frac{R + e^{b^*\psi} (1 - b^*\psi + \psi) - \sqrt{R^2 + 2e^{b^*\psi} (b^*\psi + \psi - 1)R + e^{2b^*\psi} (1 - b^*\psi + \psi)^2}}{2e^{b^*\psi}}
\]

  - Hence, autocorr. of NFA is \( \rho_b(\psi, b^*) \approx h_b(\psi, b^*) \)
Calibration

1. Common parameters

\[ \sigma \quad \text{Coefficient of relative risk aversion} \quad 2.0 \]
\[ y \quad \text{Mean endowment income} \quad 1.00 \]
\[ A \quad \text{Total absorption} \quad 0.28 \]
\[ R \quad \text{Gross world interest rate} \quad 1.059 \]
\[ \sigma_z \quad \text{Standard deviation of income (percent)} \quad 3.27 \]
\[ \rho_z \quad \text{Autocorrelation of income} \quad 0.597 \]

2. Global solution parameters

\[ \beta \quad \text{Discount factor} \quad 0.940 \]
\[ \phi \quad \text{Ad-hoc debt limit} \quad -0.51 \]

3. Local solution parameters

*Common parameters*

\[ \beta \quad \text{Discount factor} \quad 0.944 \]
\[ b \quad \text{Deterministic steady state value of NFA} \quad -0.51 \]

*Baseline calibration*

\[ \psi \quad \text{Inessential DEIR coefficient} \quad 0.001 \]

*Targeted calibration*

\[ \psi \quad \text{DEIR coefficient for 20A} \quad 0.0469 \]
\[ \psi \quad \text{DEIR coefficient for RSS} \quad 0.0469 \]
NFA autocorr. & the three local methods

- $\rho_b(\psi, b^*)$ describes mapping between debt elasticity parameter and NFA autocorr.
  - If $\psi = 0$, it has two roots given by $(1+r,1)$, so NFA is non-stationary.

- Given $(R, b^*)$, $\rho_b(\psi, b^*)$ is a U-shaped function of $\psi$, but in quantitatively relevant range is downward sloping, convex.

- Plot $\rho_b(\psi, b^*)$ as $\psi$ varies for $b^*=0$, det. ss (-0.51) and risky ss. (-0.41)

- For $0 \leq \psi \leq 0.1$, $\rho_b(\psi, b^*)$ nearly identical for 1OA, 2OA & RSS!

- Since 2nd order terms (except $h_{\sigma_z\sigma_z} \sigma_z^2$) are negligible, all three methods will have very similar 2nd & higher-order moments and IRFs, and pruning is irrelevant!
Elasticity of DEIR function & NFA dec. rule
## Comparison of long-run moments

<table>
<thead>
<tr>
<th>GLB</th>
<th>Baseline Calibration</th>
<th>Targeted Calibration</th>
</tr>
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<tr>
<td></td>
<td>2OA</td>
<td>RSS</td>
</tr>
<tr>
<td>DEIR</td>
<td>βR &lt; 1</td>
<td>DEIR</td>
</tr>
</tbody>
</table>

### Averages

| μ(c) | 0.694 | 0.701 | 0.093 | 0.692 | 0.689 | 0.689 |
| μ(nx/y) | 0.022 | 0.015 | 0.625 | 0.025 | 0.028 | 0.028 |
| μ(b/y) | -0.413 | -0.282 | -11.210 | -0.451 | -0.502 | -0.506 |

### Standard deviations relative to standard deviation of income

| σ(c)/σ(y) | 0.992 | 1.594 | 1.161 | 1.617 | 1.001 | 0.997 |
| σ(nx)/σ(y) | 0.660 | 1.327 | 1.202 | 1.346 | 0.730 | 0.730 |
| σ(nx/y)/σ(y) | 0.643 | 1.311 | 1.161 | 1.331 | 0.709 | 0.709 |
| σ(b)/σ(y) | 7.461 | 62.327 | 1.706 | 40.078 | 6.647 | 6.576 |
| σ(b)/σ(y) | 7.735 | 61.989 | 1.892 | 40.213 | 7.174 | 7.118 |

### Income correlations

| ρ(y,c) | 0.755 | 0.202 | 0.188 | 0.197 | 0.684 | 0.684 |
| ρ(y,nx) | 0.729 | 0.572 | 0.312 | 0.567 | 0.705 | 0.708 |
| ρ(y,nx/y) | 0.704 | 0.572 | 0.006 | 0.567 | 0.705 | 0.708 |
| ρ(y,b) | 0.449 | 0.128 | 0.070 | 0.124 | 0.489 | 0.488 |
| ρ(y,b/y) | 0.549 | 0.156 | 0.445 | 0.149 | 5.593 | 0.592 |

### First-order autocorrelations

| ρc | 0.840 | 0.995 | 0.996 | 0.995 | 0.929 | 0.929 |
| ρnx | 0.543 | 0.819 | 0.934 | 0.823 | 0.583 | 0.582 |
| ρnx/y | 0.551 | 0.826 | 0.995 | 0.830 | 0.591 | 0.590 |
| ρb | 0.977 | 0.999 | 0.999 | 0.999 | 0.977 | 0.977 |
| ρb/y | 0.961 | 0.998 | 0.953 | 0.998 | 0.958 | 0.959 |
Effect of higher income variability on mean NFA

As SGU (2003) showed, DEIR and bond adjustment cost are analogous up to 1OA. Hence higher $\psi$ is akin to higher adj. cost, which keeps NFA close to its mean.
Impulse response functions

a. NFA/Output Baseline

b. NFA/Output Targeted

c. Consumption Baseline

d. Consumption Targeted

e. NX/Output Baseline

f. NX/Output Targeted