Online Apppendix to
Optimal Time-Consistent Macroprudential Policy

Javier Bianchi

Minneapolis Fed, University Wisconsin & NBER

Enrique G. Mendoza

University of Pennsylvania, NBER & PIER

September, 2016

Contents

A Theoretical Analysis and Proofs

A.1 Optimal Time Consistent Planner's Problem

We explain here the analytical formulation of the Markov perfect equilibrium and show that the recursive social planner's problem under discretion is given by (12). To do this, we first provide a complete formulation of the planner's problem and then establish that solving a "relaxed" planner's problem that includes only a subset of the constraints of the time consistent planner's problemas in (12) yields equivalent outcomes.

As described in Section 2.3, we consider a social planner that chooses b_{t+1} on behalf of the representative firm-household while this agent still chooses consumption, labor supply, intermediate inputs, land holdings taking as given asset prices, and future government transfers. To derive the planner's implementability constraints, we analyze the problem of the households, which consist of choosing $\{c_t, h_t, k_{t+1}\}$ taking as given $\{q_t, T_t\}$ so as to solve the following problem::

$$\max_{\{c_t,h_t,k_{t+1}\}_{t\geq 0}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t - G(h_t)$$

$$s.t. \quad c_t + q_t k_{t+1} = k_t q_t + z_t F(k_t, h_t, v_t) - p_v v_t + T_t$$

$$\frac{b_{t+1}}{R} - \theta p_v v_t \geq -\kappa_t q_t k_t$$

The first-order conditions are

$$z_t F_h(k_t, h_t, v_t) = G'(h_t) \tag{A.1}$$

$$z_t F_v(k_t, h_t, v_t) = p_v(1 + \theta \mu_t / u'(t)) \tag{A.2}$$

$$q_t u'(t) = \beta \mathbb{E}_t \left\{ u'(t+1) \left(z_{t+1} F_k(k_{t+1}, h_{t+1}, v_{t+1}) + q_{t+1} \right) + \kappa q_{t+1} \mu_{t+1} \right\} \tag{A.3}$$

which correspond to conditions (4),(5),(7). These three conditions together with complementary slackness conditions $\mu \geq 0$ and $\mu \left(\frac{b'}{R} - \theta p^v v + \kappa q \right) = 0$ constitute the implementability constraints in the complete planner's problem, as described below. Notice that combining the household budget constraint with the government budget constraint $T_t = b_t - \frac{b_{t+1}}{R_t}$ yields the resource constraint

$$c_t + \frac{b_{t+1}}{R_t} = b_t + z_t F(1, h_t, v) - p^v v. \tag{A.4}$$

The planner's problem consists of maximizing expected lifetime utility (1) subject to (A.1)-(A.4), and complementary slackness conditions, taking as given future planner's policies $\{\mathcal{B}(b,s), \mathcal{C}(b,s), \mathcal{H}(b,s), \boldsymbol{\mu}(b,s), \mathcal{Q}(b,s)\}$.

Problem 1 *The complete recursive time consistent planner's problem is:*

$$\mathcal{V}(b,s) = \max_{c,b',q,\mu,h,v} u(c - G(h)) + \beta \mathbb{E}_{s'|s} \mathcal{V}(b',s') \tag{SP}$$

$$c + \frac{b'}{R} = b + zF(1,h,v) - p^v v \tag{SP1}$$

$$\frac{b'}{R} - \theta p^v v \geq -\kappa q \tag{SP2}$$

$$\begin{aligned} qu'(c - G(h)) = {} & \beta \mathbb{E}_{s'|s} u'(\mathcal{C}(b',s') - G(\mathcal{H}(b',s')))(\mathcal{Q}(b',s') + z' F_k(1, \mathcal{H}(b',s'), \boldsymbol{v}(b',s'))) \\ & + \kappa' \boldsymbol{\mu}(b',s')\mathcal{Q}(b',s') \end{aligned} \tag{SP3}$$

$$zF_h(1,h,v) = G'(h) \tag{SP4}$$

$$zF_v(1,h,v) = p^v \left(1 + \frac{\theta \mu}{u'(c - G(h))}\right) \tag{SP5}$$

$$\mu \geq 0 \tag{SP6}$$

$$\mu \left(\frac{b'}{R} - \theta p^v v + \kappa q\right) = 0 \tag{SP7}$$

Proposition II (Relaxed Planner Problem) *Constraints (SP4)-(SP7) of the complete planner's problem do not bind, and thus a relaxed planner's problem that is not subject to these constraints yields equivalent solutions.*

Proof: The proof proceeds by analyzing a relaxed planner's problemin which the planner is not subject to (SP4)-(SP7) and then showing that those conditions are still satisfied.

Consider the following relaxed problem (i.e., Problem 12 in the text):

$$\mathcal{V}(b,s) = \max_{c,b',q,\mu,h,v} u(c - G(h)) + \beta \mathbb{E}_{s'|s} \mathcal{V}(b',s') \tag{RP}$$

$$c + \frac{b'}{R} = b + zF(1,h,v) - p^v v \tag{RP1}$$

$$\frac{b'}{R} - \theta p^v v \geq -\kappa q \tag{RP2}$$

$$\begin{aligned} qu'(c - G(h)) = {} & \beta \mathbb{E}_{s'|s} u'(\mathcal{C}(b',s') - G(\mathcal{H}(b',s')))(\mathcal{Q}(b',s') + z' F_k(1, \mathcal{H}(b',s'), v(b',s'))) \\ & + \kappa' \boldsymbol{\mu}(b',s')\mathcal{Q}(b',s') \end{aligned} \tag{RP3}$$

Let $\lambda \geq 0$ be the multiplier on the resource constraint, $\mu^* \geq 0$ be the multiplier on the collateral constraint, $\xi \geq 0$ be the multiplier on the asset pricing implementability constraint.

Applying the envelope theorem to the first-order conditions of the RP problem yields the following conditions:

$$c :: \qquad \lambda = u'(c - G(h)) - \xi u''(c - G(h))q \tag{RP4}$$

$$b' :: \qquad \lambda = \beta R \mathbb{E}_{s'|s}[\mathcal{V}_b(b', s') + \xi \hat{\Omega}] + \mu^* \tag{RP5}$$

$$q :: \qquad \mu^* \kappa = \xi u'(c - G(h)) \tag{RP6}$$

$$h :: \qquad \lambda z F_h(1, h, v) = u'(c - G(h))G'(h) - \xi q u''(c - G(h))G'(h) \tag{RP7}$$

$$v :: \qquad zF_v(1, h, v) = p^v \left(1 + \frac{\mu^*}{\lambda}\theta\right) \tag{RP8}$$

$$KT :: \qquad \mu^* \left(\frac{b'}{R} - \theta p^v v + \kappa q\right) = 0 \tag{RP8}$$

$$EC :: \qquad \mathcal{V}_b(b, s) = \lambda \tag{RP9}$$

where $\Omega' = u''(\mathcal{C}(b', s') - G(\mathcal{H}(b', s')))\{\mathcal{Q}(b', s') + z' F_k(1, \mathcal{H}(b', s'), \mathbf{v}(b', s'))\}...$
$...\{\mathcal{C}_b(b', s') - G'(\mathcal{H}(b', s'))\mathcal{H}_b(b', s')\} + u'(\mathcal{C}(b', s') - G'(\mathcal{H}(b', s')))\{\mathcal{Q}_b(b', s') + z'[F_{kh}(1, \mathcal{H}(b', s'), \mathbf{v}(b', s'))\mathcal{H}_b(b', s') +$
$F_{kv}(1, \mathcal{H}(b', s'), \mathbf{v}(b', s'))\mathbf{v}_b(b', s')]\} + \kappa'[\boldsymbol{\mu}_b(b', s')\mathcal{Q}(b', s') + \boldsymbol{\mu}(b', s')\mathcal{Q}_b(b', s')]$.

Set $\mu = \frac{u'(c-G(h))}{\theta}\left(\frac{zF_v(1,h,v)}{p^v} - 1\right)$.[30] Rearranging (RP8), we have $\mu^* = \frac{\lambda}{\theta}\left(\frac{zF_v(1,h,v)}{p^v} - 1\right)$ and combining this with the expression for μ, we have $\mu_t = \frac{\mu_t^*}{\lambda_t} u'(c_t - G(h_t))$. Since $u'(.) > 0$, the KT conditions of the relaxed problem $\mu^* \geq 0$ and (RP8) imply (SP6) and (SP7), the KT conditions of the original problem. By definition of μ, we have that (SP5) is satisfied. Finally, substituting (RP4) into (RP7) yields the original optimality condition with respect to employment (SP4).

This completes the proof that (SP4)-(SP7) do not bind. This proves Proposition II. □

A.2 Proof of Proposition 1

Proposition 1 *[Decentralization with Debt Taxes] The constrained-efficient equilibrium can be decentralized with a state-contingent tax on debt with tax revenue rebated as a lump-sum transfer and the tax rate set to satisfy:*

$$1 + \tau_t = \frac{1}{\mathbb{E}_t u'(t+1)} \mathbb{E}_t \left[u'(t+1) - \xi_{t+1} u''(t+1)\mathcal{Q}_{t+1} + \xi_t \Omega_{t+1}\right] + \frac{1}{\beta R_t \mathbb{E}_t u'(t+1)} \left[\xi_t u''(t) q_t\right]$$

where the arguments of the functions have been shorthanded as dates to keep the expression simple.

[30] Notice the distinction between μ, which is the shadow value of relaxing the collateral constraint for individual agents, and is choice variable for the planner, and μ^*, which is the shadow value of relaxing the collateral constraint from the planner's perspective.

Define the tax as:

$$1+\tau_t = \frac{\beta R \mathbb{E}_t \left\{ u'(\mathcal{C}(b_{t+1}, z_{t+1})) - \xi_{t+1} u''(\mathcal{C}(b_{t+1}, z_{t+1}))\mathcal{Q}(b_{t+1}, z_{t+1}) + \xi_t \Omega_{t+1} \right\} + \xi_t u''(c_t) q_t}{\beta R \mathbb{E}_t u'(\mathcal{C}(b_{t+1}, z_{t+1}))} \quad \text{(A.5)}$$

We prove the proposition by showing that the decentralized equilibrium with the tax yields the same optimality conditions as the planner's constrained-efficient equilibrium. The constrained efficient equilibrium can be characterized by sequences $\{c_t, k_{t+1}, h_t, v_t, b_{t+1}, q_t, \lambda_t, \mu_t^*\}_{t=0}^{\infty}$ that satisfy (3), (4), (5), (13), (14), (15), (RP1), $k_t = 1$ together with complementary slackness conditions. The regulated decentralized equilibrium is characterized by sequences $\{c_t, k_{t+1}, b_{t+1}, h_t, v_t, q_t, \lambda_t, \mu_t\}_{t=0}^{\infty}$ that satisfy (2), (3), (4), (5), (7), (16), (RP4), $k_t = 1$ together with complementary slackness conditions. Substituting the expression for the tax (A.5) and (16), yields condition (14) and identical conditions characterizing the two equilibria.

$$u'(t) = \beta R \mathbb{E}_t \left[u'(t+1) - \xi_{t+1} u''(t+1)\mathcal{Q}_{t+1} + \xi_t \hat{\Omega}_{t+1} \right] + \xi_t u''(t) q_t + \mu_t^*$$

When the collateral constraint is not binding, we obtain the following macro-prudential debt tax:

$$\tau_t = \mathbb{E}_t \frac{-\xi_{t+1} u''(t+1)\mathcal{Q}_{t+1}}{\mathbb{E}_t u'(t+1)} \geq 0 \quad \text{(A.6)}$$

A.3 Optimal Tax on Debt Problem: Equivalence Result with a Regulator's Problem Choosing Taxes

Consider the following government's problem without commitment choosing an optimal tax on debt, given that future taxes $\mathcal{T}(B, s)$ are chosen by future governments, associated with policies $\{\mathcal{B}(b, s), \mathcal{C}(b, s), \mathcal{H}(b, s), \boldsymbol{\mu}(b, s), \mathcal{Q}(b, s)\}$. The government chooses the tax on debt to maximize utility considering the optimal response of households. That is, the government chooses a tax on debt subject to all competitive equilibrium conditions including $u'(c) = \beta R(1+\tau)\mathbb{E}\left[u'(\mathcal{C}(b', z'))\right] + \mu$

$$\mathcal{V}(b,s) = \max_{c,b',q,\mu,h,v,\tau} u(c - G(h)) + \beta \mathbb{E}_{s'|s} \mathcal{V}(b', s') \tag{A.7}$$

$$c + \frac{b'}{R} = b + zF(1,h,v) - p^v v$$

$$\frac{b'}{R} - \theta p^v v \geq -\kappa q$$

$$qu'(c - G(h)) = \beta \mathbb{E}_{s'|s} u'(\mathcal{C}(b', s') - G(\mathcal{H}(b', s')))(\mathcal{Q}(b', s') + z' F_k(1, \mathcal{H}(b', s'), \boldsymbol{v}(b', s'))) + \kappa' \boldsymbol{\mu}(b', s') \mathcal{Q}(b', s')$$

$$zF_h(1,h,v) = G'(h)$$

$$zF_v(1,h,v) = p^v \left(1 + \frac{\theta\mu}{u'(c - G(h))}\right)$$

$$\mu \geq 0$$

$$\mu\left(\frac{b'}{R} - \theta p^v v + \kappa q\right) = 0$$

$$u'(c - G(h)) = \beta R(1+\tau)\mathbb{E}\left[u'(\mathcal{C}(b', z') - G(\mathcal{H}(b', z')))\right] + \mu \tag{A.8}$$

Relative to problem (SP), problem (A.7) contains one additional restriction (A.8) and one additional policy instrument τ. Let the optimal tax that solves (A.7), and note that the MPE stationarity condition requires that $\tau(B,s) = \mathcal{T}(B,s)$

Proposition III *A sequence of allocations and prices constitutes a constrained efficient equilibrium that solves the planner's problem (12) if and only if they are the outcome of a markov perfect equilibrium where the government chooses sequentially a tax on debt.*

Proof

Consider solving the relaxed problem of maximizing the objective function dropping constraint (A.8). Notice that the resulting problem is the same as (12), after applying Proposition II. Hence, allocations and prices satisfy (12) if and only if they satisfy the optimal tax problem. In addition, notice that τ only appears in (A.8). Hence, setting $1+\tau = \frac{u'(c)}{\beta R E[u'(\mathcal{C}(b',z'))]+\mu}$ implies that condition (A.8) is satisfied. $\square$

A.4 Non-Negative Tax on Debt

We solve for the optimal tax on debt, as in problem (A.7) but now impose the additional restriction that taxes on debt that cannot be negative. This is natural because subsidies on debt require lump sum taxes. So we rule out lump-sum taxes but not lump-sum transfers. As we showed in Section 2.4, the optimal tax is non-negative when the constraint is not binding, but is possibly negative when the constraint binds (i.e. only the macroprudential component of the debt tax is non-negative, the full tax could be negative). Hence, it is possible that the non-negativity constraint on τ would be binding.

The problem of this planner is the same as in (A.7) but with an additional implementability constraint given by:

$$1 + \tau_t = \frac{u'(c_t - G(h_t)) - \mu_t}{\beta R_t \mathbb{E}\left[u'(c_{t+1} - G(h_{t+1}))\right]} \geq 1$$

The relaxed regulator's problem can be written as:

$$\mathcal{V}(b, s) = \max_{c, b', q, \mu, h, v} u(c - G(h)) + \beta \mathbb{E}_{s'|s} \mathcal{V}(b', s') \tag{A.9}$$

$$c + \frac{b'}{R} = b + zF(1, h, v) - p^v v$$

$$\frac{b'}{R} - \theta p^v v \geq -\kappa q$$

$$qu'(c - G(h)) = \beta \mathbb{E}_{s'|s} u'(\mathcal{C}(b', s') - G(\mathcal{H}(b', s')))(\mathcal{Q}(b', s') + z' F_k(1, \mathcal{H}(b', s'), \boldsymbol{v}(b', s')))$$

$$+ \kappa' \boldsymbol{\mu}(b', s') \mathcal{Q}(b', s')$$

$$\frac{u'(c - G(h))}{\beta R_t \mathbb{E}\left[u'(\mathcal{C}(b', s') - G(\mathcal{H}(b', s')))\right]} \geq 1$$

Let $\gamma^p \geq 0$ denote the lagrange multiplier on the last constraint. Following the same steps as in the derivation of the optimal tax in in Proposition (A.2) yields that the optimal tax is now given by:

$$\begin{aligned} 1 + \tau_t^{tax \geq 0} = & \frac{1}{\beta R \mathbb{E}_t u'(\mathcal{C}(b_{t+1}, z_{t+1}) - G(\mathcal{H}(b', s')))} \beta R \mathbb{E}_t \Big[u'(\mathcal{C}(b_{t+1}, z_{t+1}) - G(\mathcal{H}(b_{t+1}, z_{t+1}))) \\ & \xi_{t+1}^p \mathcal{Q}(b_{t+1}, z_{t+1}) u''(\mathcal{C}(b_{t+1}, z_{t+1})) + \gamma_{t+1}^p \left(\frac{u''(\mathcal{C}(b_{t+1}, z_{t+1}) - G(\mathcal{H}(b_{t+1}, z_{t+1})))}{\beta R \mathbb{E}_t [u'(\mathcal{C}(b_{t+1}, z_{t+1}) - G(\mathcal{H}(b_{t+1}, z_{t+1}))]} \right) \\ & + \xi_t^p \Omega_{t+1}^p + \gamma_t^p \phi_{t+1}^p \Big] - \mu_t (\varphi_t^p + 1) + \mu_t^p \\ & + \xi_t^p q_t u''(c_t) - \gamma_t^p \left(\frac{u''(c_t - G(h_t)}{\beta R \mathbb{E}_t [u'(\mathcal{C}(b_{t+1}, z_{t+1}) - G(\mathcal{H}(b_{t+1}, z_{t+1}))]} \right) \end{aligned} \tag{A.10}$$

Now suppose that both the non-negative tax constraint and the collateral constraint are slack today but might bind tomorrow with positive probability, i.e. $\gamma_t^p = 0$ and $\mu_t^* = 0$. In this case, (A.10) becomes:

$$1 + \tau_t^{tax\geq 0} = \frac{1}{\beta R \mathbb{E}_t u'(\mathcal{C}(b_{t+1}, z_{t+1}))} \Bigg[\beta R \mathbb{E}_t \Bigg[u'(\mathcal{C}(b_{t+1}, z_{t+1})) - \xi_{t+1}^p \mathcal{Q}(b_{t+1}, z_{t+1}) u''(\mathcal{C}(b_{t+1}, z_{t+1})) + \gamma_{t+1}^p \left(\frac{u''(\mathcal{C}(b_{t+1}, z_{t+1}))}{\beta R \mathbb{E}_t [u'(\mathcal{C}(\mathcal{C}(b_{t+1}, z_{t+1})))]} \right) \Bigg] \Bigg] \tag{A.11}$$

where $\phi_{t+1}^p \equiv \frac{\mu_t - u'(c_t)}{\beta^2 R_t} \mathbb{E}_t \left[\frac{u''(\mathcal{C}(b_{t+1}, z_{t+1})) \mathcal{C}_b(b_{t+1}, z_{t+1})}{(u'(\mathcal{C}(b_{t+1}, z_{t+1})))^2} \right]$

Notice that the tax has the same form as in Proposition (A.2) but now it has an additional term given by the possibly binding non-negativity constraint on future tax rates.

A.5 Derivation of Collateral Constraint

We provide a derivation of the collateral constraint (3) as an incentive compatibility constraint resulting from a limited enforcement problem. Debt contracts are signed with creditors in a competitive environment. Financial contracts are not exclusive, i.e., agents can always switch to another creditor at any point in time. Households borrow at the beginning of the period, before the asset market opens. Within period, households can divert future revenues and avoid any costs from defaulting next period when debt becomes due. At the end of the period, there are no more opportunities for households to divert revenues and repayment of previous bonds is enforced. Financial intermediaries can costlessly monitor diversion activities at time t. If creditors detect the diversion scheme, they can seize a fraction κ_t of the household assets. After defaulting, a household regains access to credit markets instantaneously and repurchases the assets that investors sell in open markets. Given this environment, a household that borrows $\tilde{d}_{t+1}$ and engages in diversion activities gains $\tilde{d}_{t+1}$ and loses $\kappa_t q_t k_t$.

Formally, let V^R and V^d be the value of repayment and default respectively, and V be the continuation value. If a household raises $\tilde{d}$ resources by borrowing $\frac{b'}{R}$ at the beginning of the period, and defaults, it gets

$$V^d(\tilde{d}, b, k, X) = \max_{b', k', c, v} u(c - G(h)) + \beta \mathbb{E}_{s'|s} V(b', k', B', s') \tag{A.12}$$

$$s.t. \quad q(B, s)k' + c + \frac{b'}{R} = \tilde{d} + q(B, s)k(1 - \kappa) + b + zF(k, h, \nu) - p_\nu \nu$$

$$-\frac{b'}{R} + \theta p_\nu \nu \leq \kappa q(B, s)k$$

where the budget constraint reflects the fact that the household regains access to credit markets and can borrow b' and the collateral constraint reflects that agents buy back the assets from investors. If the household does not default, it gets the utility from current consumption plus the continuation value of starting next period with debt b' as stated in (10).

$$V^r(b,k,X) = \max_{b',k',c,h} u(c - G(n)) + \beta \mathbb{E}_{s'|s} V(b',k',B',s') \tag{A.13}$$
$$\begin{aligned} s.t. \quad & q(B,s)k' + c + \frac{b'}{R} = q(B,s)k + b + zF(k,h,\nu) - p_\nu \nu \\ & -\frac{b'}{R} + \theta p_\nu \nu \leq \kappa q(B,s)k \end{aligned}$$

A simple inspection at the budget constraints implies that households repay if and only if $\tilde{d}_{t+1} \leq \kappa_t q_t k_t$

Notice that for the constrained-efficient equilibrium, the derivation of the feasible credit positions is analogous to the case in the decentralized equilibrium. If the planner engages in diversion, creditors can seize a fraction κ_t of assets in the economy. Moreover, households can buy back the assets at the market price q_t. This implies that the same collateral constraint applies in the constrained-efficient equilibrium in this environment.

B Computational Algorithm

B.1 Numerical Solution Method for Decentralized Equilibrium

Following Bianchi (2011), we use Coleman (1990)'s time iteration algorithm, modified to address the occasionally binding endogenous constraint, that operates directly on the first-order conditions. Formally, the computation of the competitive equilibrium requires solving for functions $\{\mathcal{B}(b,s), \mathcal{Q}(b,s), \mathcal{C}(b,s), \nu(b,s), \mathcal{H}(b,s), \mu(b,s)\}$ such that:

$$\mathcal{C}(b,s) + \frac{\mathcal{B}(b,s)}{R} = zF(1, \mathcal{H}(b,s), \nu(b,s)) + b - p_v \nu(b,s) \tag{B.1}$$

$$-\frac{\mathcal{B}(b,s)}{R} + \theta p_\nu \nu(b,s) \leq \kappa \mathcal{Q}(b,s) \tag{B.2}$$

$$u'(\mathcal{C}(b,s) - G'(\mathcal{H}(b,s))) = \beta R \mathbb{E}_{s'|s}\left[u'(\mathcal{C}(\mathcal{B}(b,s),s') - G'(\mathcal{H}(\mathcal{B}(b,s),s))\right] + \mu(b,s) \tag{B.3}$$

$$zF_n(1, \mathcal{H}(b,s), \nu(b,s)) = G'(\mathcal{H}(b,s)) \tag{B.4}$$

$$zF_\nu(1, \mathcal{H}(b,s), \nu(b,s)) = p_\nu(1 + \theta\mu(b,s)/u'(\mathcal{C}(b,s))) \tag{B.5}$$

$$\begin{aligned} qu'(c - G(h)) &= \beta \mathbb{E}_{s'|s}\{u'(\mathcal{C}(b',s') - G'(\mathcal{H}(b,s)))(\mathcal{Q}(b',s') + z'F_k(1, \mathcal{H}(b',s'), \nu(b',s'))) \\ &\quad + \kappa' \mu(b',s') \mathcal{Q}(b',s')\} \end{aligned} \tag{B.6}$$

The algorithm follow these steps:

1. Generate a discrete grid for the economy's bond position $G_b = \{b_1, b_2, ... b_M\}$ and the shock state space $G_s = \{s_1, s_2, ... s_N\}$ and choose an interpolation scheme for evaluating the functions outside the grid of bonds. For the grid for bonds, we choose a uniformly spaced grid and interpolate the functions using a piecewise linear approximation.

2. Conjecture $\mathcal{B}_k(b,s), \mathcal{Q}_k(b,s), \mathcal{C}_k(b,s), \mathcal{H}_k(b,s), \nu_k(b,s), \mu_k(b,s)$ at time $K, \forall b \in G_b$ and $\forall s \in G_s$

3. Set $j = 1$

4. Solve for the values of $\mathcal{B}_{k-j}(b,s), \mathcal{Q}_{k-j}(b,s), \mathcal{C}_{k-j}(b,s), (b,s), \mu_{k-j}(b,s)$ at time $k-j$ using (B.1)-(B.6) and $\mathcal{B}_{k-j+1}(b,s), \mathcal{Q}_{k-j+1}(b,s), \mathcal{C}_{k-j+1}(b,s)$ $\mathcal{H}_{k-j+1}(b,s), \mu_{k-j+1}(b,s) \forall\ b \in G_b$ and $\forall\ s \in G_s$:

   (a) Assume collateral constraint (B.2) is not binding. Set $\mu_{k-j}(b,s) = 0$ and solve for

$\mathcal{H}_{k-j}(b,s)$ and ν using (B.4) and (B.5). Solve for $\mathcal{B}_{k-j}(b,s)$ and $\mathcal{C}_{k-j}(b,s)$ using (B.1) and(B.3) and a root finding algorithm.

(b) Check whether $-\frac{\mathcal{B}_{k-j}(b,s)}{R} + \theta p_\nu \nu_{k-j}(b,s) \leq \kappa \mathcal{Q}_{k-j+1}(b,s)$ holds. Notice that this step uses the asset price from the previous iteration to determine whether the collateral constraint is binding. Of course, once policies and prices converge, this becomes innocuous. The advantage from this formulation is that it avoids solving through iterations for an additional market clearing price, which may or may not be unique. We conducted several robustness checks in this dimension like starting from a different initial guess for the equilibrium. Moreover, it would be straightforward to alter our method by solving for possibly multiple Q that satisfy $-\frac{\mathcal{B}_{k-j}(b,s)}{R} + \theta p_\nu \nu_{k-j}(b,s) = \kappa \mathcal{Q}_{k-j+1}$ with a certain equilibrium selection, e.g. the one that maximizes the utility of the representative agent.

(c) If constraint is satisfied, move to next grid point.

(d) Otherwise, solve for $\mu(b,s), \nu_{k-j}(b,s), \mathcal{H}_{k-j}(b,s), \mathcal{B}_{k-j}(b,s)$ using (B.2, (B.3) and(B.4) with equality.

(e) Solve for $\mathcal{Q}_{k-j}(b,s)$ using (B.6)

5. Evaluate convergence. If $sup_{B,s}\|x_{k-j}(b,s) - x_{k-j+1}(b,s)\| < \epsilon$ for $x = \mathcal{B}, \mathcal{C}, \mathcal{Q}, \mu, \mathcal{H}$ we have found the competitive equilibrium. Otherwise, set $x_{k-j}(b,s) = x_{k-j+1}(b,s)$ and $j \rightsquigarrow j+1$ and go to step 4.

B.2 Numerical Solution Method for Constrained-Efficient Equilibrium

From a methodological standpoint, the solution method we developed is related to the literature using Markov perfect equilibria to solve for optimal time-consistent policy. In particular, we extended the methods proposed in Klein et al. (2008) and Klein et al. (2007) to models with an occasionally binding collateral constraint. The algorithm we propose uses a nested fixed point algorithm. Given future policies, we solve for policy functions and value functions using value function iteration as an inner loop. In the outer loop, we update future policies given the solution to the Bellman equation. The algorithm follows these steps:

1. Generate a discrete grid for the economy's bond position $G_b = \{b_{1,}b_2, ...b_M\}$ and the shock state space $G_s = \{s_{1,}s_2, ...s_N\}$ and choose an interpolation scheme for evaluating the functions outside the grid of bonds. We use the same grid as DE and interpolate the functions using a piecewise linear approximation.

2. Guess policy functions $\mathcal{B}, \mathcal{Q}, \mathcal{C}, \nu, \mu$ at time K $\forall$ $b \in G_b$ and $\forall$ $z \in G_z$. We use as initial policies the policies of the decentralized equilibrium, and we check that we obtain the same equilibrium when starting from alternative policies.

3. For given $\mathcal{C}, \mathcal{Q}, \mathcal{H}, \nu, \mu$ solve for the value function and policy functions :

$$\begin{aligned}
V(b', s') &= \max_{c,b',\mu,h,\nu} u(c - G(h)) + \beta \mathbb{E}_{s'|s} V(b', s') && \text{(B.7)}\\
c + \frac{b'}{R} &= b + zF(k, h, \nu) - p_\nu \nu && \text{(B.8)}\\
zF_h(k, h, \nu) &= G'(h) && \text{(B.9)}\\
zF_\nu(k, h, \nu) &= p_\nu \left(1 + \frac{\theta\mu}{u'(c - G(h))}\right) && \text{(B.10)}\\
\mu\left(\frac{b'}{R} - \theta p_\nu \nu + \kappa q\right) &= 0 && \text{(B.11)}\\
\frac{b'}{R} - \theta p_\nu \nu &\geq -\kappa q && \text{(B.12)}
\end{aligned}$$

$$\begin{aligned}
qu'(c - G(h)) &= \beta \mathbb{E}_{s'|s} \{u'(\mathcal{C}(b', s') - G'(\mathcal{H}(b, s)))(\mathcal{Q}(b', s') + z'F_k(1, \mathcal{H}(b', s'), \nu(b', s'))\\
&\quad + \kappa' \mu(b', s')\mathcal{Q}(b', s')\} && \text{(B.13)}
\end{aligned}$$

This recursive problem is solved using value function iteration. The value functions and policy functions are approximated using linear interpolation whenever the bond position is not in the grid. To solve the optimal choices in each state, we first assume the collateral constraint is not binding and use a Newton type algorithm to solve the optimization problem.

If the collateral constraint is binding, we solve for every b', the combinations of c, h, ν, q, μ that satisfy these 6 conditions (B.8)-(B.13), with (B.13) holding with equality.

4. Denote by $\sigma^i, i = c, q, h, \nu, \mu$ the policy functions that solve the recursive problem in step (3) Compute the sup distance between $\mathcal{B}, \mathcal{Q}, \mathcal{C}, \nu, \mu$ and $\sigma^i, i = c, q, h, \nu, \mu$. If the sup distance is higher than 1.0e-6, update $\mathcal{B}, \mathcal{Q}, \mathcal{C}, \nu, \mu$ and solve again the recursive problem.

C Model with Separate Households and Firms

We describe here a setup in which households and firms are modeled separately and is isomorphic to the model studied in the paper with the representative firm-household as a single agent.

C.1 Household Problem

Households choose consumption, holdings of stocks, bond holdings and labor supply to maximize

$$\max_{\{c_t, h_t, b^H_{t+1}, s_{t+1}\}} \mathbb{E} \sum_{t=0}^{\infty} \beta^t u(c_t - G(h_t) + \phi_t k^H)$$

subject to

$$\begin{aligned} s_{t+1}p_t + c_t + \frac{b^H_{t+1}}{R_t} &\leq s_t(d_t + p_t) + b^H_t + w_t h_t && \text{(C.1)} \\ b^H_{t+1} &\geq -\kappa_t q_t k^H && \text{(C.2)} \end{aligned}$$

where ϕ captures preference for housing, s_t represents the holdings of firm shares and p_t represents the price of firm shares. We assume that the stock of housing owned by households is constant.

The first-order conditions are:

$$\begin{aligned} p_t u'(t) &= \beta \mathbb{E}_t u'(t+1)(d_{t+1} + p_{t+1}) && \text{(C.3)} \\ G'(h_t) &= w_t && \text{(C.4)} \\ u'(t) &= \beta R_t \mathbb{E}_t u'(t+1) + R\mu^H_t, && \text{(C.5)} \\ \mu^H_t(b^H_{t+1} + \kappa_t q_t k^H) &= 0 && \text{(C.6)} \\ \mu^H_t &\geq 0 && \text{(C.7)} \end{aligned}$$

where μ^H is the non-negative Lagrange multiplier on the household borrowing constraint.

C.2 Firms

The problem of the firm is to choose capital, labor, intermediate inputs, dividends and bond holdings to maximize equity value, which can be expressed as:

$$\max_{\{d_t, b^F_{t+1}, k^F_{t+1}, n_t, v_t\}} \quad \sum_{t=0}^{\infty} u'(c_t - G(h_t))d_t$$

$$d_t + q_t k^F_{t+1} + \frac{b^F_{t+1}}{R_t} \leq k^F_t q_t + F(z_t, k^F_t, n_t, v_t) - w_t n_t - p^v_t v_t + b^F_t \tag{C.8}$$

$$b^F_{t+1} - \theta p^v_t v_t \geq -\kappa_t q_t k^F_t \tag{C.9}$$

First order conditions are:

$$u'(t)q_t = \mathbb{E}_t u'(t+1)(F_k(t+1) + q_{t+1}) + \beta \mathbb{E}_t \mu^F_{t+1} q_{t+1} \tag{C.10}$$

$$F_n(t) = w_t \tag{C.11}$$

$$F_v(t) = p^v \left(1 + \theta \frac{\mu^F_t}{u'(t)}\right) \tag{C.12}$$

$$u'(t) = \beta R_t \mathbb{E}_t u'(t+1)) + R\mu^F_t, \tag{C.13}$$

$$\mu^F_t (b^F_{t+1} + \theta p^v_t v_t + \kappa_t q_t k^F_t) = 0 \tag{C.14}$$

$$\mu^F_t \geq 0 \tag{C.15}$$

where μ^F is the non-negative Lagrange multiplier on the firm collateral constraint.

C.3 Market Clearing and Competitive Equilibrium

Market clearing requires:

$$h_t = n_t \tag{C.16}$$

$$s_t = 1 \tag{C.17}$$

$$k^F_{t+1} + k^H = \bar{K} \tag{C.18}$$

Notice that (C.13) and (C.5) imply that $\mu^H_t = \mu^F_t$.

A competitive equilibrium in this economy is given by sequences of allocations $\{c_t, h_t, b^H_{t+1}, s_{t+1}, k^H, d_t, b^F_{t+1}, k^F_{t+1}, n_t, v_t, \mu^H_t\}$ and prices $\{q_t, w_t\}$ such that conditions (C.1)-(C.18) hold.

C.4 Firm-Household problem

$$\max_{\{c_t,b_{t+1},k_{t+1},n_t,v_t\}} \mathbb{E}\sum_{t=0}^{\infty}\beta^t u(c_t - G(n_t) + \phi_t k^H)$$

$$\begin{aligned} c_t + k_{t+1}q_t + \frac{b_{t+1}}{R_t} &\leq k_t q_t + F(z_t,k_t,n_t,v_t) + b_t && \text{(C.19)} \\ b_{t+1} - \theta p_v v &\geq -\kappa q_t(k^H + k_t) && \text{(C.20)} \end{aligned}$$

The first-order conditions are:

$$\begin{aligned} u'(t)q_t &= \beta\mathbb{E}_t u'(t+1)(F_k(t+1) + q_{t+1}) + \beta\mathbb{E}_t\mu_{t+1}q_{t+1} && \text{(C.21)} \\ F_n(t) &= G'(n_t) && \text{(C.22)} \\ u'(t) &= \beta R_t\mathbb{E}_t u'(t+1) + R_t\mu_t, && \text{(C.23)} \\ \mu_t(b_{t+1} - \theta p_v v + \kappa_t q_t(k^H + k_t)) &= 0 && \text{(C.24)} \\ \mu_t &\geq 0 && \text{(C.25)} \end{aligned}$$

where μ_t denotes the Lagrange multiplier on the consolidated collateral constraint.

A competitive equilibrium in this economy is given by a sequence of allocations $\{c_t, n_t, b_{t+1}, k^F_{t+1}, \mu_t\}$ and prices $\{q_t, w_t\}$ such that conditions (C.19)-(C.25) hold.

C.5 Equivalence

Proposition IV *If $\{c_t, h_t, b^H_{t+1}, s_t, k^H, d_t, b^F_{t+1}, k^F_{t+1}, n_t, v_t, w_t, q_t, \mu^H_t\}$ is a competitive equilibrium allocation in the economy with separate households and firms, then $\{c_t, n_t, b_{t+1}, k^F_{t+1}, v_t, w_t, q_t, \mu_t\}$ is a competitive equilibrium in an economy with a representative firm-household with $b_{t+1} = b^H_{t+1} + b^F_{t+1}, k_{t+1} = k^F_{t+1}$.*

Conversely, if $\{c_t, n_t, b_{t+1}, k^F_{t+1}, v_t, w_t, q_t, \mu_t\}$ is a competitive equilibrium allocation in an economy with a representative firm-household, then there exists prices $\{w_t, q_t, \mu^H_t\}$ such that $\{c_t, h_t, b^H_{t+1}, s_t, k^H, d_t, b^F_{t+1}\}$, $\{k^F_{t+1}, n_t, v_t\}$ is a competitive equilibrium allocation in the economy with separate households and firms, with $b_{t+1} = b^H_{t+1} + b^F_{t+1}, k_{t+1} = k^F_{t+1}$.

Proof: " $\rightarrow$" Suppose $\{c_t, h_t, b^H_{t+1}, s_{t+1}, k^H, d_t, b^F_{t+1}, k^F_{t+1}, n_t, v_t\}$ is a competitive equilibrium for an economy with separate households and firms. Then we have to show that conditions (C.1)-(C.15) imply that (C.19)-(C.25) also hold. This follows from simple inspection: Budget constraint of the household firm (C.19) follows by combining (C.1),(C.17) and (C.8). The firm-household collateral constraint (C.20) follows from combining (C.2),(C.9) and $b_{t+1} = b^H_{t+1} + b^F_{t+1}$. Labor market condition (C.22) follows from (C.4) and (C.11). Euler equations and complementary slackness conditions (C.23)-(C.25) follow from (C.5)-(C.7).

□

Proof: " $\leftarrow$" Suppose $\{c_t, n_t, b_{t+1}, k^F_{t+1}\}$ is a competitive equilibrium allocation in an economy with a representative firm-household. Then, we have to show that (C.19)-(C.25) imply that equations (C.1)-(C.15) also hold. This follows from simple inspection after constructing prices as follows $w = G'(h)$, q_t so that $p_t u'(t) = \beta \mathbb{E}_t u'(t+1)(d_{t+1} + p_{t+1}), u'(t)q_t = \beta \mathbb{E}_t u'(t+1)(F_k(t+1) + q_{t+1}) + \beta \mathbb{E}_t \mu_{t+1} q_{t+1}$ Set b^H_{t+1}, b^F_{t+1} such that $b^H_{t+1} \leq \kappa q_t k^H, b^F_{t+1} - \theta p^v_t v_t \geq -\kappa_t q_t k^F_t$ and $b^F_{t+1} = b_{t+1} - b^H_{t+1}$. Set $s_{t+1} = 1$ and $d_t = F(z_t, k^F, n_t, v_t) - w_t n_t - p^v_t v_t + b^F_t - \frac{b^F_{t+1}}{R_t}$. Market clearing in labor market (C.16) and optimality of labor demand (C.11) follows from $w = G'(h)$. and (C.22). Euler equation and complementary slackness condition (C.5)-(C.7) follow from (C.23)-(C.25).

D Investment

This section shows that the qualitative insights of the model with capital in fixed supply extend to a model with investment and capital adjustment cost. In order to have k_{t+1} units of capital ready for production in period $t+1$, an agent with k_t^o units of used capital needs to employ $\phi(k_{t+1}, k_t^o)$ units of the consumption good at date t.

There is a competitive market for used capital, where agents can buy and sell capital at the price q_t^o, after production has taken place. To simplify the exposition, we assume no intermediate inputs and a borrowing constraint $\frac{b_{t+1}}{R_t} \geq -\kappa_t q_t \bar{K}$ where $\bar{K}$ is the aggregate capital stock at steady state and q is the price of newly installed capital readily available to produce in the following period.

The budget constraint is as follows

$$c_t + \frac{b_{t+1}}{R_t} + q_t^o k_t^o + \phi(k_{t+1}, k_t^o) + q\tilde{k}_{t+1} \leq b_t + z_t F(k_t, h_t) + q_t^o k_t + q k_{t+1}$$

According to this, households sell old capital k_t and newly installed capital k_{t+1}, and use bonds and production to buy newly installed capital $\tilde{k}_{t+1}$, old capital k_t^o, consume and issue new bonds. This allows for the possibility of $k_t^o \neq k_t$ and $k_{t+1} \neq \tilde{k}_{t+1}$. However, market clearing in the used capital market requires $k_t^o = k_t, k_{t+1} = \tilde{k}_{t+1}$.

Recursive representation— Let $X = (B, K, s)$ denote the aggregate state of the economy. The recursive optimization problem of agents is given by:

$$V(b,k,X) = \max_{c,k^o,\tilde{k}',b',h,k_{t+1}} u(c - G(h)) + \beta \mathbb{E}_{s'|s} V(b', k', X') \quad \text{(D.1)}$$

$$c + \frac{b'}{R_t} + \phi(k', k^o) + q^o(X)k^o + q(X)\tilde{k}' = b + zF(k,h) + q^o(X)k + q(X)k' \quad \text{(D.2)}$$

$$\frac{b'}{R_t} \geq -\kappa q(X)\bar{K} \quad \text{(D.3)}$$

Optimality conditions are:

$$u'(c_t - G(h_t)) = \beta R \mathbb{E}_t[u'(c_{t+1} - G(h_{t+1}))] + \mu \quad \text{(D.4)}$$

$$G'(h) = F_h(k,h) \quad \text{(D.5)}$$

$$q^o(X) = \phi_2(\tilde{k}'^o) \quad \text{(D.6)}$$

$$q_t = \phi_1(k_{t+1}, k_t^o) \quad \text{(D.7)}$$

$$u'(c_t - G(h_t))q_t = \beta \mathbb{E}_t[u'(c_{t+1} - G(h_{t+1}))\{q_{t+1}^o + z_{t+1}F_k(k_{t+1}, h_{t+1})\}] \quad \text{(D.8)}$$

Market clearing implies that the resource constraint is

$$c+\frac{b'}{R}+\phi(k',k^{o})=b+zF(k,h) \tag{D.9}$$

D.1 Optimal Time Consistent Planner's Problem

As in the planner's problem of the paper, the planner chooses directly borrowing on behalf of the households, and lets all markets clear competitively. That is, the planner chooses allocations and prices subject to a set of implementability constraints given by (D.5)-(D.9).

Under discretion, the planner takes future policies for capital $\mathcal{K}$, consumpion $\mathcal{C}$, bonds $\mathcal{B}$ as given and solves the following problem:

$$\begin{aligned}
\mathcal{V}(b,k,s) &= \max_{c,k',b',h} u(c-G(h))+\beta\mathbb{E}_{s'|s}\mathcal{V}(b',k',s') \\
c+\frac{b'}{R}+\phi(k',k^{o}) &= b+zF(k,h) \\
\frac{b'}{R} &\geq -\kappa\phi_1(k',k)k_t \\
G'(h) &= F_h(k,h) \\
u'(c-G(h))\phi_1(k',k) &= \beta\mathbb{E}_{s'|s}[u'(\mathcal{C}(b',k',X')-G(\mathcal{H}(b',k',X'))) \{\phi_2(\mathcal{K}(b',k',X'),k')) \\
&\quad +z'F_k(k',\mathcal{H}(b',k',X'))\}]
\end{aligned}$$

Using first-order conditions and envelope condition, the Euler equation for bonds is:

$$\begin{aligned}
u'(c_t-G(h_t)) &= \beta R\mathbb{E}_t\big[u'(\mathcal{C}(b_{t+1},k_{t+1})-G(\mathcal{H}(b_{t+1},k_{t+1})) \\
&\quad -\xi_{t+1}\phi_1(\mathcal{K}(b_{t+1},k_{t+1},X),k_{t+1})u''(\mathcal{C}(b_{t+1},k_{t+1},X)-G(\mathcal{H}(b_{t+1},k_{t+1},X)))+\xi_t\Omega_t\big] \\
&\quad +\xi_t\phi_1(k_{t+1},k_t)u''(c_t-G(h_t))+\mu_t^*
\end{aligned}$$

where $\Omega\equiv\mathbb{E}_{s'|s}[u''(\mathcal{C}(b',k',X')-G(\mathcal{H}(b',k',X')))\{\phi_2(\mathcal{K}(b',k',X'),k'))+z'F_k(\bar{K},\mathcal{H}(b',k',s'))\}(\mathcal{C}_b(b',k',s')$ $-G'(\mathcal{H}(b',k',s'))\mathcal{H}_b(b',k',s'))+u'(\mathcal{C}(b',k',s')-G(\mathcal{H}(b',k',s')))(\phi_{21}(\mathcal{K}(b',k',s'),k')\mathcal{K}_b(b',k',s')$ $+z'\{F_{kk}(\bar{K},\mathcal{H}(b',k',s'))\mathcal{K}_b(b',k',s'))+F_{kh}(\bar{K},\mathcal{H}(b',k',s'))\mathcal{H}_b(b',k',s'))\}]$.

Suppose that the implementability constraint is slack today but binds tomorrow, i.e $\xi_t=0$ but $\xi_{t+1}\geq 0$. This yields

$$\begin{aligned}
u'(c_t-G(h_t)) &= \beta R\mathbb{E}_t[u'(\mathcal{C}(b_{t+1},k_{t+1},X)-G(\mathcal{H}(b_{t+1},k_{t+1},X))) && \text{(D.10)}\\
&\quad -\xi_{t+1}\phi_1(\mathcal{K}(b_{t+1},k_{t+1},X),k_{t+1})u''(\mathcal{C}(b_{t+1},k_{t+1},X)] && \text{(D.11)}
\end{aligned}$$

Just like in the condition (14) of the paper, there is a positive wedge between the marginal cost of borrowing from the planner and households when the constraint is not binding at t but is expected to bind at $t + 1$.

E Commitment

This section provides more details about the analysis under commitment. There are two sections. Section E.1 derives some theoretical results and E.2 provides a numerical analysis.

E.1 Theoretical Results

Under commitment, the planner chooses at time 0 once and for all $\{c_t, b_{t+1}, q_t, h_t, v_t, \mu_{t+1}\}_{t\geq 0}$ to solve the following problem:

$$\max_{\{c_t, b_{t+1}, q_t, h_t, v_t, \mu_t\}_{t\geq 0}} \mathbb{E}_t \sum_{t=0}^{\infty} \beta^t u(c_t - G(h_t))$$

$$\begin{aligned}
c_t + \frac{b_{t+1}}{R} &\leq b_t + z_t F(1, h_t, v_t) - p_t^v v_t && \text{(E.1)}\\
z_t F_h(1, h_t, v_t) &= G'(h_t) && \text{(E.2)}\\
z_t F_v(1, h_t, v_t) &= p_t^v \left(1 + \frac{\theta \mu_t}{u'(c_t - G(h_t))}\right) && \text{(E.3)}\\
\frac{b_{t+1}}{R} - \theta p_t^v v_t &\geq -\kappa_t q_t && \text{(E.4)}\\
u'(c_t - G(h_t)) q_t &= \beta \mathbb{E}_t \Big[u'(c_{t+1} - G(h_{t+1}))(q_{t+1} + z_{t+1} F_k(1, h_{t+1}, v_{t+1})) \\
&\quad + \kappa_{t+1} \mu_{t+1} q_{t+1} \Big] && \text{(E.5)}\\
\mu_t \left(\frac{b_{t+1}}{R} - \theta p_t^v v_t + \kappa_t q_t\right) &= 0 && \text{(E.6)}\\
\mu_t &\geq 0 && \text{(E.7)}
\end{aligned}$$

Let λ_t, ζ_t^h, ζ_t^v, μ_t^*, ξ_t, ν_t and χ_t denote the lagrange multipliers on constraints (E.1)-(E.7) respectively. First-order conditions with respect to c_t, b_{t+1}, q_t, h_t, v_t, and μ_t are:

$$c_t :: \qquad \lambda_t = u'(t) - \xi_t u''(t) q_t + \xi_{t-1} u''(t)(q_t + z_t F_k(t) + \kappa_t \mu_t q_t) \tag{E.8}$$

$$b_{t+1} :: \qquad \lambda_t = \beta R_t \mathbb{E}_t \lambda_{t+1} + \mu_t^* + \mu_t \nu_t \tag{E.9}$$

$$q_t :: \qquad \xi_t = \xi_{t-1}(1 + \kappa_t \mu_t) + \frac{\kappa_t(\mu_t \nu_t + \mu_t^*)}{u'(t)} \tag{E.10}$$

$$h_t :: \qquad z_tF_h(t) = G'(h_t) + \frac{1}{\lambda_t}\left[\zeta_t^h[z_tF_{hh}(t) - G''(h)] + \zeta_t^v z_tF_{vh}(t) - \xi_{t-1}u'(t)z_tF_{kh}(t)\right] \tag{E.11}$$

$$v_t :: \qquad z_tF_v(t) = \frac{1}{\lambda_t}\left[p_t^v[\lambda_t + \theta\mu_t^* + \theta\mu_t\nu_t] + \zeta_t^h z_tF_{hv}(t) + \zeta_t^v z_tF_{vv}(t) - \xi_{t-1}u'(t)z_tF_{kv}(t)\right] \tag{E.12}$$

$$\mu_t \;:: \qquad \xi_{t-1}\kappa_tq_tu'(t) + \chi_t + \zeta_t^v p^v\theta = -\nu_t\left(\frac{b_{t+1}}{R} - \theta p_t^v\nu_t + \kappa_tq_t\right) \tag{E.13}$$

Notice that conditions (E.8)-(E.10) correspond to conditions (19)-(21 in Section 2.6.

Complementary slackness conditions is:

$$\mu_t^*\left(\frac{b_{t+1}}{R} - \theta p_t^v v_t + \kappa_tq_t\right) = 0 \tag{E.14}$$

$$\chi_t\mu_t = 0 \tag{E.15}$$

The Euler equation for bonds is:

$$\begin{aligned} u'(t) &= \beta R\mathbb{E}_t[u'(t+1) - \xi_{t+1}u''(t+1)q_{t+1} + \xi_tu''(t+1)(q_{t+1}(1+\kappa_{t+1}\mu_{t+1}) + z_{t+1}F_k(t+1))\;] \\ &\quad +\xi_tu''(t)q_t - \xi_{t-1}u''(t)(q_t(1+\kappa_t\mu_t) + z_tF_k(t)) - \frac{\zeta_t^\nu p_t^v\theta\mu_tu''(t)}{u'(t)^2} + \mu_t^* + \nu_t\mu_t \end{aligned} \tag{E.16}$$

Following the same steps as in A.2 and assuming for illustration purposes that the implementability constraints (E.11), (E.12), and (E.13) are not binding, the macro-prudential tax on debt under commitment $\tau_t^{MP,C}$ is given by:

$$\tau_t^{MP,C} = \frac{-\mathbb{E}_t\frac{\kappa\mu_{t+1}}{u'(c_{t+1})}u''(c_{t+1})q_{t+1} + \xi_{t-1}(\mathbb{E}_tu''(c_{t+1})z_{t+1} - z_tu''(c_t))}{\mathbb{E}_tu'(c_{t+1})} \tag{E.17}$$

Compared to the tax in the markov perfect equilibrium (17), the tax under commitment features another term that relates to previous commitments as given by the second term on the right hand side of (E.17).

E.2 Numerical Results

A numerical solution of the model under commitment requires to find appropriate state variables to keep track of previous promises. Notice that ξ follows an increasing sequence over time and hence is not an appropriate state variable for a numerical solution.

Building on Kydland and Prescott (1980), one can transform the planner's problem to express it recursively expanding the states to include consumption and asset prices. For simplicity, we consider here a production function of the form $F(k) = zk$, a collateral constraint that depends on the market value of aggregate assets and only productivity shocks. The recursive problem for $t > 0$ can be expressed as follows

$$
\begin{aligned}
V(b, q, c, z) \;&=\; \max_{b', q(z'), c(z')} u(c) + \beta \mathbb{E}_{z'} V(b', q(z'), c(z'), z') \\
c \; + b'/R \;&\leq\; b + z \\
b' \;&\geq\; -\kappa(s) q \\
q u'(c) \;&\geq\; \beta \mathbb{E} c(z')(z' + q(z'))
\end{aligned}
\tag{E.18}
$$

with states $(b, q, c, z) \in A$ where A is the largest feasible set for (b, q, c, z), i.e., the largest set such that one can find a sequence b_{t+1}, c_t, q_t satisfying all the constraints.

At $t = 0$, the planner is not bound by past promises of consumption and asset prices. The time 0 problem consists of choosing $q(b, s), c(b, s)$ that maximize $V(b, q, c, s)$

$$
\max_{c,q} V(b, q, c, s) \tag{E.19}
$$

To solve the model, we construct a grid for the three endogenous state variables b, q, c of dimension NB, NQ and NC respectively and a grid of dimension NS for the exogenous state variables. Notice that since the planner is choosing asset prices and consumption for each possible value of the shock tomorrow, the dimension of the control space for each combination of state variables is $NS \times NB \times NQ \times NC + 1$. To keep the numerical solution manageable, we use relatively coarse grids of $NS = 2, NB = 20, NQ = 10, NC = 20$.

Figure 9 shows the value function, and policy functions for asset prices, consumption, and bonds at $t = 0$ and compares them with the decentralized equilibrium. That is, we first solve (E.18) and obtain the policy function $b'(b, q, c, s)$ and value function $V'(b, q, c, s)$, and then find c^*, q^* that solve (E.19). Figure 9 plots the solution for c^*, q^* together with the associate value function, and the bond policy that solves (E.18) for c^*, q^* that solve (E.19) .

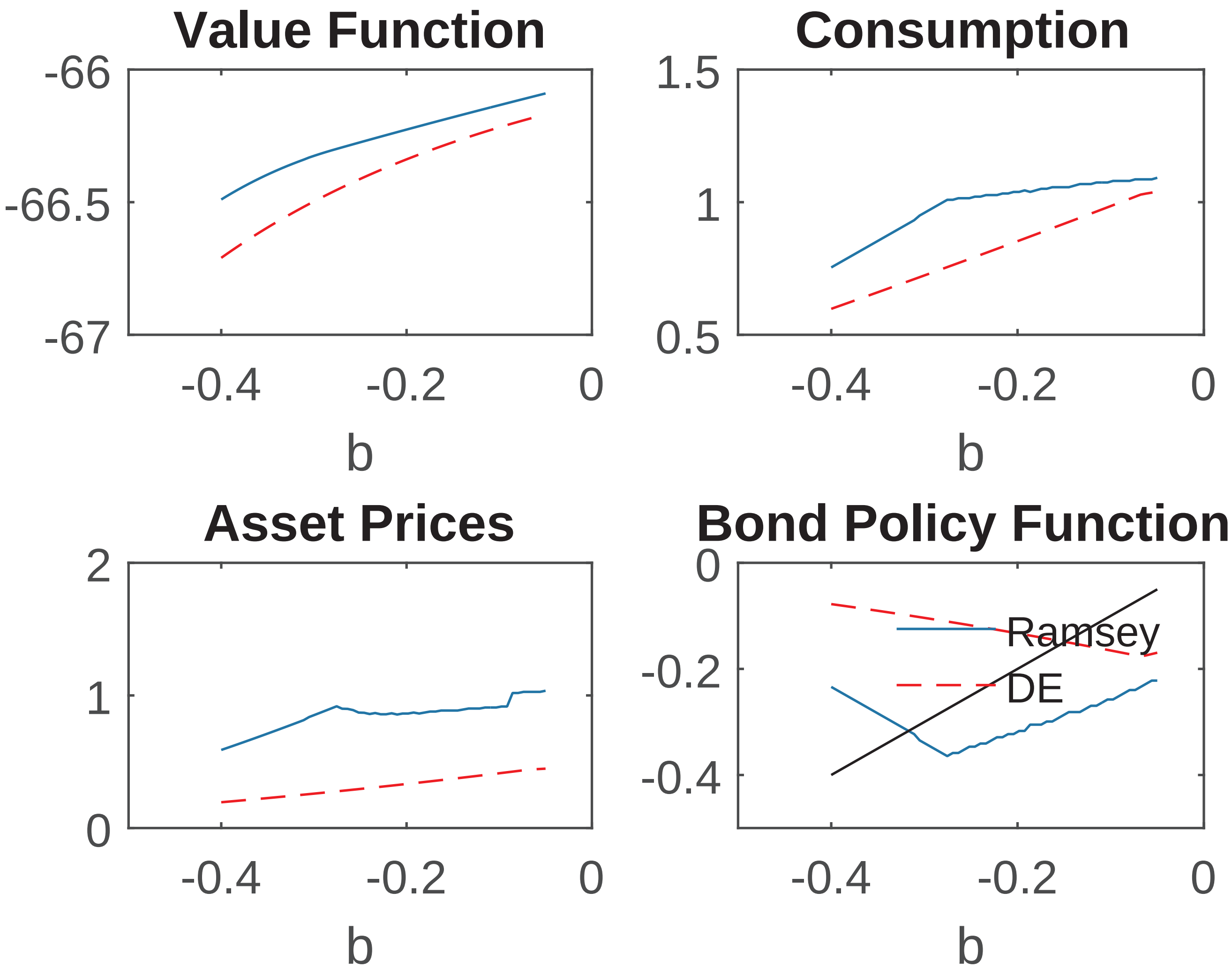


Figure 9: Policy Functions under Commitment vs Decentralized Equilibrium

Note: Dashed lines represent optimal macroprudential policy under commitment.

F Data Appendix

F.1 Data Sources

- Net Foreign Asset Position (NFA): *Flow of Funds*
- Total Credit: *Survey of Terms of Business Lending* and *Flow of Funds*
- Intermediate Inputs: United Nations *UNdata*
- GDP: OECD National Accounts Statistics

F.2 Frequency and Duration of Financial Crises

To construct estimates of the duration and frequency of financial crises, we applied the methodology proposed by Forbes and Warnock (2012) to identify the timing and duration of sharp changes in financial conditions. A financial crisis is defined as an event in which the cyclical component of the linearly-detrended current account is above two-standard deviations from its mean. Since the current account is the overall measure of financing of the economy vis-a-vis the rest of the world, this unusually large current accounts represent unusually large drops in foreign financing. The starting (ending) dates of the events are set in the year within the previous (following) two years in which the current account first rose (fell) above (below) one standard deviation. Using the data for all OECD countries over the 1984Q1-2012Q4 period, we obtained financial crises with a frequency of 4 percent and a mean duration of 1 year.[31]

Table 4 indicates the list of all crises events identified with this methodology and Figures F.1-F.3 show the data for the current account for each country.

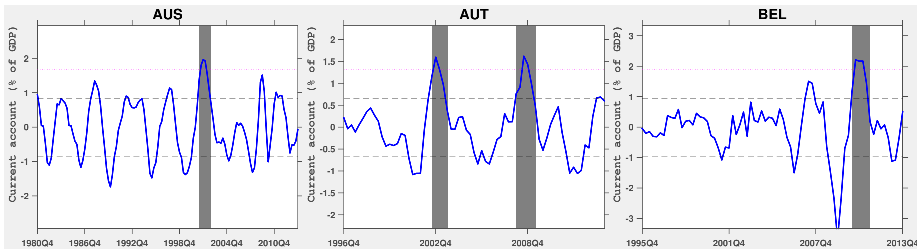
[31]The data are quarterly, and the sample length varies across countries, with shorter samples for emerging economies . Because of this, some well-known events are not identified because they are outside the sample period (e.g. data for Chile start in 2003, which means that the 1982 financial crisis is excluded).

Table 4: Financial Crises Episodes.

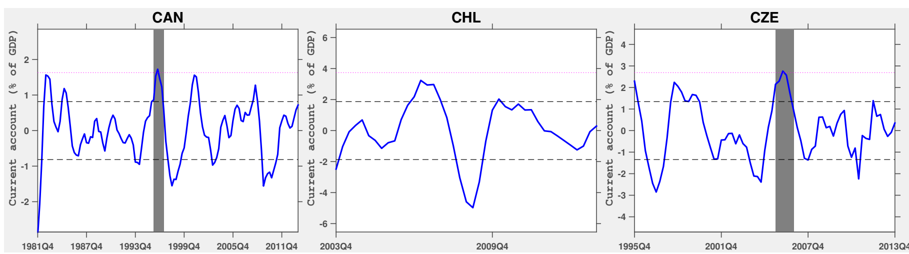
Australia	[2001Q2 - 2002Q3]
Austria	[2002Q3 - 2003Q2], [2008Q1 - 2009Q1]
Belgium	[2010Q2 - 2011Q2]
Canada	[1996Q1 - 1997Q1]
Chile	
Czech Republic	[2005Q3 - 2006Q3]
Denmark	[2005Q4 - 2006Q3], [2010Q3 - 2011Q2]
Estonia	[2009Q3 - 2010Q4]
Finland	[1995Q1 - 1996Q2]
France	[1999Q1 - 2000Q1]
Germany	[1989Q1 - 1990Q4]
Greece	
Hungary	[2001Q4 - 2002Q3], [2009Q4 - 2011Q1]
Iceland	
Ireland	[2003Q2 - 2004Q4]
Israel	[2006Q2 - 2007Q3], [2010Q1 - 2010Q4]
Italy	
Japan	[2007Q2 - 2008Q2]
Korea, Republic of	[2004Q1 - 2005Q1]
Luxembourg	[2005Q1 - 2005Q3]
Mexico	[1983Q2 - 1984Q3], [1995Q4 - 1997Q2]
Netherlands	[2006Q1 - 2007Q1]
New Zealand	[2009Q3 - 2010Q4]
Norway	[2000Q3 - 2002Q1]
Poland	
Portugal	[2003Q1 - 2004Q3]
Slovak Republic	[1995Q4 - 2000Q4]
Slovenia	
Spain	
Sweden	[2006Q4 - 2008Q4]
Switzerland	[2010Q1 - 2011Q1]
Turkey	[1994Q4 - 1995Q2], [2001Q4 - 2002Q3], [2009Q2 - 2010Q2]
United Kingdom	
United States	[1991Q1 - 1992Q2], [2009Q2 - 2010Q2]

Figure F.1: Financial Crises

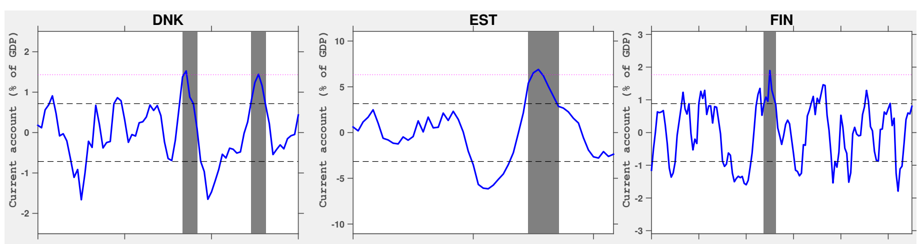
(a) Australia **(b)** Austria **(c)** Belgium



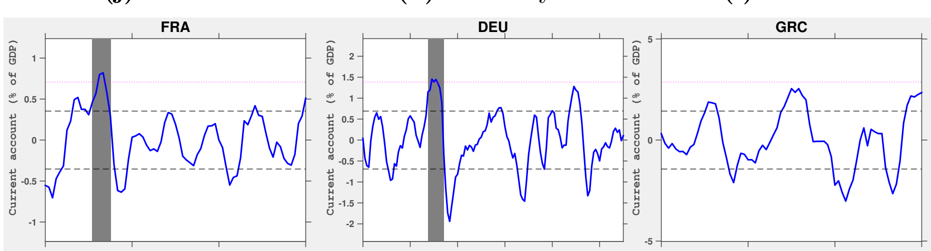
(d) Canada **(e)** Chile **(f)** Czech



(g) Denmark **(h)** Estonia **(i)** Finland

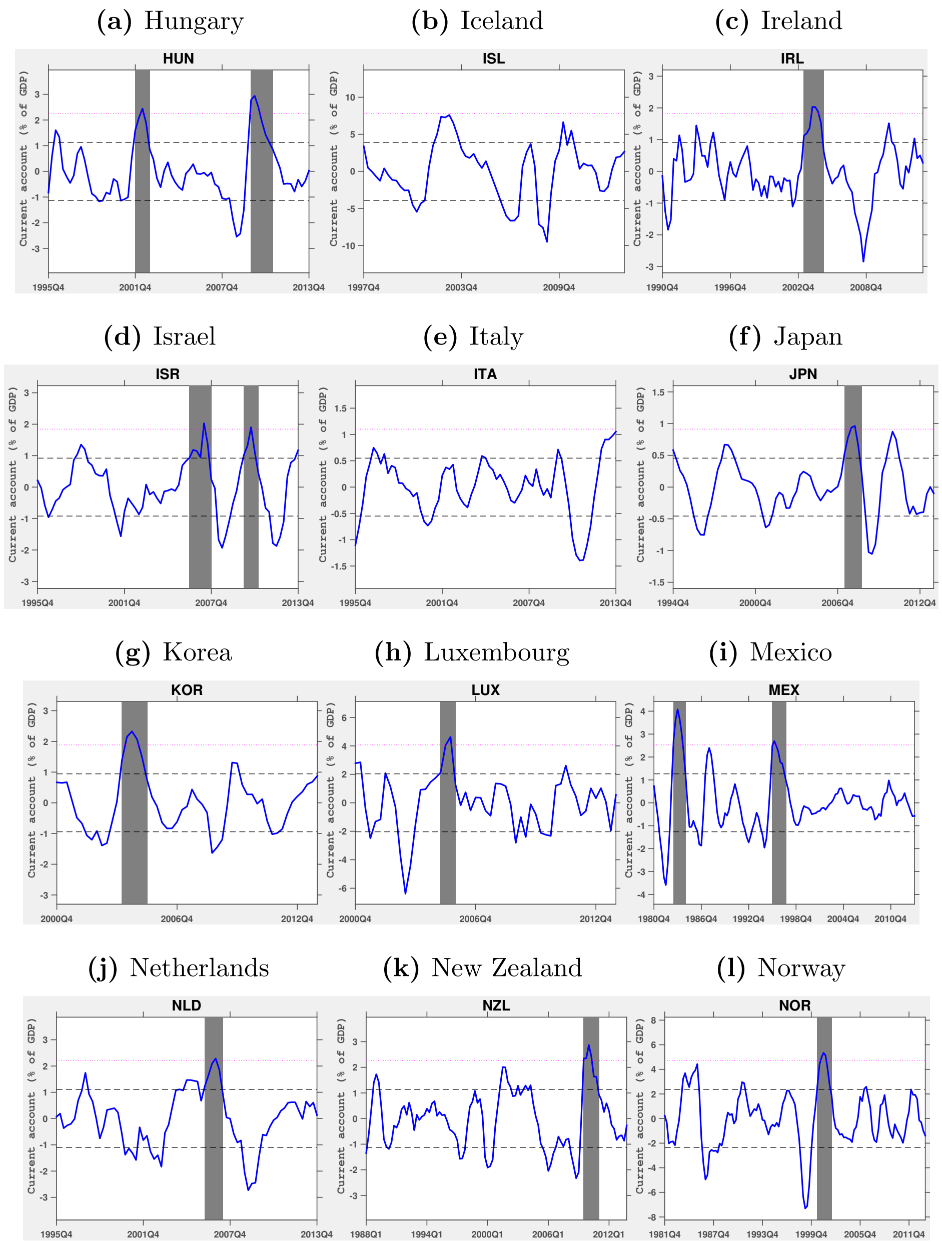


(j) France **(k)** Germany **(l)** Greece



Ratio of cumulated net capital inflows over the last four quarters to cumulated GDP over the last four quarters. Shaded areas describe periods of Financial Crises

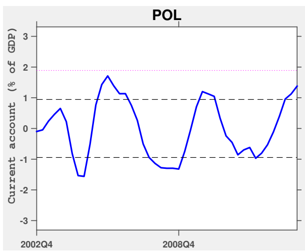
Figure F.2: Financial Crises



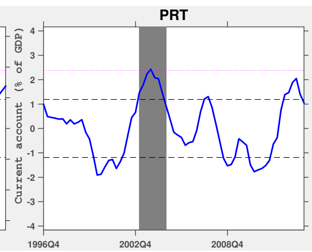
Ratio of cumulated net capital inflows over the last four quarters to cumulated GDP over the last four quarters. Shaded areas describe periods of Financial Crises

Figure F.3: Financial Crises

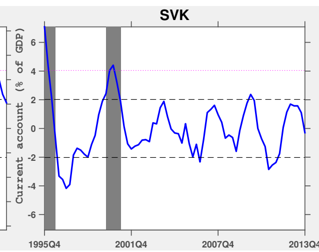
(a) Poland



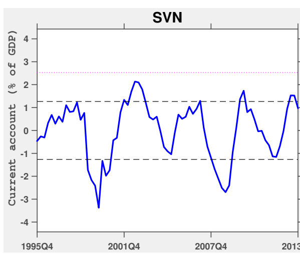
(b) Portugal



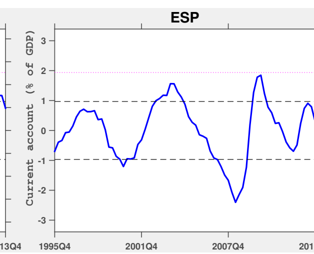
(c) Slovak Republic



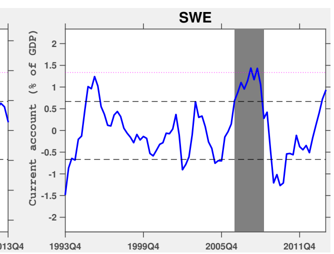
(d) Slovenia



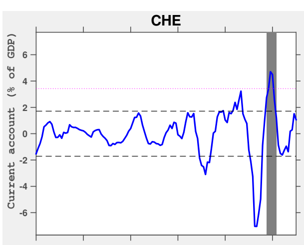
(e) Spain



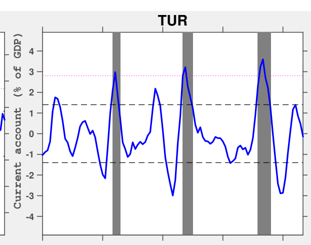
(f) Sweden



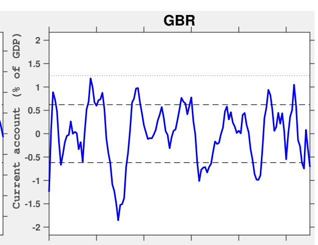
(g) Switzerland



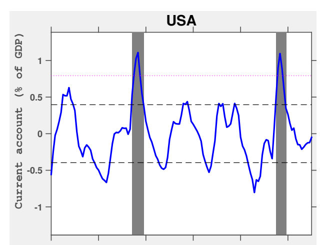
(h) Turkey



(i) United Kingdom



(j) United States



Ratio of cumulated net capital inflows over the last four quarters to cumulated GDP over the last four quarters. Shaded areas describe periods of Financial Crises

G Great Recession Experiment

In section 3.2, we conduct ed an event analysis designed to show how average financial crises look in DE and conduct counterfactual experiments to show the effectiveness of macroprudential policy. In this section, we examine instead the DE model's predicted time-series dynamics leading up to and including the global financial crisis event using a window spanning the 2000-2009 period. We compare these dynamics with the observed dynamics in the United States and Europe, and with a counterfactual of what the event would have looked like under the SP's optimal policy.[32] The results for three of the model's key variables (asset prices, output and the current account) are shown in Figure 1.

To generate the simulated data for the DE and SP we need to set an initial condition for b, and values for the realizations of TFP, R and κ for the ten years in the event window. The initial condition for b is set equal to the private NFA-GDP ratio of the United States observed in 2000, which was -11.6 percent. The values of the interest-rate shocks are set to their observed realizations during 2000-2009 and the values of the TFP shocks are set so as to match the observed deviation from linear trend of the U.S. real GDP in the same period—we use interpolation over the realizations included in the Markov approximation of the AR(1) process of TFP). The values of κ are set to κ^H for 2000-2008 and κ^L for 2009. The DE and SP decision rules for bonds, together with the recursive functions that map the values of bonds and the shocks into equilibrium prices and allocations, are then used to generate the plots shown in the first column of Figure 1.

Comparing the DE model's crisis dynamics with the U.S. and European data, Panels (a)-(c) of the Figure 1 show that the model does quite well at tracking the dynamics of asset prices, particularly for the United States, which is the country used to set the initial conditions and shock realizations for the model simulations. The timing of the crash in the asset price is off by one year, because we set the change to κ^L and the lowest realization of TFP in 2009, to be consistent with the fact that the lowest deviation from trend in GDP was observed in 2009. We could calibrate to 2008 instead and then the crash in the asset price would be in the same year as in the data. The current account in the DE and the U.S. data share the feature that the financial crisis is associated with a sharp current account reversal. The reversal, however, is much large in the model than in the data, which is partly due to the fact that debt in the model is only one-period debt, while actual U.S. net foreign assets include significant positions in long-term instruments.[33] For the same reason, the large current account reversal implies a decline in private consumption larger than what was observed in U.S. data.

As figure G.1 shows, the policy not only prevents the asset price crash, but in addition it

[32]Data for Europe represents simple average of European Union—the source is Eurostat.

[33]The model also abstracts from government policies that were put in place to offset the credit crunch (see e.g. Gertler and Kiyotaki (2010) and Bianchi (2012)).

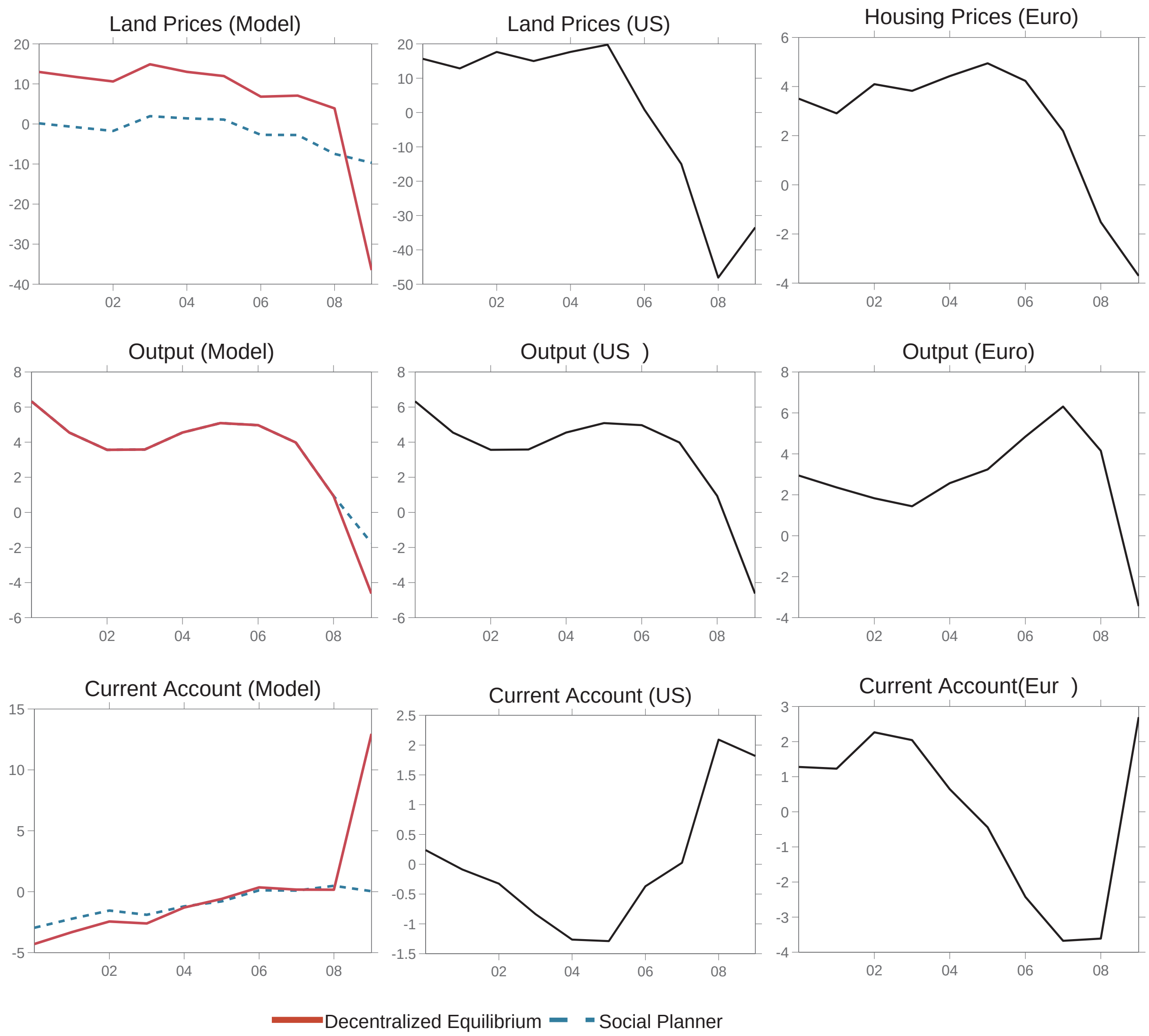


Figure G.1: Comparison of Crises Dynamics

Note: In the model, Land Prices and output expressed as a percentage deviation of mean values for decentralized equilibrium. Data values are expressed as deviation from a linear trend over the period 1984-2010.

produces lower and more stable asset prices for the entire ten-year period. The output dynamics are identical across the two economies before the crisis, because they experience the same TFP shocks calibrated to replicate the path of output, but when the crisis hits output falls less in the SP because the collateral constraint is less binding, and hence implies a smaller cutback in working capital financing. The DE shows slightly larger current account deficits than SP from 2002 to 2004, from then until the crisis hits the two are about the same, and then when the crisis hits the SP avoids the current account reversal completely. The larger initial deficits of the DE reflect the incentive to overborrow that private agents have because of the effect of the pecuniary externality. The differences in current accounts are small, which means that debt positions pre-crisis are not

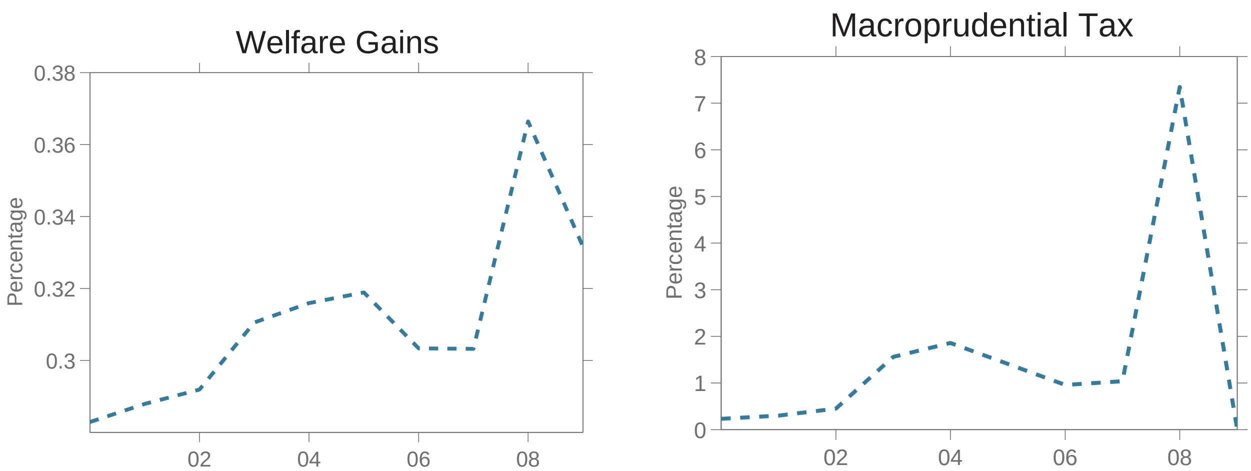


Figure G.2: Macroprudential Tax and Welfare Gains

all that different, but as we show later, small differences in debt positions between the DE and SP result in large differences in macro outcomes, because the calibrated model features a strong financial amplification mechanism.

Figure G.2 shows the time-series dynamics of the optimal tax and the welfare gains (i.e. these are values conditional on each year instead of the averages shown in Table 2). The tax increases first gradually and then sharply to about 7 percent just before the crisis. The welfare gain of the optimal policy follows a very similar pattern, and reaches a maximum of 36 basis points the year before the crisis.

H Early Warnings

We examine whether it is feasible to construct a parsimonious statistical framework that yields accurate "early warnings" of financial crises by conducting an experiment similar to the one Boissay et al. (2015) proposed. We produce a 500,000-observations time-series simulation of the DE, which includes roughly 20,000 crisis events (since the probability of crises is 4 percent). This yields a time-series of the one-step-ahead probability of observing a crisis at t+1 conditional on date t. Then we select a cutoff value such that the model issues a crisis warning when the probability exceeds the cutoff. The criterion for setting the cutoff is that the warnings be statistically accurate, in the sense that Type-1 or Type-2 errors are in the 95 percentile.[34] Type-1 errors occur when the model does not issue a warning at t but a crisis occurs at $t+1$ in the simulated data (i.e. the model failed to predict a crisis). Type-2 errors occur when the model issues a warning at t but a crisis does not occur at $t+1$ (i.e. it wrongly predicted a crisis). At the 95 percentile for Type-1 (Type-2) errors, there should not be more than 5 percent of errors of that type. Table 4 shows that the model produces warnings with this accuracy if the probability cutoffs are set to 1.8 percent for Type-1 errors and 8.6 percent for Type-2 errors.

The above represents the "best" early warning system attainable, in the sense that the warnings are based on the true model's crisis probabilities. Since in practice these probabilities are unobservable, however, they cannot be used directly to build early-warning indicators, as Boissay et al. (2015) noted. Hence, we examine whether logit regressions using the model's fundamentals as explanatory variables can do as well as the model in terms of the fractions of Type-1 and Type-2 errors generated when issuing crisis warnings. In these regressions, the dependent variable is a

[34] Boissay et al. applied the cutoff only to Type-2 errors, but in principle it can be applied to both.

Table 5: Type-1 and Type-2 Errors in Crises Warnings

	Model	Logit	
	Probability	All Regressors	Credit-Output
(i) 5 percent of Type-1 errors in the model			
Type 1	5.0	4.4	3.4
Type 2	63.7	69.1	73.2
(ii) 5 percent of Type-2 errors in the model			
Type 1	82.8	84.0	81.7
Type 2	5.0	7.9	9.2

Note: All values are expressed in percent. Warning probability cutoffs are 1.8 and 8.6 percent in scenarios (i) and (ii) respectively. "All Regressors" includes credit, asset prices and all exogenous shocks as explanatory variables

binary variable set to 0 when a crisis does not occur and 1 when it occurs (which is observable), and the independent variables enter in logs and with a one-period lag.

Table 4 shows Type-1 and Types-2 errors in crisis warnings obtained from two logit regressions using the 1.8 and 8.6 percent cutoffs produced by the model. One uses as regressors all of the model's state variables (TFP, interest rates, κ, and the bond position) together with GDP and asset prices, and the other uses only the ratio of total credit (bonds plus working capital) to GDP. Both of these regressions are good early-warning systems, because the fractions of Type-1 and Type-2 errors they produce are similar to the ones produced by the model, although the logit with all the regressors does slightly better. We also estimated alternative regressions with subsets of the regressors used in the first logit model, but they all produced larger fractions of Type-1 and Type-2 errors. This is in line with the finding of Boissay et al. (2015), showing that a logit regression using only the debt-GDP ratio approximates well model-based errors.

I Sensitivity

The supply side channel driven by working capital plays an important role for determining the large gains from macroprudential policy. Reducing θ by 25% reduces the welfare gains by about 33 %, i.e. welfare gains fall from 0.3 to 0.2 percentage points of permanent consumption. Hence, we conclude that large gains from macroprudential policy arise due to adverse effects of crises on economic activity, besides the gains from higher consumption smoothing. This links our results to Schmitt-Grohe and Uribe (2013) who show that capital controls achieve substantial gains by reducing average unemployment via reductions in volatility, and hence departing from standard Lucas welfare calculations.

In the second sensitivity experiment we modify the model to relax the assumption of a perfectly elastic supply of funds at an exogenous interest rate R_t. We argued that this is a natural assumption for many of the economies to which we calibrate our model, and is also a convenient assumption theoretically to abstract from redistribution effects. To see the robustness of our results to this assumption, we introduce a real interest rate which varies with aggregate bond holdings. In particular, we assume that the net interest rate is now given by $r(B') = r_t - \varrho(e^{-(B'-\bar{B})} - 1)$, where $\bar{B}$ denotes the average value of bond holdings and set $\varrho = 0.05$.[35] With $\varrho > 0$, the interest rate increases with the debt of the economy. In principle, this could work to attenuate the Fisherian deflation and the pecuniary externality, because of the endogenous self-correcting mechanism increasing the cost of borrowing as debt increases, but we found that quantitatively this did not result in large changes relative to the baseline.[36] This is shown in the last row of Table 6 for a value of $\varrho = 0.05$. With this value of ϱ, the real interest rate reaches a minimum of -1.5 percent in the simulations which is around the minimum value observed in the data between 1980 and 2012.

Finally, we consider an increase in the discount factor from $\beta = 0.95$ to $\beta = 0.96$. This makes agents less willing to borrow and the economy becomes relatively less exposed to crises. As a result, there are lower gains from macroprudential policy. Notice that higher patience also makes agents relatively more precautionary about future fluctuations, which in principle could make macroprudential policy more desirable. Quantitatively, however, the first effect dominates, as can be seen in Table 6.

[35] This specification is proposed by Schmitt-Grohe and Uribe (2003) to avoid the problem with the unit root in net foreign assets that arises when using perturbation methods to solve small open economy models. In our model, its purpose is only to approximate what would happen if the interest rate could respond to debt choices in a richer general equilibrium model.

[36] The average macroprudential debt tax is slightly higher because now the planner also internalizes how borrowing affects the interest rate, which is taken as exogenous by individual agents. The welfare gains of moving from the constrained efficient equilibrium with an exogenous interest rate to the decentralized equilibrium with the endogenous interest rate are about the same as in the baseline, which again suggests that the exogeneity of the interest rate does not have significant effects on the quantitative results.

Table 6: Sensitivity Analysis

	Macroprudential Debt Tax	Welfare Gains	Crisis Prob. DE	Crisis Prob. SP	Asset Price Drop DE	Asset Price Drop SP	Equity Premium DE	Equity Premium SP
Baseline	3.6	0.30	4.0	0.02	−43.9	−5.4	4.8	0.8
Low WK ($\theta = 0.12$)	3.6	0.21	5.4	0.02	−37.1	−5.6	4.1	1.0
Endogenous R	3.7	0.28	3.88	0.00	−43.1	−6.6	4.8	0.7
Higher Patience	3.1	0.20	2.65	0.02	−51.0	−10.4	4.62	0.54

J Asset Pricing

We report here more details on the asset pricing implications of the models and the implications of macroprudential policy. Figure J.1 shows plots of six key asset pricing variables as functions of B in the DE and SP economies when TFP and the interest rate take their average values and $\kappa = \kappa^H$ (this is in contrast with Figure 2, which plotted policy functions for a "bad" state with low TFP and κ^L). The variables plotted are the expected return on assets, the price of assets, the Sharpe ratio, the volatility of returns, the risk premium, and the price of risk.

In this Figure DE experiences higher risk premia, return volatilities, risk prices and Sharpe ratios due to the fact that the DE is significantly more risky than the SP economy. Moreover, differences with SP become larger for lower values of B since this implies that it is more likely that the collateral constraint will bind at $t+1$. On the other hand, expected return are higher for SP. The higher risk premia in DE should in principle push asset prices down by reducing excess returns. At equilibrium, however, this effect is more than offset by the first-order effect of the SP's debt tax, which by arbitrage of returns between assets and bonds this tax increases the expected return on assets. In turn, higher excess returns contribute to explain the uniformly lower asset prices of the SP relative to the DE *for all* the domain of B in panel (b), in line with eq. (8).

It is also important to note in Figure J.1 the significant nonlinearities in the asset pricing variables within the DE itself (and keeping in mind we are looking at these variables for realizations of shocks in a "good" state as of date t). In particular, in the region with a positive probability of a crisis at $t+1$, the Sharpe ratio, return volatility, risk premium and price of risk are steep decreasing functions of B, while they are virtually flat in the stable credit region. Similarly, asset prices are a steeper function of B in the positive crisis probability region than in the stable credit region.

Table 7 reports statistics that characterize the main properties of asset pricing behavior in the DE and SP. The Table shows expected excess returns ($\mathbb{E}_t[R^q_{t+1}]$) in column (1) and its two components, namely the after-tax risk free rate and the equity premium (R^{ep}_t) in columns (2) and (3) respectively. Using eq. (9), R^{ep}_t is decomposed into the two components that result from the effect of collateral constraints that bind at t (column (4)) or are expected to bind at $t+1$ (column (5)), and the standard risk premium component given by the covariance between the stochastic discount factor and asset returns (column (6)). The equity premium in column (3) is equal to the sum of these three components. In addition, the Table reports the market price of risk (column (7)), the log standard deviation of returns (column (8)) and the Sharpe ratio (column (9)). All of these statistics are reported for the unconditional long-run distributions of each economy as well as for distributions conditional on the collateral constraint being binding and not binding.

The unconditional equity premium is significantly higher in the DE than in the SP by about

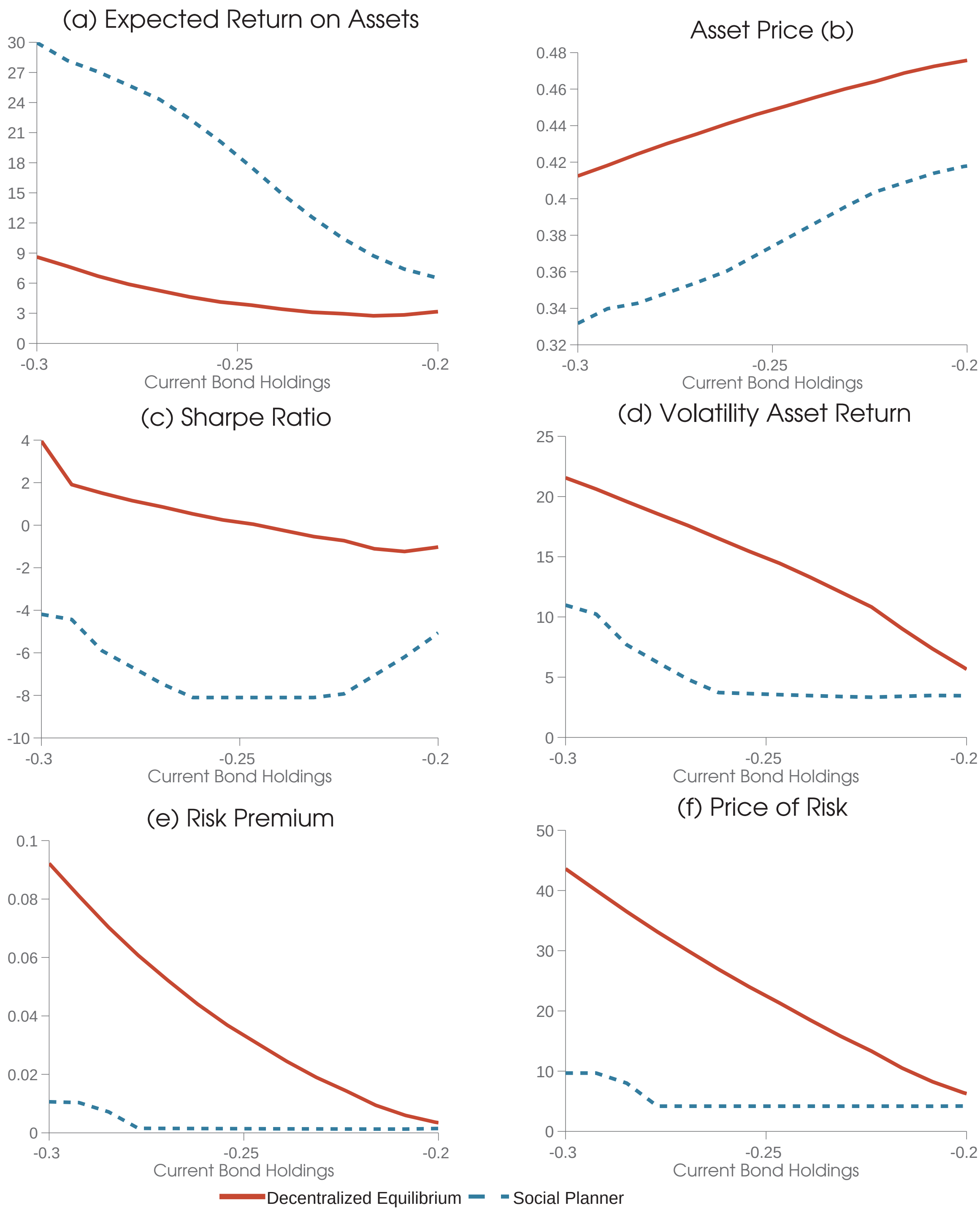


Figure J.1: Asset Pricing Variables in "Good" States of Nature

400 basis points (4.8 v. 0.8 percent).[37] This difference is due to both higher risk premium (1.5 vs. 0.2 percent) and higher liquidity premium (4.7 vs. 1.8 percent). Moreover, the difference in equity

[37] the sizable equity premium in the DE contrasts sharply with existing findings (e.g. Heaton and Lucas (1996)) showing that credit frictions without the Fisherian deflation mechanism do not produce large premia.

Table 7: Asset Pricing Moments

	(1) Expected Return	(2) Risk-free Plus Tax	(3) Equity Premium	(4) Liquidity Premium	(5) Collateral Effect	(6) Risk Premium	(7) Price of Risk	(8) $\sigma_t(R^q_{t+1})$	(9) SR_t
				Decentralized Equilibrium					
Unconditional	6.0	1.2	4.8	4.7	1.4	1.5	14.6	9.1	0.5
Constrained	85.6	1.2	84.4	84.1	0.0	0.2	4.1	6.2	13.7
Unconstrained	1.3	1.2	0.1	0.0	1.5	1.6	15.3	9.3	0.0
				Social Planner					
Unconditional	4.1	3.3	0.8	1.8	1.2	0.2	5.2	3.8	0.2
Constrained	6.9	-21.8	28.7	28.5	0.0	0.2	5.1	3.8	7.6
Unconstrained	3.9	5.0	-1.2	0.0	1.3	0.2	5.2	3.8	-0.3

Note: The Sharpe ratio for each row is computed as the average of all corresponding equilibrium realizations of excess returns divided by the standard deviation of excess returns. All figures except the Sharpe ratios are in percent.

premium can be attributed to both higher volatility (14.6 vs 5.2) and higher price of risk (14.6 vs 5.2). Higher price of risk in DE reflects the fact that consumption, and therefore the pricing kernel, fluctuate significantly more in the former than in the latter. Moreover, financial crises episodes introduce higher volatility in asset prices and asset returns. Finally, the unconditional Sharpe ratio of the DE is 0.5 v. 0.2 in the SP, implying that risk-taking is “overcompensated” in the competitive equilibrium relative to the compensation it receives under the optimal policy. The differences in asset pricing statistics between DE and SP apply to constrained and unconstrained region, as Table 7 shows.

K Comparison with the work of Jeanne & Korinek

Bianchi and Mendoza (BM) and Jeanne and Korinek (JK) developed models of optimal macro-prudential policy in which assets valued at market prices serve as collateral, and hence in both the competitive equilibrium is distorted by pecuniary externalities that can be tackled with debt taxes.[38] This Section of the Appendix reviews the differences between the two studies and provides a comparative analysis with numerical examples and formal proofs illustrating important differences in the optimal policy problems.

K.1 Differences in Model Structure and Results

The models proposed by BM and JK differ in several parts of their structure and in the formulation of the optimal policy problems. In terms of model structure, the models differ in that in the BM setup individual assets are used as collateral, production is endogenous, and working capital loans subject to the collateral constraint are used to pay for a fraction of the cost of inputs. In contrast, in the JK setup aggregate assets are used as collateral by individual borrowers and output is an exogenous stochastic process. In addition, the credit constraints differ in that in the JK model the constraint is specified as the sum of the fraction of assets pledgeable as collateral plus an exogenous constant.

The differences in the structure of the models imply that the BM setup has three key features absent from the JK model: (a) agents value asset holdings for their role in relaxing the collateral constraint when they formulate their optimal plans, (b) the constraint generates inefficiencies in factor allocations and production, and (c) financial crises have effects on both aggregate supply and demand. The quantitative findings are also different, because BM find that there is strong financial amplification affecting output, consumption and debt via the Fisherian deflation of asset prices and large pecuniary externalities, which result in optimal macroprudential policy having large effects on the probability and magnitude of crises. In contrast, as explained in Section 3.4 of the paper, in the quantitative analysis conducted by JK, the constant term in the credit limit dwarfs the fraction of the value of assets that serve as collateral, and the probability of crises equals the exogenous probability of a low-output regime. As a result, they find that debt taxes cannot affect the probability of crises and have small effects on their magnitude.

[38] Work on both projects started in the late 2000s, with the first working papers issued in 2010: Bianchi, J. and E.G. Mendoza (2010), Overborrowing, Financial Crises and 'Macro-prudential' Taxes, NBER WP No. 16091, June 2010. Jeanne, O. and A. Korinek (2010), Managing Credit Booms and Busts: A Pigouvian Taxation Approach, NBER WP 16377, September 2010. The last revision by BM is the paper to which this Appendix belongs, and the last revision by JK is Jeanne and Korinek (2012), http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.352.4554&rep=rep1&type=pdf.

K.2 Comparative Analysis of Planner Problems

The BM and JK studies pose social planner problems that differ critically in whether the planner has the ability to influence asset prices when the collateral constraint binds, which in turn results in differences in how optimal policy is constructed. In the planner's problem studied by JK, the planner's date-t debt choice is not allowed to affect date-t asset prices when the constraint binds. The planner only considers how its debt choice affects prices at $t+1$. Effectively, today's borrowing capacity is predetermined with respect to today's debt choice. In contrast, in BM's formulation the planer sets its date-t debt choice considering its effects on asset prices at both t and $t+1$. This is important because, as discussed in Section 2 of the main text, the time-inconsistency problem of macroprudential policy under commitment originates precisely in the planner's ability to affect asset prices contemporaneously when the constraint binds, and this ability is also an important feature of the optimal time-consistent policy. The different planning problems result in different optimality conditions (compare eq. (14) in the main text of this paper v. eq. (19) in JK (2010)), and different optimal tax results (compare Prop. 1 and eq (17) in this paper v. eq. (21) in JK (2010)). We provide below theoretical results and numerical examples demonstrating that the two planner problems are not equivalent and that the solution of the planner's problem in the JK setup can be suboptimal in the BM model.

In order to compare the JK and BM planner problems, one can interpret the former as one that uses a "reduced-form pricing function" to value collateral. For the two to be equivalent, the two planner problems should yield analogous optimality conditions, which would also imply analogous optimal debt tax expressions. This could be the case if two assumptions that JK impose on the reduced-form pricing function are satisfied. First, the asset price is increasing in net worth (p. 10 of Jeanne and Korinek (2010)) states that "one can see that if the price of the asset is increasing with aggregate net liquid wealth the social planner raises saving"). Second, Assumption 1 from p. 9 of Jeanne and Korinek (2010), which states that the reduced-form pricing function is differentiable and satisfies $\kappa \frac{\partial \hat{q}(b,z,c)}{\partial c} < 1$. These, however, are assumptions about an endogenous equilibrium outcome, namely the asset pricing function, and in fact their validity turns out to be difficult to establish. In particular, we show below that the two planner problems are not equivalent and that Assumption 1 is invalid in general for the CRRA utility function that both JK and BM use, including for the parameterization considered in JK's work. Moreover, since Assumption 1 is a non-parametric assumption, it would need to be verified numerically by solving first for the equilibrium price of the "true" Markov perfect equilibrium without reduced-form pricing function for a particular state space and parameterization.

K.2.1 The two planner problems

We start by re-writing simplified versions of the planning problems in JK and BM using a common notation. As in Appendix A3 in Jeanne and Korinek (2010), the JK planner's problem is represented by the following Lagrangian (with choice variable b_{t+1} and ignoring the constant term in the JK borrowing constraint for simplicity):

$$L = \mathbb{E}_t \sum_{t=0}^{\infty} \beta^t \left[u\left(z_t + b_t - \frac{b_{t+1}}{R} \right) + \mu_t \left(\frac{b_{t+1}}{R} + \kappa \bar{q}(b,z) \right) \right] \tag{K.1}$$

μ is the Lagrange multiplier on the collateral constraint, and $\bar{q}$ is defined as the price of the asset *when the constraint is binding* (see p. 12 of Jeanne and Korinek (2010)).

The reduced-form pricing function $\bar{q}(b,z)$ is taken as given by the planner, but in the planner's optimum it satisfies the Euler equation for assets of the competitive equilibrium. It is implicit in this treatment that the planner does not have incentives to alter the price implied by the function $\bar{q}$ when the constraint binds at t. This planner cares for how the debt choice affects "tomorrow's" asset price ($\partial \bar{q}'(b',z')/\partial b' \neq 0$) but not for how it affects "today's" price (i.e. $\partial \bar{q}(b,z)/\partial b' = 0$, trivially since b' is not an argument of $\bar{q}(b,z)$). Since this is a property of an endogenous equilibrium object (the pricing function), it would need to be proved or at least verified numerically. Intuitively, because this property makes the borrowing capacity of date t predetermined with respect to the choice of b_{t+1}, the planner is prevented from choosing optimally on this margin when the constraint binds at t, because the planner does not realize how that debt choice can be used to influence date-t prices and borrowing capacity.

In BM planner's problem, the planner does care about the effects of b' on asset prices when the credit constraint binds "today." The planner's problem was formulated in Problem 2 (p. 14) using the standard, explicit approach to formulate optimal policy problems without commitment as Markov perfect equilibria, but here we rewrite it in simpler form as follows. Take as given a conjecture of the policy rule for bonds of future planners $\mathcal{B}(b,z)$, and the associated consumption allocations $\mathcal{C}(b,z)$ and asset prices $\mathcal{Q}(b,z)$, the BM planner's problem is characterized by the following Bellman equation:

$$\begin{aligned} \mathcal{V}(b,z) &= \max_{c,b',q} u\left(\frac{b'}{R} - b + z \right) + \beta \mathbb{E}_{z'|z} \mathcal{V}(b',z') \\ \frac{b'}{R} &\geq -\kappa q \\ u'(c)q &= \beta \mathbb{E}_{z'|z} u'\left(b' + z' - \frac{\mathcal{B}(b',z')}{R} \right) \left(\mathcal{Q}(b',z') + z' \right) \end{aligned} \tag{K.2}$$

Note that using the above pricing condition, the collateral constraint of this problem can be re-written as:

$$\frac{b'}{R} \geq -\kappa \tilde{q}(b, b', z)$$

where

$$\tilde{q}(b, b', z) \quad = \frac{\beta \mathbb{E}_{z'|z} u'\left(b' + z' - \frac{\mathcal{B}(b',z')}{R}\right)\left(\mathcal{Q}(b', z') + z'\right)}{u'(c)} \tag{K.3}$$

The above constraints make evident the key difference between the BM and JK planner problems: In the BM case, the planner internalizes that the choice of b' influences current consumption and, through the resulting effect on the stochastic discount factor, it affects current asset prices and hence current borrowing capacity. In contrast, in the JK planner's problem (K.1) the date-t asset price and borrowing capacity are predetermined. At equilibrium it is true that in the BM setup $\tilde{q}(b, b', z) = q(b, z)$, but this condition is not imposed *before* actually solving the time-consistent planner's problem. This treatment is also more general, because it can capture solutions in which numerically it turns out that the effects of the debt choice on asset prices when the constraint binds are negligible and solutions in which they are not, and it allows us to establish properties like the sign of the macroprudential debt tax from primitives like the concavity of the utility function.

K.2.2 Non-equivalence of the planner problems

For the JK planner's problem (K.1) to be equivalent to the BM planner's problem (K.2), it must be that optimal policies solve (K.1) *if and only if* they solve (K.2). Before discussing the mathematical analysis showing that this does not hold in general, we provide the economic intuition behind the result why the equivalence can fail in *both* directions. First, the solution to JK planner's problem may still be time-inconsistent (i.e is not a solution to the BM problem). To see this, consider the reduced-form $\bar{q}(b, z)$ that the JK planner takes as given. At any point in time, the planner may have incentives to deviate from the implied asset price $\bar{q}(b, z)$ so as to affect current asset prices, as formally established below. The JK planner takes borrowing capacity of the current period as predetermined, and does not realize that it has a choice despite having a binding constraint. In contrast, BM's planner faces a borrowing capacity given by $\tilde{q}(b, b', z)$, and hence it knows that when choosing b' it can indeed affect asset prices. Second, using the pricing function of the true MPE that solves the planner's problem in BM as a reduced-form price in the JK setup might still leave the JK planner with the incentive to deviate. To see this, recall that the optimal time-consistent policy internalizes how additional borrowing raises the asset price, while the reduced-form planner takes it as given and hence perceives a lower marginal benefit from borrowing, which could lead it to borrow less.

One may be tempted to conjecture that the JK planner's problem supports the time-consistent solution of the BM setup, because the planner at time t takes as given the policy functions in future periods. Note, however, that this is necessary but not sufficient to ensure time consistency. It is not sufficient in particular for ensuring that the Markov stationarity condition of the MPE holds, because the assumption that the planner takes the policy functions of future planners as given does not guarantee that the optimal choices of future planners will be consistent with those policy functions that the current planner assumed. This would need to be proven in order to demonstrate that the JK planner's solution satisfies the Markov stationarity condition. But in fact, we show below that Markov stationarity fails precisely because the JK planner is not allowed to internalize the effect of the choice of b_{t+1} on q_t.

The rationale for arguing that under Assumption 1 of JK the planner problems are equivalent is based on the following heuristic argument. The two approaches would be equivalent if at equilibrium when the collateral constraint binds at t the planner faces no relevant debt choice. Jeanne and Korinek (2012) state that this is the case if Assumption 1 holds: "This assumption guarantees that when the social planner reduces aggregate debt, the collateral constraint (14) is relaxed." In turn, this would imply that there is only one optimal debt choice for the planner that can satisfy the constraint with equality, as excess borrowing capacity would be a monotonically decreasing function of debt. In this case, since at equilibrium $\tilde{q}(b, b', z) = q(b, z)$, it is "inessential" to impose the borrowing constraint $\frac{b'}{R} \geq -\kappa q(b, z)$ as opposed to $\frac{b'}{R} \geq -\kappa \tilde{q}(b, b', z)$.

The above argument, however, has two important drawbacks. First, in the optimal time-consistent planner's problem that BM study, the planner faces more than one debt choice that can satisfy the collateral constraint with equality, and these different choices differ in terms of social value. Second, Assumption 1 is invalid under the CRRA preferences that both BM and JK use. We prove these two results and illustrate them numerically. These findings imply that the way in which the collateral constraint is treated *is* essential and leads to suboptimal choices for the JK planner, because it is prevented from internalizing how b' affects the current equilibrium price when the constraint binds.

If the two planner problems are equivalent, it should be possible to demonstrate that the same optimality conditions characterize the planners' optimal plans under both solutions. We show next that this is not the case. Using Proposition 2 in Jeanne and Korinek (2012), and rearranging the Kuhn-Tucker conditions, the optimality conditions of JK's planner's problem are characterized by recursive functions $c(\cdot), q(\cdot), \mu(\cdot)$ such that:[39]

[39] Note also that JK denote the Lagrange multiplier of the credit constraint by λ whereas here we denoted it μ.

$$
\begin{aligned}
u'(c(b,z)) &= \mu(b,z) + \beta R \mathbb{E}_t \left[u'(c(b',z')) + \mu(b',z')\kappa \frac{\partial q}{\partial b'} \right] && \text{(K.4)} \\
u'(c(b,z))q(b,z) &= \beta \mathbb{E} u'(c(b'(b,z)),z'))[z' + q(b'(b,z),z')] && \text{(K.5)} \\
\frac{b'(b,z)}{R} + c(b,z) &= b + z && \text{(K.6)} \\
\mu(b,z)(b'(b,z) - \kappa q(b,z)) &= 0, \quad b'(b,z) \geq -\kappa q(b,z) && \text{(K.7)}
\end{aligned}
$$

In contrast, the optimality conditions of the BM planner's problem are given by

$$
\begin{aligned}
u'(c(b,z)) &= \beta R \mathbb{E}_t \Big\{ u'(\mathcal{C}(b_{t+1},z_{t+1})) - \frac{\kappa \mu_{t+1}}{u'(c_{t+1})} u''(\mathcal{C}(b_{t+1},z_{t+1}))\mathcal{Q}(b_{t+1},z_{t+1}) && \text{(K.8)} \\
&\quad + \frac{\kappa \mu(b,z)}{u'(c(b,z))} [u''(c(b,z))q(b,z) + \mathcal{Q}_b(b_{t+1},z_{t+1})u'(\mathcal{C}(b_{t+1},z_{t+1})) \\
&\quad \frac{\kappa \mu_t}{u'(c(b,z))} u''(\mathcal{C}(b_{t+1},z_{t+1}))\mathcal{C}_b(b_{t+1},z_{t+1})(\mathcal{Q}(b_{t+1},z_{t+1}) + z_{t+1})] + \mu(b,z) \Big\} \\
u'(c(b,z))q(b,z) &= \beta \mathbb{E} u'(\mathcal{C}(b'(b,z)),z'))[\mathcal{Q}(b'(b,z),z') + z'] && \text{(K.9)} \\
\frac{b'(b,z)}{R} + c(b,z) &= b + z && \text{(K.10)} \\
\mu(b,z)(b'(b,z) - \kappa q(b,z)) &= 0, \quad b'(b,z) \geq -\kappa q(b,z) && \text{(K.11)}
\end{aligned}
$$

There are obvious differences in the optimality conditions (4)-(7) v. (K.8)-(K.11) both when the collateral constraint binds and when it does not. When the collateral constraint binds at t and it has zero probability of binding at $t+1$, the JK planner faces a wedge between the marginal benefit of c and c' given by μ, indicating again that it takes the date-t borrowing capacity as predetermined. On the other hand, the wedge faced by the BM planner shows three additional terms that reflect the effects of the debt choice on current asset prices. The first term is $\frac{\kappa\mu(b,z)}{u'(c(b,z))}u''(c(b,z))q(b,z)$, which shows that one more unit of debt today rises current consumption and, via the resulting effect on the stochastic discount factor, rises the value of collateral and relaxes the collateral constraint. The second term is $\mathcal{Q}_b(b_{t+1},z_{t+1})u'(\mathcal{C}(b_{t+1},z_{t+1}))$, which shows that one more unit of debt reduces asset prices tomorrow, and the lower expected capital gains reduce today's asset price and tighten the constraint. The third term is $\frac{\kappa\mu_t}{u'(c_t)}u''(\mathcal{C}(b_{t+1},z_{t+1}))\mathcal{C}_b(b_{t+1},z_{t+1})(\mathcal{Q}(b_{t+1},z_{t+1}) + z_{t+1})$, which shows that one more unit of debt reduces consumption tomorrow, and again via the stochastic discount factor rises asset prices today and relaxes the constraint. For the two planner problems to be equivalent, these effects should vanish, but this does not hold in general because the borrowing capacity is not predetermined, it depends on the date-t debt choice b_{t+1} via its effects on consumption at t and $t+1$ and hence on the stochastic discount factor and asset prices at t, and thus the planner faces a relevant choice about consumption and debt when the constraint binds at date t.

When the collateral constraint is not binding at date t but may bind at $t+1$, the wedges in the planners' Euler equations have a similar intuition, but they are different expressions. In JK, the wedge is $\beta R\mathbb{E}_t\kappa\frac{\partial q(b_{t+1},z_{t+1})}{\partial b}\mu_{t+1}$ whereas in BM the wedge is $-\beta R\mathbb{E}_t\frac{\kappa\mu(\mathcal{B}(b,z),z')}{u'(\mathcal{C}(\mathcal{B}(b,z),z'))}u''(\mathcal{C}(b_{t+1},z_{t+1}))\mathcal{Q}(b_{t+1},z_{t+1})$. These wedges indicate that both planners internalize the effect of the debt choice of date t on the value of collateral at t+1 if the constraint becomes binding, and hence these are the wedges that determine the macroprudential component of the debt taxes that decentralize the planner problems (see equation (21) in Jeanne and Korinek (2010) and Proposition 1 in this paper). But the expressions are very different in that it is straightforward to prove that the wedge in the BM planner is strictly positive because of the concavity of the utility function, which demonstrates that the BM macroprudential debt tax is unambiguously positive, while the JK wedge depends on the assumed derivative of an equilibrium object (the pricing function). Jeanne and Korinek (2012) argue that the tax is positive *if* $\frac{\partial q(b',z')}{\partial b}>0$, but the latter is not proved (in p. 10 they write "...one can see that if the price of the asset is increasing with aggregate net liquid wealth...the social planner raises saving above the laissez-faire level, strictly so if there is a risk that the collateral constraint will bind in the next period").

If the two planner problems were equivalent, the wedges related to μ_{t+1} and the macroprudential debt taxes would be equivalent too. The above arguments show, however, that the two planner problems are not equivalent. To fully prove that the taxes are the same, one would need to prove that this holds:

$$\beta R\mathbb{E}_t\kappa\frac{\partial q(b',z')}{\partial b}\mu(b',z')=-\beta R\mathbb{E}_t\frac{\kappa\mu(\mathcal{B}(b,z),z')}{u'(\mathcal{C}(\mathcal{B}(b,z),z')}u''(\mathcal{C}(b_{t+1},z_{t+1}))\mathcal{Q}(b_{t+1},z_{t+1})$$

using $q(b,z)=\frac{\beta\mathbb{E}u'(c(b'(b,z)),z'))[z'+q(b'(b,z),z')]}{u'(c(b,z))}$. As noted before, this requires that (a) the incentives to influence date-t asset prices with the debt choice when the constraint binds at t vanish, and (b) that the assumed derivative of the reduced-form pricing function in the JK planner be identical to the derivative implicit in the equilibrium pricing kernel of the BM planner. A necessary condition for these properties to hold would be that the heuristic argument that there is no relevant debt choice when the constraint binds is correct.

K.2.3 Numerical example of non-equivalence

We now provide a numerical example showing that the solutions to the JK and BM planner problems are not equivalent, because in this example the BM planner faces a relevant debt choice when the constraint binds. The asset prices and bond positions used for this example were solved using the parameter values from Jeanne and Korinek (2010) and their specification of the borrowing

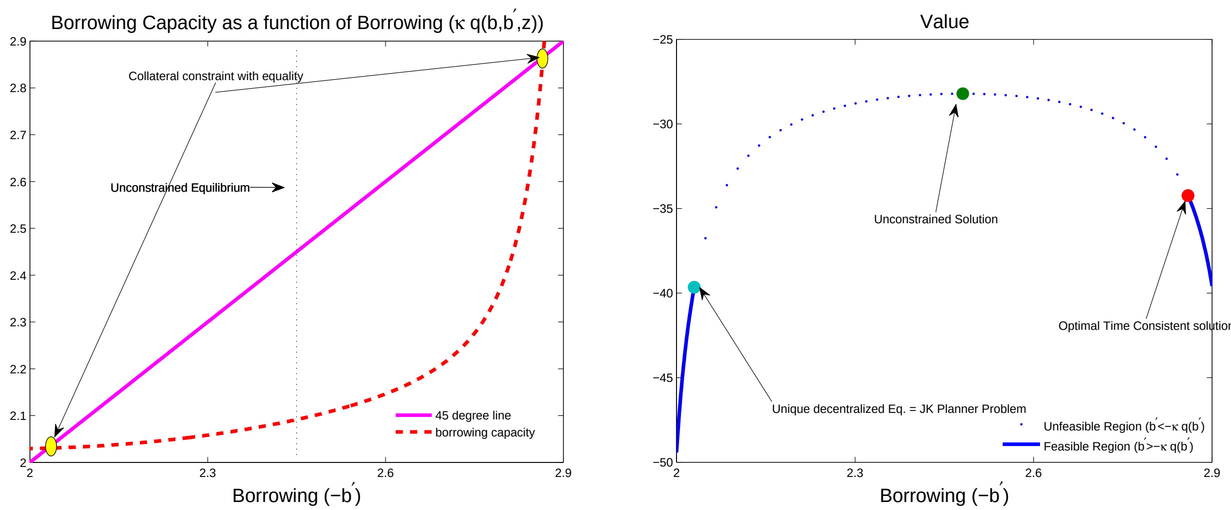


Figure K.1: Borrowing and Welfare for the BM and JK Planners

constraint.[40]

Figure 1 shows two plots. The left panel plots borrowing capacity (i.e. $\kappa\tilde{q}(b, b', z)$) as a function of the bond choice b' for given b, z, where $\tilde{q}$ is as defined in ($K.3$), together with the 45 degree line. A given b' satisfies the credit constraint if the borrowing capacity is greater or equal than the debt along the 45 degree line. The panel on the right plots the social welfare of the planner associated with each borrowing choice. There is a discontinuity for borrowing choices for which borrowing capacity is below the 45 degree line, because these are not feasible choices since they violate the credit constraint.

As indicated in the left panel, there are two values of the debt choice that satisfy the collateral constraint with equality, one higher and one lower than the debt amount that would be chosen in the absence of the credit constraint. Thus, the BM planner faces two possible debt choices to choose from when the constraint binds, while the JK planner is effectively forced to choose the one with less debt. Since the BM planner attains higher utility with the higher debt, this is its optimal choice. If the price implied by this optimal choice were given to the JK planner, it would choose the unconstrained solution, but this would then deliver a price that is different from the optimal time-consistent solution of the BP planner and it would violate the collateral constraint. Thus, in this example the solution of the planner's problem in JK is suboptimal and time-inconsistent.

These results are based on one parameterization, but they are more general. They hold also for the parameterizations in the BM papers. In addition, we prove below that the results also hold in general in a two-period model with CRRA preferences, and show a numerical example based on log utility.

[40] Parameter values are $\beta = 0.96, R = 1.03, \sigma = 2, z_L = 0.969 z_H = 1, \pi_L = 0 : 05\kappa = 0.046$, dividend parameter , $\alpha = 0.2$ and the extra term in the credit constraint $\psi = 1.97$.

K.2.4 Two-period model: numerical example & general results

We conclude by showing that, in general, the choices of JK's planner are suboptimal relative to the optimal time-consistent planner studied by BM in a simple two-period model. For simplicity, there is no uncertainty and both the rate of time preference and the interest rate are zero. Households start with an endowment w and k_1 units of an asset. This asset is in fixed unit supply and is traded at a price q. Assets purchased in the first period, k_2, deliver z units of consumption in period 2. Households can borrow up to a fraction κ of the market value of aggregate assets.

The households' optimization problem in the decentralized equilibrium is:

$$
\begin{aligned}
& \max_{b,k_2,c_1,c_2} u(c_1) + u(c_2) \\
& s.t \\
c_1 + b + qk_2 &= w + qk_1 \\
c_2 &= k_2 z + b \\
b &\geq -\kappa q
\end{aligned}
$$

Hence, the decentralized equilibrium without policy intervention satisfies these optimality conditions:

$$
\begin{aligned}
u'(c_1) &= u'(c_2) + \mu \\
u'(c_1)q &= u'(c_2)z \\
k &= 1 \\
\mu(b + q) &= 0, \mu \geq 0
\end{aligned}
$$

Notice $c_2 \geq c_1$ must hold in equilibrium since $\mu \geq 0$.

Since $k_2 = k_1 = 1$, the JK planner's problem is:

$$
\begin{aligned}
& \max_b u(w - b) + u(z + b) \\
b &\geq -\kappa q
\end{aligned}
$$

where $q = \frac{u'(z+b)z}{u'(w-b)}$ but is taken as given by the planner.

Again since $k_2 = k_1 = 1$, the BM planner's problem is:

$$
\begin{aligned}
& \max_b u(w - b) + u(z + b) \\
b &\geq -\kappa \frac{u'(z + b)z}{u'(w - b)}
\end{aligned}
$$

We assume log utility in order to derive tractable closed-form analytical results and prove that the BM planner faces two relevant debt choices when the constraint binds. Following this, we prove that Assumption 1 of Jeanne and Korinek (2012) fails in general for CRRA utility, not just the log case.

If κ is large enough, the economy is unconstrained and $c_1 = c_2 = \frac{w+z}{2}$, and $b^{unc} = \frac{w-z}{2}$. Hence, we focus on values of κ such that the collateral constraint binds. In particular, we show numerical results for these parameter values $\kappa = 0.4, w = 0.2, z = 1.4$.

Figure 2 is the analog of Figure 1 but for the two-period model. The panel in the left shows borrowing capacity and the one in the right social welfare, both as functions of the debt choice. The unconstrained level of borrowing (i.e. without the credit constraint) is $b^{unc} = -0.6$. As before, the JK planner yields one level of debt that satisfies the collateral constraint with equality, $b_1^{\max} = -0.17$. For the BM planner, there are two debt levels that satisfy the collateral constraint with equality: $b_1^{\max} = -0.17$ and $b_2^{\max} = -0.84$, with associated asset prices $q_1 = 0.41$ and $q_2 = 1.68$ respectively.[41]

As in Figure 1, the panel on the right of Figure 2 shows that the optimal time-consistent solution of the BM planner delivers higher welfare than the JK planner. As mentioned earlier, by taking the borrowing capacity as given, the JK planner does not internalize that by borrowing $b_2^{\max} = -0.84$ it can boost the asset price and relax the collateral constraint. Taking $q_2 = 1.68$ as given, the JK planner would like to choose the unconstrained level of borrowing, which would be inconsistent with that value of the asset price and would violate the collateral constraint. The JK planner chooses $b_1^{\max} = -0.17$, which is suboptimal.

We finish with two propositions. The first one generalizes the results illustrated above numerically for the log-utility case. The second one proves that Assumption 1 in JK is not valid in general for CRRA utility.

Proposition V *If $(z - \kappa z)^2 - 4\kappa z w > 0$, the BM planer has two debt choices for which the collateral constraint holds with equality.*

Proof: With log utility, the debt that satisfies the collateral constraint with equality is given by $b = -\kappa \frac{(w-b)z}{z+b}$. This is a quadratic equation which has two roots if $(z - \kappa z)^2 - 4\kappa z w > 0$. The two debt solutions are:

$$b = \frac{\kappa z - 1 \pm \sqrt{(z - \kappa z)^2 - 4\kappa z w}}{2}$$

[41] This may suggest that there could be multiple equilibria in the decentralized economy, but this is not the case in this example. In particular, $b_2^{\max}$ is not a decentralized equilibrium, because $b_2^{max} < b^{unc}$, which in turn implies that $c_1 > c_2$ and thus violates the optimality condition $\mu \geq 0$. Thus, the decentralized equilibrium without policy intervention is unique.

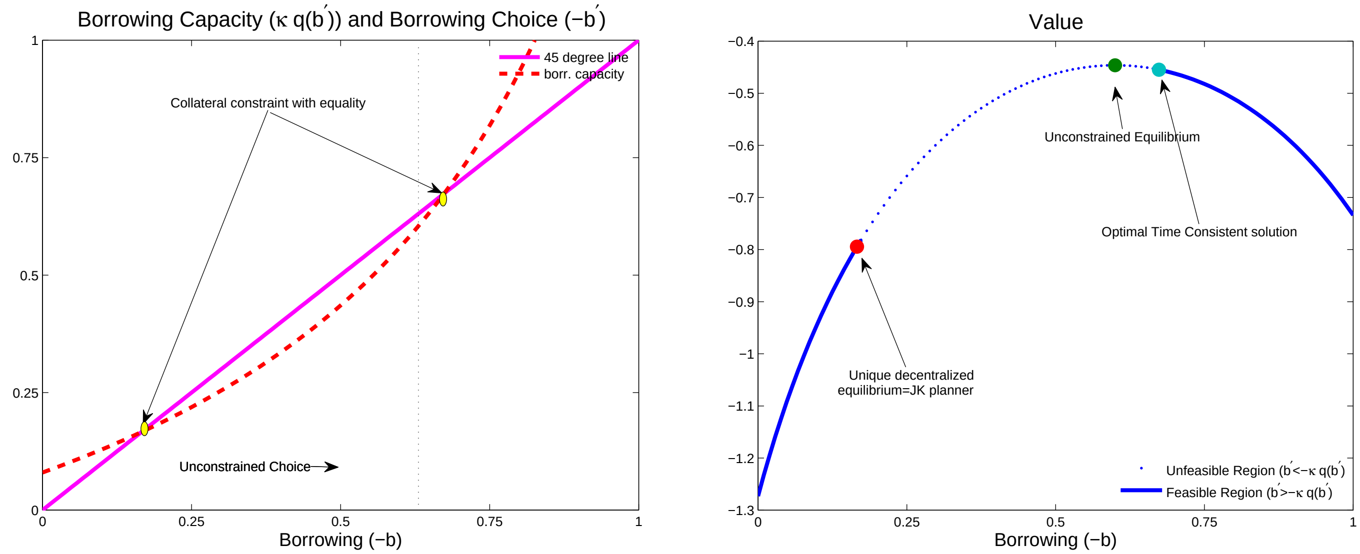


Figure K.2: Two-Period Model: Borrowing & Welfare for the BM and JK Planners.

Proposition VI *Let* $\hat{q}(b, c_1) = \frac{u'(c_2)}{u'(c_1)} z_2$ *and* $u(.) = \frac{c^{1-\sigma}}{1-\sigma}$*, then Assumption 1 of Jeanne and Korinek (2012) (i.e.* $\kappa \frac{\delta \hat{q}(b,c_1)}{\delta c_1} < 1$ *) is violated because there exists* c_1 *such that* $\kappa \frac{\delta \hat{q}(b,c_1)}{\delta c_1} \geq 1$ *for all* $\kappa > 0$*.*

Proof: At equilibrium, since $k_1 = k_2 = 1$, the period budget constraints of the two period model imply that feasible allocations satisfy:

$$\begin{aligned} c_2 &= z_2 + b' \\ b' &= z_1 - c_1 + b \\ c_1 &= z_1 - b' + b \end{aligned}$$

$$c_1 + c_2 = z_1 + z_2 + b$$

Hence,

$$\hat{q}(b, c_1) = \frac{u'(z_2 + z_1 - c_1 + b)}{u'(c_1)} z_2$$

With the CRRA utility function, we have that the pricing function is:

$$\hat{q}(b, c_1) = \left(\frac{c_1}{c_2}\right)^{\sigma} z_2$$

From the feasible consumption allocation and the pricing function with CRRA utility we obtain that:

$$\frac{\partial \left(\frac{c_1}{c_2}\right)}{\partial c_1} = \frac{\partial \left(\frac{c_1}{z_2 + z_1 - c_1 + b}\right)}{\partial c_1} = \frac{z_2 + z_1 + b}{c_2^2}$$

and thus,

$$\begin{aligned}
\frac{\partial \hat{q}(b,c_1)}{\partial c_1} &= \sigma \left(\frac{c_1}{c_2}\right)^{\sigma-1} z_2 \left(\frac{z_2+z_1+b}{c_2^2}\right) \\
&= \sigma c_1^{\sigma-1} z_2 \left(\frac{z_2+z_1+b}{c_2^{1+\sigma}}\right) \\
&= \sigma c_1^{\sigma-1} z_2 \left(\frac{z_2+z_1+b}{((z_2+z_1+b)-c_1)^{1+\sigma}}\right)
\end{aligned}$$

It follows from the above that, for any $\sigma > 0$,

$$\lim_{c_1 \to z_1+z_2+b} \frac{\partial \hat{q}(b,c_1)}{\partial c_1} = \infty$$

Thus, the derivative of the pricing function with respect to c_1 grows infinitely large as c_1 approaches its maximum feasible amount z_1+z_2+b (or equivalently as c_2 approaches zero from above), which is finite. Hence, Assumption 1 from JK is violated

Furthermore, for any $\sigma > 1$, this also holds:

$$\lim_{c_1 \to 0, z_1+z_2+b \neq 0} \frac{\partial \hat{q}(b,c_1)}{\partial c_1} = 0$$

Hence, the derivatives of the pricing function approach zero and infinity as c_1 approaches its minimum and maximum feasible allocations respectively. Since the Inada condition of CRRA preferences imply that c_1 and c_2 are strictly positive at equilibrium, and since the pricing function is strictly convex, this also implies that, for $\sigma > 1$ there exists a finite, feasible value $c_1^* < z_1+z_2+b$ such that $\frac{\delta \hat{q}(b,c_1^*)}{\delta c_1} = 1/\kappa$, and hence for any $c_1 \geq c_1^*$ Assumption 1 fails.[42]

[42] Since consumption allocations are positive and $\frac{\delta \hat{q}(b,c_1)}{\delta c_1} = \sigma \left(\frac{c_1}{c_2}\right)^{\sigma-1} z_2 \left(\frac{z_2+z_1+b}{c_2^2}\right)$, clearly the first derivative of the pricing function is positive. The convexity then follows from noting that $\frac{\delta^2 \hat{q}(b,c_1)}{\delta c_1^2} > 0$, because increasing the feasible c_1 reduces the feasible c_2, so that both $\frac{c_1}{c_2}$ and $\frac{z_2+z_1+b}{c_2^2}$ rise, and thus the derivative of the pricing function rises with c_1.

References

Bianchi, J. (2011). 'Overborrowing and Systemic Externalities in the Business Cycle'. *American Economic Review*, vol. 101, no. 7, 3400–3426.

Bianchi, J. (2012). 'Efficient bailouts?' NBER Working Paper No. 18587.

Bianchi, J. and Mendoza, E. G. (2015). 'Optimal time-consistent macroprudential policy'. Mimeo, Minneapolis Fed.

Coleman, W. (1990). 'Solving the stochastic growth model by policy-function iteration'. *Journal of Business & Economic Statistics*, vol. 8, no. 1, 27–29.

Gertler, M. and Kiyotaki, N. (2010). 'Financial intermediation and credit policy in business cycle analysis'. Handbook of Monetary Economics, forthcoming.

Heaton, J. and Lucas, D. (1996). 'Evaluating the effects of incomplete markets on risk sharing and asset pricing'. *Journal of Political Economy*, vol. 104, no. 3, 443–487.

Klein, P., Krusell, P., and Rios-Rull, V. (2008). 'Time-consistent public policy'. *The Review of Economic Studies*, vol. 75, no. 3, 789–808.

Klein, P., Quadrini, V., and Rios-Rull, V. (2007). 'Optimal Time–consistent Taxation with International Mobility of Capital'. *The B.E. Journal of Macroeconomics*, vol. 5, no. 1.

Kydland, F. E. and Prescott, E. C. (1980). 'Dynamic optimal taxation, rational expectations and optimal control'. *Journal of Economic Dynamics and Control*, vol. 2, 79–91.

Schmitt-Grohe, S. and Uribe, M. (2003). 'Closing small open economies'. *Journal of International Economics*, vol. 61, 163–185.