LIMIT OF RANDOM MEASURES ASSOCIATED WITH THE INCREMENTS OF A BROWNIAN SEMIMARTINGALE

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SUMMARY:

We consider a Brownian semimartingale X (the sum of a stochastic integral w.r.t. a Brownian motion and an integral w.r.t. Lebesgue measure), and for each n an increasing sequence $T(n,i)$ of stopping times and a sequence of positive $\mathcal{F}_{T(n,i)}$–measurable variables $\Delta(n,i)$ such that $S(n,i) := T(n,i) + \Delta(n,i) \leq T(n,i+1)$. We are interested in the limiting behavior of processes of the form $U_t^n(g) = \sqrt{\delta_n}\ \sum_{i:S(n,i)\leq t}[g(T(n,i),\xi_i^n) - \alpha_i^n(g)]$, where δ_n is a normalizing sequence tending to 0 and $\xi_i^n = \Delta(n,i)^{-1/2}(X_{S(n,i)} - X_{T(n,i)})$ and $\alpha_i^n(g)$ are suitable centering terms and g is some predictable function of (ω,t,x). Under rather weak assumptions on the sequences $T(n,i)$ as n goes to infinity, we prove that these processes converge (stably) in law to the stochastic integral of g w.r.t. a random measure B which is, conditionally on the path of X, a Gaussian random measure. We give some applications to rates of convergence in discrete approximations for the p-variation processes and local times.

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1 Introduction

1) Consider a triangular array $(\xi_i^n)_{1\leq i\leq n}$ of $I\!R^d$–valued variables and, with any function g on $I\!R^d$, associate the processes

$$U_t^n(g)=n^{-1/2}\sum_{1\leq i\leq [nt]}[g(\xi_i^n)-\alpha_i^n(g)], \tag{1.1}$$

where $\alpha_i^n(g)$ are suitable centering terms. Finding limit theorems for $U^n(g)$ is an old problem, solved in many special cases: e.g. the ξ_i^n's are rowwise i.i.d., or rowwise mixing, or are the increments of martingales... In a series of recent papers [4], [10], [11], Fujiwara and Kunita have investigated the properties of the limit $U^n(g)$ as a function of g: indeed for suitably chosen centering terms, $g\mapsto U_t^n(g)$ is linear; then in the simplest case of rowwise i.i.d. the limit appears to be of the form

$$U(g)_t=\int_{[0,t]\times I\!R^d} g(x)B(ds,dx), \tag{1.2}$$

where B is a Gaussian random measure, and more precisely a white noise conditioned on the fact that $B([0,t]\times I\!R^d)=0$ for all t (this is just a somewhat sophisticated version of the usual Donsker's Theorem).

2) In this paper we consider a richer situation. We start with a standard d–dimensional Brownian motion $W=(W^i)_{1\leq i\leq d}$ on the standard Wiener space $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geq 0},P)$ and the $(\xi_i^n)_{1\leq i\leq d}$ are increments of W. More precisely, for each n we have a strictly increasing sequence of stopping times $(T(n,i),i\geq 1)$, and associated positive variables $\Delta(n,i)$, and we set $S(n,i)=T(n,i)+\Delta(n,i)$ and

$$\xi_i^n=\Delta(n,i)^{-1/2}(W_{S(n,i)}-W_{T(n,i)}). \tag{1.3}$$

Denote by ρ the Gaussian measure $\mathcal{N}(0,I_d)$ on $I\!R^d$. We also consider functions g: $\Omega\times I\!R_+\times I\!R^d\to I\!R^q$ which are "predictable", and instead of (1.1) we are interested in the asymptotic behavior of the processes

$$U_t^n(g)=\sqrt{\delta_n}\sum_{i:S(n,i)\leq t}\left(g(T(n,i),\xi_i^n)-\int\rho(dx)\ g(T(n,i),x)\right). \tag{1.4}$$

where δ_n is a normalizing sequence going to 0 as $n\to\infty$.

We need a series of hypotheses for $U^n(g)$ to converge to a non-trivial limit. First about g:

Assumption K: g is a function: $\Omega\times I\!R_+\times I\!R^d\to I\!R^q$, with

i) it is predictable, i.e. $\mathcal{P}\otimes\mathcal{R}^d$–measurable, where $\mathcal{P}$ is the predictable σ–field on $\Omega\times I\!R_+$,

ii) $t\mapsto g(\omega,t,x)$ is continuous,

iii) there is a non-decreasing adapted finite-valued process $\gamma=(\gamma_t)$ having

$$|g(\omega,t,x)|\leq\gamma_t(\omega)(1+|x|^{\gamma_t(\omega)}). \tag{1.5}$$

□

Second, there are assumptions on the times $T(n,i)$ and $\Delta(n,i)$: the increments of W should be taken on non-overlapping intervals, that is $S(n,i)\leq T(n,i+1)$. Further, for technical reasons we need $S(n,i)$ to be $\mathcal{F}_{T(n,i)}$–measurable: this is a serious restriction, but something of this sort cannot be totally avoided (take for instance $\Delta(n,i)$ to be such that $\xi_i^n=0$ identically in (1.3), to see that without strong assumptions on $\Delta(n,i)$ we cannot hope for non-trivial limits for (1.4)). Hence we assume the

Assumption A1: For each $n \in \mathbb{N}^\star$ we are given $\mathcal{T}_n = (T(n,i), \Delta(n,i)) : i \in \mathbb{N})$ with:

(i) The sequence $T(n,i)$ is an increasing family of stopping times with $T(n,0) = 0$ and $\lim_i \uparrow T(n,i) = \infty$.

(ii) Each $\Delta(n,i)$ is a $(0,\infty)$–valued $\mathcal{F}_{T(n,i)}$–measurable random variable, such that $S(n,i) := T(n,i) + \Delta(n,i) \leq T(n,i+1)$. □

We also need some nice asymptotic behavior of the sequence $(\mathcal{T}_n)$ in relation with the normalizing constants δ_n in (1.4). This is expressed through the following random "empirical measures" on $\mathbb{R}_+$, where ε_a denotes the Dirac mass with support $\{a\}$:

$$\mu_n = \delta_n \sum_{i \geq 0, S(n,i) < \infty} \varepsilon_{S(n,i)}, \tag{1.6}$$

$$\mu_n^\star = \sum_{i \geq 0, S(n,i) < \infty} \sqrt{\Delta(n,i)\delta_n}\ \varepsilon_{S(n,i)}. \tag{1.7}$$

Assumption A2: μ_n and $\mu_n^\star$ vaguely converge in probability to some random Radon measures μ and $\mu^\star$. □

Both (A1) and (A2) are satisfied in the so-called *regular case*, where $T(n,i) = i/n$, $\Delta(n,i) = 1/n$ and $\delta_n = 1/n$: then $\mu = \mu^\star$ is Lebesgue measure. In general the convergence of μ_n implies the relative compactness of the sequence $\mu_n^\star$ (in probability, for the vague topology), and also its convergence (in probability) to $\mu^\star = 0$ when μ is a.s. singular w.r.t. Lebesgue measure.

3) Our first main result, under (A1) and (A2), is the existence of a random martingale measure B on $\mathbb{R}_+ \times \mathbb{R}^d$, defined on an extension of the original space $(\Omega, \mathcal{F}, P)$, such that for any g having (K), $U_t^n(g)$ converges in law to $U_t(g) = \int g(s,x) 1_{[0,t]}(s)\ B(ds,dx)$. The measure B is called the *tangent measure* to W along the sequence $(\mathcal{T}_n)$, and its precise description in terms of W, μ, $\mu^\star$ is given later.

However the statement is simple in the regular case, and goes as follows (all unexplained notions below are recalled in Sections 2 and 3):

Theorem 1.1 *Assume that we are in the regular case, (or more generally that (A1) and (A2) hold with $\mu = \mu^\star =$ Lebesgue measure). There is a random measure B on $\mathbb{R}_+ \times \mathbb{R}^d$, defined on a very good extension of the Wiener space, which is a white noise with intensity measure $dt \times \rho(dx)$ conditioned on having $B([0,t] \times \mathbb{R}^d) = 0$ for all t, and which satisfies*

$$\int x 1_{[0,t]}(s)\ B(ds,dx) = W_t, \tag{1.8}$$

and such that for every g satisfying (K) the processes $U^n(g)$ converge stably (in the sense of Renyi) *in law to the process*

$$U_t(g) = \int g(s,x) 1_{[0,t]}(s)\ B(ds,dx). \tag{1.9}$$

That (1.8) should hold comes from the fact that if $g(x) = x$ then $U_t^n(g) = W_{[nt]/n}$. Taking $g = 1$, hence $U_t^n(g) = 0$, shows that one must have $B([0,t] \times \mathbb{R}^d) = 0$.

Related results have appeared in various guises in the literature: for instance they come naturally when one studies the error term in approximation for stochastic integrals or differential equations: see Rootzen [14], which contains a discussion of the interest of stable convergence in this context, or Kurtz and Protter [12]. The main applications we have in mind concern statistical problems related to estimation of the variance coefficient with discrete observations for diffusion processes, in the spirit of Dohnal [3] or Genon-Catalot and Jacod [6]. This is why we have considered schemes $\mathcal{T}_n$ based on stopping times rather than deterministic times (see also the applications relating to local time, in Section 9).

4) Our second main results will be obtained as a consequence of the first one, and concerns m-dimensional "Brownian semimartingales" of the form

$$X_t = x_0 + \int_0^t a_s dW_s + \int_0^t b_s ds, \qquad x_0 \in I\!R^m, \tag{1.10}$$

with the following:

Assumption H: a and b are predictable locally bounded processes, with values in $I\!R^m \otimes I\!R^d$ and $I\!R^m$ respectively, and $t \mapsto a_t$ is continuous. $\square$

In this setting we study the limit of processes like $U^n(g)$ in (1.4), with different centering terms, and X instead of W in the definition (1.3) of ξ_i^n. The limit can still be expressed as a suitable integral w.r.t. the tangent measure B to W, and also as $\int g(s,x) 1_{0,t]}(s)\ B^X(ds,dx)$ with another random measure B^X called the *random measure tangent to* X along $(\mathcal{T}_n)$.

5) The paper is organized as follows. Part I (Sections 2-5) concerns the Brownian case: Section 2 is devoted to some preliminary results on extensions of spaces and random measures; in Section 3 we describe the tangent random measure to W and state the result, which is proved in Sections 4 and 5. Part II is about Brownian semimartingales of the form (1.10): results are gathered in Section 6, and proofs are given in Sections 7 and 8. Finally Section 9 is devoted to some simple applications (rates of convergence for q-variations, approximation of local times, etc...).

PART I: THE BROWNIAN CASE

2 Extension of spaces and martingale measures

In this section we start with some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$. We gather a number of results on extensions of this space and martingale measures: some are new, and some are more or less well known but we have been unable to find precise statements for them in the literature. We state them in a general context, but very often we assume the following hypothesis, which is met by the Wiener space:

Assumption B: All martingales on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ are continuous, and the σ–field $\mathcal{F}_0$ is P–trivial. $\square$

2.1 Extension of filtered spaces

We call *extension of* $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ a filtered probability space $(\overline{\Omega}, \overline{\mathcal{F}}, (\overline{\mathcal{F}})_t, \overline{P})$ constructed as follows: starting with an auxiliary filtered space $(\Omega', \mathcal{F}', (\mathcal{F}'_t))$ and a transition probability $Q_\omega(d\omega')$ from $(\Omega, \mathcal{F})$ into $(\Omega', \mathcal{F}')$, we set $(\overline{\Omega}, \overline{\mathcal{F}}) = (\Omega, \mathcal{F}) \otimes (\Omega', \mathcal{F}')$, $\overline{\mathcal{F}}_t = \cap_{s>t} \mathcal{F}_s \otimes \mathcal{F}'_s$ and $\overline{P}(d\omega, d\omega') =$

$P(d\omega)Q_\omega(d\omega')$. We also assume that each σ–field $\mathcal{F}'_{t-}$ is separable (this is an *ad-hoc* definition, sufficient for our purposes here).

According to [7] (see Lemma (2.17)), the extension is called *very good* if all martingales on $(\Omega,\mathcal{F},(\mathcal{F}_t),P)$ are also martingales on $(\overline{\Omega},\overline{\mathcal{F}},(\overline{\mathcal{F}}_t),\overline{P})$ or, equivalently, if $\omega\mapsto Q_\omega(A')$ is $\mathcal{F}_t$–measurable for every $A'\in\mathcal{F}'_t$.

A process Z on the extension is called an *$\mathcal{F}$–conditional martingale* (resp. *Gaussian process*) iff for P–almost all ω the process $Z(\omega,.)$ is a martingale (resp. a Gaussian process) on the space $(\Omega',\mathcal{F}',(\mathcal{F}'_t),Q_\omega)$. A locally square–integrable martingale on the extension is called $(\mathcal{F}_t)$*–localizable* if there exists a localizing sequence of stopping times (T_n) *relative to* $(\mathcal{F}_t)$.

Lemma 2.1 *Let M be a right–continuous adapted process on a very good extension, each M_t being $\overline{P}$-integrable. Then M is an $\mathcal{F}$–conditional martingale iff M is an $(\overline{\mathcal{F}}_t)$–martingale orthogonal to all bounded $(\mathcal{F}_t)$–martingales.*

Proof. Let $t\leq s$, and U and U' be bounded measurable functions on $(\Omega,\mathcal{F}_t)$ and $(\Omega',\mathcal{F}'_t)$ respectively, and Z be a bounded $(\mathcal{F}_t)$–martingale. We have

$$\overline{E}(UU'Z_sM_s)=\int P(d\omega)U(\omega)Z_s(\omega)\int Q_\omega(d\omega')U'(\omega')M_s(\omega,\omega'), \tag{2.1}$$

$$\overline{E}(UU'Z_tM_t)=\int P(d\omega)U(\omega)Z_t(\omega)\int Q_\omega(d\omega')U'(\omega')M_t(\omega,\omega'). \tag{2.2}$$

If M is an $\mathcal{F}$–conditional martingale, for P–almost all ω we have $\int Q_\omega(d\omega')U'(\omega')M_s(\omega,\omega')=\int Q_\omega(d\omega')U'(\omega')M_t(\omega,\omega')$, and the latter is $\mathcal{F}_t$–measurable as a function of ω because the extension is very good. Using the fact that Z is a martingale on $(\Omega,\mathcal{F},(\mathcal{F}_t),P)$ we have $\overline{E}(UU'Z_sM_s)=\overline{E}(UU'Z_tM_t)$, hence ZM is a martingale on the extension: then M is a martingale (take $Z=1$), orthogonal to all bounded $(\mathcal{F}_t)$–martingales.

Conversely assume that M is a martingale, orthogonal to all bounded $(\mathcal{F}_t)$–martingales. Take a bounded $\mathcal{F}_s$–measurable function V, and consider the $(\mathcal{F}_t)$–martingale $Z_t=E(V|\mathcal{F}_t)$, which has $Z_s=V$. By hypothesis the left-hand sides of (2.1) and (2.2) are equal, and in the right-hand side of (2.2) we can replace Z_t by $Z_s=V$ because the last integral is $\mathcal{F}_t$–measurable in ω. Then (taking $U=1$) we have for all V as above:

$$\int P(d\omega)V(\omega)\int Q_\omega(d\omega')U'(\omega')M_s(\omega,\omega')=\int P(d\omega)V(\omega)\int Q_\omega(d\omega')U'(\omega')M_t(\omega,\omega').$$

So for P–almost all ω, $Q_\omega(U'M_t(\omega,.))=Q_\omega(U'M_s(\omega,.))$. Because of the separability of the σ–fields $\mathcal{F}'_{t-}$ and of the right-continuity of M, we have this relation P–almost surely in ω, simultaneously for all $t\leq s$ and all $\mathcal{F}'_{t-}$–measurable variable U': this gives the result. $\square$

Below $\langle M,N\rangle$ is the usual predictable bracket of the two locally square–integrable martingales M and N, with the convention $\langle M,N\rangle_0=\overline{E}(M_0N_0)$. If $M=(M^i)_{1\leq i\leq d}$ is d-dimensional its transpose is M^T and MM^T, resp. $\langle M,M^T\rangle$, is the d^2-dimensional process with components M^iM^j, resp. $\langle M^i,M^j\rangle$. A process Z is called $(\mathcal{F}_t)$–locally square–integrable if there is a localizing sequence (T_n) of $(\mathcal{F}_t)$–stopping times such that each $Z^2_{T_n\wedge t}$ is integrable.

Lemma 2.2 *Assume (B) and let Z be a continuous q–dimensional $\mathcal{F}$–conditional Gaussian martingale on a very good extension, which moreover is $(\mathcal{F}_t)$–locally square–integrable* (by Lemma 1 it is an $(\mathcal{F}_t)$–localizable locally square–integrable martingale, and $\langle Z,Z^T\rangle$ exists).

a) There is a version of $\langle Z,Z^T\rangle$ which is $(\mathcal{F}_t)$–predictable, hence which does not depend on ω'.

b) Z is $\mathcal{F}$–conditionally centered iff $\overline{E}(Z_0)=0$, in which case the $\mathcal{F}$–conditional law of Z is characterized by the process $\langle Z, Z^T\rangle$ (i.e., for P–almost all ω, the law of $Z(\omega,.)$ under Q_ω depends only on the function $t\mapsto \langle Z, Z^T\rangle_t(\omega)$.

Proof. By $(\mathcal{F}_t)$–localization we may and will assume that Z is square–integrable. Set $F_t(\omega)=\int Q_\omega(d\omega')Z_t(\omega,\omega')$ and $G_t(\omega)=\int Q_\omega(d\omega')(Z_tZ_t^T)(\omega,\omega')$.

a) There is a P–full set A such that if $\omega\in A$, under Q_ω, the process $Z(\omega,.)$ is both Gaussian and a martingale, hence it is a process with independent and centered increments: so $F_t(\omega)=F_0(\omega)$ and $(Z_tZ_t^T)(\omega)-G_t(\omega)$ is a martingale. By Lemma 2.1, ZZ^T-G is an $(\overline{\mathcal{F}}_t)$–martingale, while $G_0=\overline{E}(Z_0Z_0^T|\mathcal{F})=\overline{E}(Z_0Z_0^T|\mathcal{F}_0)=\overline{E}(Z_0Z_0^T)=\langle Z,Z^T\rangle_0$ (use the very good property of the extension and the fact that $\mathcal{F}_0$ is P–trivial). Further since G is continuous $(\mathcal{F}_t)$–adapted it is $(\mathcal{F}_t)$–predictable, hence is a version of $\langle Z,Z^T\rangle$.

b) Similarly $F_t=F_0=\overline{E}(Z_0)$, so the necessary and sufficient condition is trivial. Further if $\omega\in A$ and $F_t(\omega)=0$ for all t, the law of $Z(\omega,.)$ under Q_ω is characterized by the covariance $\int Q_\omega(d\omega')(Z_tZ_s^T)(\omega,\omega')=G_{s\wedge t}(\omega)$, hence the last claim. □

Lemma 2.3 *Assume (B), and let Z be a continuous q–dimensional local martingale on a very good extension, with the following: $\overline{E}(Z_0)=0$,and Z is orthogonal to all $(\mathcal{F}_t)$–martingales, and $\langle Z,Z^T\rangle$ has an $(\mathcal{F}_t)$–predictable version. Then Z is an $\mathcal{F}$–conditional centered Gaussian martingale.*

Proof. Since $\langle Z,Z^T\rangle$ is $(\mathcal{F}_t)$–predictable, it is $(\mathcal{F}_t)$–locally integrable, and as in the previous lemma we may and will assume that Z is in fact square–integrable. Since Z is orthogonal to all $(\mathcal{F}_t)$–martingales, the same is true of $M:=ZZ^T-\langle Z,Z^T\rangle=2Z\cdot Z^T$. Lemma 2.1 applied to Z and to M shows that for P–almost all ω, under Q_ω the process $Z(\omega,.)$ is a continuous martingale with deterministic bracket $\langle Z,Z^T\rangle(\omega)$, hence it is a Gaussian martingale, centered by Lemma 2.2–b because $\overline{E}(Z_0)=0$: hence the result. □

2.2 Martingale measures

1) First we recall some facts about martingale measures: see Walsh [15] for a complete account. Let again $(\Omega,\mathcal{F},(\mathcal{F}_t),P)$ be a filtered probability space. A (finite) L^2–valued *martingale measure* B on $I\!R^d$ is a collection $(B(A)_t: t\in I\!R_+, A\in\mathcal{R}^d)$ of random variables and a sequence (T_n) of stopping times increasing to $+\infty$, such that for all $n\in I\!N$:

$$\left.\begin{array}{l}\text{(i) for all } A\in\mathcal{R}^d,\ t\mapsto B(A)_t \text{ is a square-integrable martingale,}\\ \text{(ii) for all } t\in I\!R_+,\ A\mapsto B(A)_t \text{ is a } L^2\text{–valued random measure.}\end{array}\right\}\tag{2.3}$$

The measure is called *continuous* if each $t\mapsto B(A)_t$ is a.s. continuous. The (random) *covariance measure* is

$$\nu(\omega;[0,t]\times A\times A')=\langle B(A),B(A')\rangle_t(\omega).\tag{2.4}$$

In general $[0,t]\times A\times A'\mapsto\nu(\omega;[0,t]\times A\times A')$ cannot be extended as a (signed) measure $\nu(\omega;.)$ on $I\!R_+\times I\!R^d\times I\!R^d$. However, it has the following:

Property P: (i) Each process $\nu([0,.]\times A\times A')$ is càdlàg predictable.

(ii) $A\mapsto\nu([0,t]\times A\times A')$ is an L^2–valued measure on $(I\!R^d,\mathcal{R}^d)$.

(iii) It is *symmetric positive definite*, in the sense that $\nu((s,t]\times A\times A')=\nu((s,t]\times A'\times A)$ and that for all $n\in I\!N$, $a_i\in I\!R$, $A_i\in\mathcal{R}^d$, then $t\mapsto\sum_{1\le i,j\le n}a_ia_j\nu([0,t]\times A_i\times A_j)$ is a.s. increasing.

(iv) $E[\nu([0,T_n]\times A\times A))<\infty$ for all $A\in\mathcal{R}^d$, for some localizing sequence (T_n) of stopping times. □

Following Walsh [15], we say that B (or ν) is *worthy* if there is a positive random measure $\eta(\omega,.)$ on $I\!R_+\times I\!R^d\times I\!R^d$ which satisfies (P) and such that $|\nu|\ll\eta$ (i.e., for all $s\leq t$, $A,A'\in\mathcal{R}^d$, $|\nu([0,t]\times A\times A')-\nu([0,s]\times A\times A')|\leq\eta((s,t]\times A\times A')$). In this case, there is a version of ν which extends as a (signed) measure on $I\!R_+\times I\!R^d\times I\!R^d$.

If B is worthy, we can define a *stochastic integral process* $f\star B_t=\int f(.,s,x)1_{[0,t]}(s)\;B(ds,dx)$ for every predictable function f on $\Omega\times I\!R_+\times I\!R^d$ having $\int f(s,x)f(s,x')1_{[0,t]}(s)\;\eta(ds,dx,dx')<\infty$ a.s. for all t. Stochastic integrals are characterized by the fact that $f\star B_t=B(A)_t$ if $f(\omega,s,x)=1_A(x)$, that $f\mapsto f\star B$ is a.s. linear, and that $f\star B$ is a locally square–integrable martingale with

$$\langle f\star B,f'\star B\rangle_t=\int f(s,x)f(s,x')1_{[0,t]}(s)\;\nu(ds,dx,dx'). \tag{2.5}$$

Recall also that a *white noise* on $I\!R_+\times I\!R^d$ with *intensity measure* m (a positive σ–finite measure on $I\!R_+\times I\!R^d$) is a Gaussian family of centered variables $\phi=(\phi(A):A\in\mathcal{R}_+\otimes\mathcal{R}^d)$ with $\phi(A)$ and $\phi(A')$ independent when $A\cap A'=\emptyset$, and such that $E[\phi(A)^2]=m(A)$. Obviously m characterizes the law of ϕ, and if $m([0,t]\times I\!R^d)<\infty$ for all t, then $B(A)_t:=\phi([0,t]\times A)$ defines an L^2–valued martingale measure on $I\!R^d$ for the filtration $\mathcal{F}_t=\cap_{s>t}\sigma(B(A)_r:r\leq s,A\in I\!R^d)$, with deterministic covariance measure $\nu([0,t]\times A\times A')=m([0,t]\times(A\cap A'))$. In this case ν is worthy.

2) Consider now a very good extension $(\overline{\Omega},\overline{\mathcal{F}},(\overline{\mathcal{F}}_t),\overline{P})$ of $(\Omega,\mathcal{F},(\mathcal{F}_t),P)$. By definition an $\mathcal{F}$–*conditional Gaussian measure* is an L^2–valued martingale measure on the extension, such that each finite family $(B(A_1),\cdots,B(A_n)))$ is an $\mathcal{F}$–conditional Gaussian process. Further, it is an $\mathcal{F}$–*conditional centered Gaussian measure* if moreover each $B(A)$ is also an $\mathcal{F}$–conditional centered martingale.

Proposition 2.4 *Let B be an $\mathcal{F}$–conditional Gaussian measure on a very good extension.*

a) There is a unique decomposition $B=B'+B$", where B' is an L^2–valued martingale measure on $(\Omega,\mathcal{F},(\mathcal{F}_t),P)$ and B'' is an $\mathcal{F}$–conditional centered Gaussian measure. The corresponding covariance measures ν, ν', ν'' have $\nu=\nu'+\nu''$.

b) Under (B), there is a version of ν which does not depend on ω', and the $\mathcal{F}$–conditional law of B is characterized by B' and ν (or ν'').

Proof. Using (2.3)-(i), by $(\mathcal{F}_t)$–localization we may and will assume that each $B(A)$ belongs to the space $\overline{\mathcal{H}}^2$ of all square-integrable martingales on the space $(\overline{\Omega},\overline{\mathcal{F}},(\overline{\mathcal{F}}_t),\overline{P})$, which we endow with the Hilbert norm $\|M\|^2=\overline{E}(M_\infty^2)$. Let $\mathcal{H}^2$ be the closed subspace of all elements of $\overline{\mathcal{H}}^2$ that are martingales on $(\Omega,\mathcal{F},(\mathcal{F}_t),P)$.

a) Call $B'(A)$ the orthogonal projection of $B(A)$ in $\overline{\mathcal{H}}^2$, on $\mathcal{H}^2$. Since $M\mapsto M_t$ is continuous from $\overline{\mathcal{H}}^2$ into $L^2(\overline{P})$, the collection $B'=(B'(A)_t:t\geq0,A\in\mathcal{R}^d)$ is an L^2–valued measure martingale on $(\Omega,\mathcal{F},(\mathcal{F}_t),P)$. Set $B''=B-B'$, which is an L^2–valued measure martingale on $(\overline{\Omega},\overline{\mathcal{F}},(\overline{\mathcal{F}}_t),\overline{P})$, and also clearly an $\mathcal{F}$–conditional Gaussian measure. Since $B''(A)$ is orthogonal to $\mathcal{H}^2$, Lemma 2.1 yields that it is an $\mathcal{F}$–conditional martingale. Further $B'(A)_0=\overline{P}(B(A)_0|\mathcal{F}_0)=\overline{E}(B(A)_0|\mathcal{F})$ since we have a very good extension. Then $E[B''(A)_0]=0$, and it follows from Lemma 2.2 that B'' is an $\mathcal{F}$–conditional centered Gaussian measure.

We have thus a decomposition $B=B'+B''$. Now, for any such decomposition $B''(A)$ is orthogonal to $\mathcal{H}^2$ by Lemma 2.1, while $B'(A)\in\mathcal{H}^2$, hence uniqueness. The orthogonality of any $B'(A)$ with any $B''(A')$ readily yields $\nu=\nu'+\nu''$.

b) Since ν is $(\mathcal{F}_t)$–predictable in the sense of P-(i) and since a version of ν'' is given by

$\nu''([0,t]\times A\times A')=\int Q_\omega(d\omega')\ (B''_t(A)B''_t(A'))(\omega,\omega')$ (see the proof of Lemma 2.2), we see that ν does not depend on ω'. The second claim follows from Lemma 2.2-b. □

Proposition 2.5 *Let $\nu=(\nu(\omega;[0,t]\times A\times A'):t\geq 0,A,A'\in\mathcal{R}^d)$ satisfy (P) and be worthy. There is an $\mathcal{F}$–conditional centered Gaussian measure on a very good extension of $(\Omega,\mathcal{F},(\mathcal{F}_t),P)$, having ν for covariance measure.*

Proof. Let $\mathcal{E}$ be a countable algebra generating the Borel σ–field $\mathcal{R}^d$. Set $\Omega'=\mathbb{R}^{\mathbb{Q}_+\times\mathcal{E}}$, with the "canonical process" $B'=(B'(A)_t:t\in\mathbb{Q}_+,A\in\mathcal{E})$, and $\mathcal{F}'_t=\cap_{s>t}\sigma(B'(A)_r:r\leq s,A\in\mathcal{E})$ and $\mathcal{F}'=\bigvee_{t>0}\mathcal{F}'_t$. Then $\mathcal{F}'$ and all $\mathcal{F}'_{t-}$ are separable. Using (P-iii) we see that there is a unique probability measure Q_ω on $(\Omega',\mathcal{F}')$ under which B' is a centered Gaussian process with covariance $Q_\omega[B'(A)B'(A')]=\nu(\omega;[0,t\bigwedge s]\times A\times A')$. Further, (P-i) implies that $Q_\omega(d\omega')$ is a transition probability from $(\Omega,\mathcal{F})$ into $(\Omega',\mathcal{F}')$, and also from $(\Omega,\mathcal{F}_t)$ into $(\Omega',\mathcal{F}'_t)$ for all t. Therefore the extension $(\overline{\Omega},\overline{\mathcal{F}},(\overline{\mathcal{F}}_t),\overline{P})$ of $(\Omega,\mathcal{F},(\mathcal{F}_t),P)$ based upon $(\Omega',\mathcal{F}',(\mathcal{F}'_t),Q_\omega)$ (see §2.1) is very good.

Under Q_ω, the process $(B'(A)_t)_{t\in\mathbb{Q}}$ is also a martingale along $\mathbb{Q}_+$; hence if we set $B''(A)_t=\limsup_{s\in\mathbb{Q},s>t,s\to t}B'(A)_s$ we obtain a process $B''(A)$ indexed by $\mathbb{R}_+$ which is again a centered Gaussian martingale under each Q_ω. Further (P-iv) yields $\overline{P}(B''(A)_t^2)<\infty$, hence by Lemma 2.1, for each $A\in\mathcal{E}$, $(B''(A)_{T_p\wedge t})_{t\geq 0}$ is a square–integrable martingale on the extension.

Now we use the existence of a positive random measure η having (P) and dominating ν: if $A_n\in\mathcal{E}$ decreases to $\emptyset$, then $\overline{E}(B''(A_n)^2_{T_p\wedge t})\leq\overline{E}[\eta([0,T_p]\times A_n\times A_n)]\to 0$ as $n\to\infty$. Thus $A\mapsto B''(A)_{T_p\wedge t}$ is an L^2–valued measure on $(\mathbb{R}^d,\mathcal{E})$. At this point we can repeat the argument of Walsh [15] to the effect of constructing $B(A)$ for $A\in\mathcal{R}^d$ as the stochastic integral of the function 1_A w.r.t. the martingale measure B'' on $(\mathbb{R}^d,\mathcal{E})$. The family $B=(B(A)_t:t\geq 0,A\in\mathcal{R}^d)$ constructed in this way clearly satisfies (2.3), and $B(A)=B''(A)$ if $A\in\mathcal{E}$.

Moreover if $A\in\mathcal{R}^d$ there is a sequence $A_n\in\mathcal{E}$ with $B''(A_n)_{T_p\wedge t}\to B(A)_{T_p\wedge t}$ in $L^2(\overline{P})$: we deduce first that (2.4) holds if $A\in\mathcal{R}^d$ and $A'\in\mathcal{E}$, and repeating the same argument and using the symmetry in (P)-(iii) gives (2.4) for all $A,A'\in\mathcal{R}^d$, that is ν is the covariance measure of B; we deduce next that, since each $B''(A_n)$ is orthogonal to all $(\mathcal{F}_t)$–martingales by Lemma 2.1, the same is true of $B(A)$ and therefore by Lemma 2.1 again $B(A)$ is an $\mathcal{F}$–conditional martingale. Furthermore by taking a subsequence we can even suppose that the convergence $B''(A_n)_t\to B(A)_t$ holds P-a.s. for all $t\geq 0$, hence Q_ω–a.s. for P–almost all ω: since $(B''(A^1_n),\ldots,B''(A^p_n))$ is a centered Gaussian process under Q_ω for $A^i_n\in\mathcal{E}$, it follows that $(B(A^1_n),\ldots,B(A^p_n))$ is also a centered Gaussian process under Q_ω for P–almost all ω, if $A^i_n\in\mathcal{R}^d$. Hence $(B(A^1),\ldots,B(A^p))$ is an $\mathcal{F}$–conditional centered Gaussian martingale for all $A^i\in\mathcal{R}^d$, and we are finished. □

Proposition 2.6 *Assume (B), and let B be a worthy $\mathcal{F}$–conditional centered Gaussian measure on a very good extension, with covariance measure ν (not depending on ω'). Let $f:\ \Omega\times\mathbb{R}_+\times\mathbb{R}^d\to\mathbb{R}^q$ be predictable and integrable w.r.t. B. Then $f\star B$ is an $\mathcal{F}$–conditional centered Gaussian martingale, orthogonal to all $(\mathcal{F}_t)$–martingales, and its $\mathcal{F}$–conditional law is determined by its bracket which does not depend on ω'):*

$$\langle f\star B,f^T\star B\rangle_t=\int f(s,x)f^T(s,x')1_{[0,t]}(s)\ \nu(ds,dx,dx'). \tag{2.6}$$

Proof. All claims are obvious when $f(\omega,t,x)=(1_{A_1}(x),\ldots,1_{A_q}(x))$ (use Lemma 2.2 for the last property), and follow by linearity for all "simple" functions.

In the general case the bracket is given by (2.6) (see (2.5)) and thus by $(\mathcal{F}_t)$–localization we can and will assume that $f\star B$ is square–integrable. There is a sequence (f_n) of simple functions such that $f_n\star B_t\to f\star B_t$ $\overline{P}$–a.s. and in $L^2(\overline{P})$ for all t. Then repeating the final argument of the previous proof, we obtain that $f\star B$ is an $\mathcal{F}$–conditional centered Gaussian martingale, orthogonal to all $(\mathcal{F}_t)$–martingales. The last claim again comes from Lemma 2.2. □

Remarks: 1) An $\mathcal{F}$–conditional Gaussian measure is not a Gaussian measure, unless its covariance measure ν is deterministic.

2) If B is an $\mathcal{F}$–conditional centered Gaussian measure, it is not true in general that for P–almost all ω, $B(\omega,.)$ is a Gaussian martingale measure on $(\Omega',\mathcal{F}',(\mathcal{F}'_t),Q_\omega)$. However, when this is true, in Proposition 2.6 $f\star B(\omega,.)$ is also the "Wiener" integral of the deterministic function $(s,x)\mapsto f(\omega,s,x)1_{[0,t]}(s)$ w.r.t. the Gaussian measure $B(\omega,.)$, relative to Q_ω. □

3 The main result

1) In the rest of the paper $(\Omega,\mathcal{F},(\mathcal{F}_t),P)$ is the d–dimensional standard Wiener space, with the canonical process W. Recall that $\rho=\mathcal{N}(0,I_d)$. We write $\rho(f)=\int f(x)\rho(dx)$, and $\rho(x1_A)=\int x1_A(x)\rho(dx)$, and $\rho(f_t)(\omega)=\int f(\omega,t,x)\rho(dx)$, etc...

In order to define the tangent martingale measure, we need the following Lemma, which will be proved in Section 4:

Lemma 3.1 *Assume (A1) and (A2). Let λ be the Lebesgue measure on $\mathbb{R}_+$, and μ^{ac} be the absolutely continuous part of μ w.r.t. λ. There are two nonnegative predictable processes θ, $\theta^\star$ such that*

$$\mu^{ac}([0,t])=\int_0^t\theta_s ds,\qquad \mu^\star([0,t])=\int_0^t\theta^\star_s ds, \tag{3.1}$$

$$\theta^{\star 2}_s\le\theta_s. \tag{3.2}$$

Definition 1: A tangent measure to W along the sequence $(\mathcal{T}_n)$ satisfying (A1) and (A2) is an $\mathcal{F}$–conditional Gaussian measure B on $\mathbb{R}^d$, defined on a very good extension $(\overline{\Omega},\overline{\mathcal{F}},(\overline{\mathcal{F}}_t),\overline{P})$ of the filtered space $(\Omega,\mathcal{F},(\mathcal{F}_t),P)$, such that $\overline{E}(B(A)_0)=0$ for all $A\in\mathcal{R}^d$, that

$$\langle W,B(A)\rangle_t=\rho(x1_A)\ \mu^\star([0,t]) \tag{3.3}$$

for all $A\in\mathcal{R}^d$, and having the covariance measure

$$\nu([0,t]\times A\times A')=(\rho(A\cap A')-\rho(A)\rho(A'))\ \ \mu([0,t]). \tag{3.4}$$

□

The following provides an equivalent definition for a tangent measure, and proves that it exists and is "essentially" unique in the sense that its $\mathcal{F}$—conditional law is uniquely determined (by application of Proposition 2.4).

Proposition 3.2 *a) B is a tangent measure iff it is an $\mathcal{F}$–conditional Gaussian measure whose decomposition $B=B'+B''$ of Proposition 2.4 has, with ν'' covariance measure of B'':*

$$B'(A)=\rho(x^T1_A)\ \theta^\star\cdot W, \tag{3.5}$$

$$\nu''([0,t]\times A\times A')=(\rho(A\cap A')-\rho(A)\rho(A'))\,\mu([0,t])-\rho(x^T1_A)\rho(x1_A)\ \int_0^t\theta^{\star 2}_s ds. \tag{3.6}$$

b) There exists a tangent measure, and all of them are worthy.,

Proof. a) Let $B = B' + B''$ be the decomposition of the tangent measure B. Then $B'(A)$ is a local martingale on $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$, hence $B'(A) = \alpha^T \cdot W$ for some predictable d–dimensional process α, while $B''(A)$ is orthogonal to W: thus $\langle W, B'(A)\rangle = \langle W, B(A)\rangle$, and (3.1) and (3.3) yield $\alpha_t = \rho(x1_A)\theta^\star_t$ for λ–almost all t and (3.5) follows. The covariance measure ν' of B' is trivially given by the last term in (3.6) (with the + sign), so $\nu = \nu' + \nu''$ gives (3.6).

Conversely assume (3.5) and (3.6). Again $\langle W, B'(A)\rangle = \langle W, B(A)\rangle$, hence (3.4) holds, and (3.3) follows from (3.6) and $\nu = \nu' + \nu''$.

b) The formula (3.5) clearly defines an L^2–valued martingale measure on $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ ($\theta^\star$ is integrable w.r.t. W by (3.1) and (3.2)). We apply Proposition 2.5 to obtain an $\mathcal{F}$–conditionally centered Gaussian measure B'' on a very good extension, with covariance measure ν'' given by (3.6): for this we need to show that ν'' satisfies (P) and is worthy. Recalling that every càdlàg adapted process on the Wiener space is predictable, we have (P-i), while (P-ii) and the symmetry in (P-iii) are obvious. We have (P-iv) because the increasing predictable process $\mu([0,\cdot])$ is locally bounded. If $A_i \in \mathcal{R}^d$, $a_i \in I\!R$, and $f = \sum_{1\le i\le n} a_i 1_{A_i}$ and $\mu^s = \mu - \mu^{ac}$, (3.6) yields

$$\sum a_i a_j \nu''([0,t]\times A_i \times A_j) = \left(\rho(f^2) - \rho(f)^2\right)\ \mu^s([0,t]) + \int_0^t \left(\theta_s(\rho(f^2) - \rho(f)^2) - \theta^{\star 2}_s \rho(x^T f)\rho(xf)\right)\ ds. \qquad (3.7)$$

Observe that the orthogonal projection in $L^2(\rho)$ of the function f on the linear space spanned by the orthogonal vectors $(1, x_1, \ldots, x_d)$ is $g = \rho(f) + \sum_{1\le i\le d}\ x_i\rho(x_i f)$, hence $\rho(f^2) - \rho(f)^2 - \rho(x^T f)\rho(xf) = \rho(f^2) - \rho(g^2) \ge 0$. Taking (3.2) into account, we deduce that (3.7) is non-decreasing in t and thus (P-iii) holds. For the worthyness, we observe that $|\nu''| \le 2\eta$, where η is the positive random measure having $\eta([0,t]\times A \times A') = (\rho(A\cap A') - \rho(A)\rho(A'))\ \mu([0,t])$. That η satisfies (P) is obvious.

At this point we have the existence of B'', and $B = B' + B''$ has all properties of (a). Then B is a tangent measure, and its covariance measure ν is given by (3.4) and has $|\nu| \le \eta$, hence it is worthy. □

If g satisfies (K), then $\int g^T(s,x)g(s,x')1_{[0,t]}(s)\eta(ds,dx,dx') < \infty$ (with η as in the previous proof), hence g is integrable w.r.t. B and the brackets are:

$$\langle g\star B, g^T \star B\rangle_t = \int \left(\rho(g_s g_s^T) - \rho(g_s)\rho(g_s^T)\right) 1_{[0,t]}(s)\mu(ds). \qquad (3.8)$$

Note also that

$$g\star B' = (\rho(gx^T)\theta^\star)\cdot W, \qquad \langle g\star B', W^T\rangle_t = \int \left(\rho(g_s x^T)\theta^\star_s\right)\ ds \qquad (3.9)$$

(approximate g by simple functions, or use the characterization (2.5) of stochastic integrals), and,

$$\langle g\star B'', g^T\star B''\rangle_t = \int \left(\rho(g_s g_s^T) - \rho(g_s)\rho(g_s^T)\right) 1_{[0,t]}(s)\mu(ds) - \int \left(\rho(g_s x^T)\rho(x g_s T)\theta^{\star 2}_s\right)\ ds. \qquad (3.10)$$

In view of Proposition 2.6, this implies that the $\mathcal{F}$–conditional law of $g\star B$ is determined by $g\star B'$ and either (3.8) or (3.10).

2) Before stating the main result, we should recall what stable convergence means. This notion was introduced by Renyi [13]; see also Aldous and Eagleson [1], or [9] §VIII-5-c for a complete account. Let Y_n be a sequence of random variables on $(\Omega, \mathcal{F}, P)$, taking values in a metric space

E, and let Y be an E–valued variable defined on an extension $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{P})$. We say that Y_n converges stably in law to Y if

$$E(Zf(Y_n)) \rightarrow \overline{E}(Zf(Y)) \tag{3.11}$$

for every continuous bounded function f on E and every bounded measurable function Z on $(\Omega, \mathcal{F})$. This implies the convergence in law of Y_n to Y.

Consider also the following subset I of $\mathbb{R}_+$, whose complement is at most countable:

$$I = \{t \geq 0 : \mu(\{t\}) = 0 \ \ P\text{-a.s.}\}. \tag{3.12}$$

Theorem 3.3 *Assume (A1) and (A2), and let B be a tangent measure to W along the sequence $(\mathcal{T}_n)$. Let g satisfy (K), and $U^n(g)$ be given by (1.4).*

a) If μ has a.s. no atom, the processes $U^n(g)$ converge stably in law (for the Skorokhod topology) to $g \star B$.

b) For all $t_1, \ldots, t_p$ in I, the variables $U^n_{t_1}(g), \ldots, U^n_{t_p}(g))$ converge stably in law to $(g \star B_{t_1}, \ldots, g \star B_{t_p})$.

Remark 2: When μ has no atom, Lemma 2.2 applied to $Z = g \star B''$ shows that for P–almost all ω, the process $Z(\omega, .)$ is Q_ω–a.s. continuous. Then $g \star B$ is a.s. continuous (since $g \star B'$ is clearly so; in fact B is a continuous martingale measure). If μ has atoms, then $g \star B$ jumps at each time the bracket (3.8) jumps; now, by (1.4) and since $\delta_n \to 0$, the jumps of $U^n(g)$ tend uniformly to 0, so we cannot have convergence in law of $U^n(g)$ to $g \star B$ in the Skorokhod sense. □

Remark 3: In case $\mu = \mu^\star = \lambda$, Theorem 1.1 is a part of Theorem 3.3 (a part only, because the statement in Theorem 1.1 does not completely characterizes the random measure B). In this case ν is “continuous in time” and deterministic, so B is a centered Gaussian measure, whose law is determined by ν. Now, if one starts with a white noise $\widetilde{B}$ on $\mathbb{R}_+ \times \mathbb{R}^d$ with intensity measure $dt \otimes \rho(dx)$, a simple conditioning on Gaussian random vectors shows that, conditionally on having $\widetilde{B}([0,t] \times \mathbb{R}^d) = 0$ for all t, the covariance of $\widetilde{B}$ is given by (3.3) with $\mu = \lambda$. Further the bracket (3.10) for $g(\omega, t, x) = x$ is null because $\mu = \mu^\star = \lambda$, and (3.9) gives $x \star B' = W$, hence $x \star B = W$: that is, B satisfies all requirements of Theorem 1.1.

More generally, when $\theta^\star \equiv 1$ then $B''(\omega, .)$ is under Q_ω a white noise with intensity measure $\mu(\omega, dt) \otimes \rho(dx)$, conditioned on $1 \star B''(\omega) = 0$ and $x \star B''(\omega) = 0$. When $\theta^\star \equiv 0$ it is a white noise with the same intensity measure, conditioned on $1 \star B''(\omega) = 0$.

4 Discretization schemes

In all this section we are given a sequence $(\mathcal{T}_n)$ satisfying (A1) and (A2).

Proof of Lemma 3.1. a) We will first prove that a.s., for all $s < t$:

$$\mu^\star((s,t]) \leq \sqrt{t-s}\ \sqrt{\mu((s,t])}. \tag{4.1}$$

To this effect, up to taking a subsequence, we may assume that for all fixed ω outside a null set we have $\mu_n \to \mu$ and $\mu^\star_n \to \mu^\star$ weakly. Since $\Delta(n,i) \leq t$ if $S(n,i) \leq t$, we have

$$\begin{aligned} \mu^\star_n((s,t)) &= \sum_{i: s < S(n,i) < t} \sqrt{\Delta(n,i)\delta_n} \\ &\leq \sqrt{\delta_n t} + \sum_{i: s < S(n,i-1), S(n,i) < t} \sqrt{\Delta(n,i)\delta_n} \end{aligned}$$

$$\leq \sqrt{\delta_n t} + \left(\sum_{i:s<S(n,i-1)<t} \delta_n\right)^{1/2} \left(\sum_{i:s<S(n,i-1),S(n,i)<t} \Delta(n,i)\right)^{1/2}$$
$$\leq \sqrt{\delta_n t} + \sqrt{\mu_n((s,t))}\ \sqrt{t-s}.$$

Since $\delta_n \to 0$, and $\mu^\star((s,t)) \leq \liminf_n \mu_n^\star((s,t))$ and $\limsup_n \mu_n([s,t]) \leq \mu([s,t])$, we get $\mu^\star((s,t)) \leq \sqrt{t-s}\ \sqrt{\mu([s,t])}$. This implies first that $\mu^\star$ has no atom, and secondly that (4.1) holds.

b) Let $\lambda' = \lambda + \mu$, so that $\lambda = \alpha \cdot \lambda'$ and $\mu = \beta \cdot \lambda'$ for two nonnegative predictable processes α, β with $\alpha + \beta = 1$ (recall that all adapted cadlag processes on the Wiener space are predictable). By applying the martingale construction of Radon-Nikodym derivatives, we deduce from (4.1) that $\mu^\star$ has the form $\mu^\star = \gamma \cdot \lambda'$ for some γ satisfying $\gamma \leq \sqrt{\alpha\beta}$.

First $1_{\{\alpha>0\}} \cdot \lambda' = ((1/\alpha)1_{\{\alpha>0\}}) \cdot \lambda$. Then $\mu^{ac} = ((\beta/\alpha)1_{\{\alpha>0\}}) \cdot \lambda$, that is a version of θ in (3.1) is $\theta = (\beta/\alpha)1_{\{\alpha>0\}}$. Next, since $\alpha = 0$ implies $\gamma = 0$ we get $\mu^\star = ((\gamma/\alpha)1_{\{\alpha>0\}}) \cdot \lambda$, hence a version of $\theta^\star$ is $\theta^\star = (\gamma/\alpha)1_{\{\alpha>0\}}$. Since $\gamma^2 \leq \alpha\beta$, (3.2) readily follows. $\square$

2) Next, we show that it is not a restriction to suppose, in addition to (A1) and (A2), the following:

Assumption A3: All stopping times of the schemes $\mathcal{T}_n$ are finite–valued, and the total mass of μ is infinite. $\square$

Indeed, we wish to prove results of the form (3.11) with $Y_n = U^n(g)$ and f being continuous for the Skorokhod topology. As is well know, for this it is enough to consider functions f that depend only on the restriction of the path of the process to any finite interval. That is, we really have to consider the processes $U^n(g)$ on (arbitrary) finite intervals.

So fix $T \in I$ (see (3.12)) and define a new scheme $\mathcal{T}'_n$ as follows: Replace the times $T(n,i) \geq T$ by the times $T + j\delta_n$ for $j \in \mathbb{N}$, and re–order so as to obtain a new strictly increasing sequence $T'(n,i)$ of stopping times, then set

$$\Delta'(n,i) = \begin{cases} \Delta(n,i) \bigwedge (T - T(n,i)) & \text{if } \ T(n,i-1) < T \\ \delta_n & \text{otherwise.} \end{cases}$$

This defines new schemes $\mathcal{T}'_n = (T'(n,i), \Delta'(n,i) : i \geq 1)$ which satisfy (A1). The measures μ'_n and $\mu'^\star_n$ associated with $\mathcal{T}'_n$ by (1.6) ad (1.7) coincide with μ_n and $\mu_n^\star$ on $[0,T)$, and the three measures μ'_n, $\mu'^\star_n$ and $\delta_n \sum_{i\geq 1} \varepsilon_{T+i\delta_n}$ coincide on (T,∞). Since $T \in I$, the sequence $(\mathcal{T}'_n)$ satisfies (A2) with $\mu' = 1_{[0,T)} \cdot \mu + 1_{[T,\infty)} \cdot \lambda$ and $\mu'^\star = 1_{[0,T)} \cdot \mu^\star + 1_{[T,\infty)} \cdot \lambda$, hence (A3) as well. Further, it follows that the $\mathcal{F}$–conditional distributions of the restriction to $[0,T] \times \mathbb{R}^d$ of the tangent measures along $(\mathcal{T}_n)$ and $(\mathcal{T}'_n)$ coincide.

Now the processes $U'^n(g)$ associated with $\mathcal{T}'_n$ by (1.4) have $U'^n(g) = U^n(g)$ on $[0,T)$. Then if we can prove Theorem 3.3 for $(\mathcal{T}'_n)$, and since T is arbitrary large, we deduce Theorem 3.3 for $(\mathcal{T}_n)$.

Thus it is no restriction to assume (A3), in addition to (A1) and (A2).

3) As stated in Remark 2, we do not have functional convergence of the $U^n(g)$'s when μ has atoms. And even if μ has no atom we have problems in proving the stable convergence if the support of μ has "holes". To solve these problems, we add fictitious point to fill in the holes, and also change time to "smooth" out the atoms of μ. This amounts to modify the limiting measures μ and $\mu^\star$ according to the following.

For any right-continuous non-decreasing function F: $\mathbb{R}_+ \to \overline{\mathbb{R}}_+$ we call F^{-1} its right-continuous inverse (taking values in $\overline{I\!R}_+$ again). We write $F(\infty)=\lim_{t\to\infty}F(t)$. Let D be the (random) topological support of μ, and set

$$\left.\begin{array}{ll} F(t)=\mu([0,t]), & F^\star(t)=\mu^\star([0,t]) \\ F'(t)=F(t)+\int_0^t 1_{D^c}(s)ds, & F''(t)=\inf(s>0:\ s+F'^{-1}(s)>t). \end{array}\right\} \tag{4.2}$$

$$\left.\begin{array}{ll} \Phi(t)=t-F''(t), & A=\{t:\ \Phi(t+\varepsilon)>\Phi(t)\ \forall\varepsilon>0\}, \\ R(t)=F(\Phi(t))+t-u_t, & \text{where}\ \ u_t=\inf(s\geq t:\ s\in A), \\ R^\star(t)=F^\star(\Phi(t)). & \end{array}\right\} \tag{4.3}$$

Lemma 4.1 *a) Each $\Phi(t)$ is an $(\mathcal{F}_t)$–stopping time, and the processes Φ, R, $R^\star$ are continuous, non-decreasing, adapted to the filtration $(\mathcal{F}_{\Phi(t)})_{t\geq 0}$, and $R(\infty)=\Phi(\infty)=\infty$.*

b) There are $(\mathcal{F}_{\Phi(t)})$–predictable processes ϕ, ψ, $\psi^\star$ such that a.s.

$$\Phi(t)=\int_0^t\phi(s)ds,\qquad R(t)=\int_0^t\psi(s)ds,\qquad R^\star(t)=\int_0^t\psi^\star(s)ds, \tag{4.4}$$

$$0\leq\phi\leq 1_A,\qquad 1_{A^c}\leq\psi\leq 1,\qquad 0\leq\psi^\star\leq\sqrt{\phi\psi}. \tag{4.5}$$

Proof. i) As said before, F'' and F''^{-1} are continuous and strictly increasing, and $F''(t)-F''(s)\leq t-s$ is obvious when $s\leq t$, hence $0\leq\Phi(t)-\Phi(s)\leq t-s$: therefore Φ has the form (4.4), with $0\leq\phi\leq 1_A$. Further

$$\{\Phi(t)\leq s\}=\{F''(t)\geq t-s\}=\{t\geq t-s+F'^{-1}(t-s)\}=\{F'^{-1}(t-s)\leq s\}=\{F'(s)\leq t-s\}\in\mathcal{F}_s$$

because F' is $(\mathcal{F}_t)$–adapted. This yields that $\Phi(\infty)=\infty$ and that $\Phi(t)$ is an $(\mathcal{F}_t)$–stopping time for each t, hence Φ is $(\mathcal{F}\Phi(t))$–predictable (recall that Φ is continuous) and there is an $(\mathcal{F}_{\Phi(t)})$–predictable version of ϕ as well.

ii) The following chain of equivalences is obvious: $F'(r-)\leq v\leq F'(r)\Leftrightarrow r=F'^{-1}(v)\Leftrightarrow F''^{-1}(v)=v+r\Leftrightarrow F''(v+r)=v\Leftrightarrow\Phi(v+r)=r$. Further F' is strictly increasing, and $F'(r)-F'(r-)=F(r)-F(r-)$. Therefore if $u'_t=\sup(s\leq t:s\in A)$ (with $\sup(\emptyset)=0$), we readily deduce from (4.3) that

$$\left.\begin{array}{ll} R(t)=F(\Phi(t)-)+t-u'_t, & F(\Phi(t)-)\leq R(t)\leq F(\Phi(t)), \\ u_t=\Phi(t)+F'(\Phi(t)), & u'_t=\Phi(t)+F'(\Phi(t)-). \end{array}\right\} \tag{4.6}$$

Therefore R is non–decreasing, and $R(\infty)=\infty$ because $\Phi(\infty)=\infty$, and $F(\infty)=\infty$) by (A3), and R is linear with slope 1 on each interval $[u'_t,u_t]$. If $s<t$ and $u_s\leq u'_t$, we also have by (4.6):

$$\begin{array}{rcl} R(u'_t)-R(u_s) & = & F(\Phi(t)-)-F(\Phi(s)) \\ & \leq & F'(\Phi(t)-)-F'(\Phi(s))\ =\ u'_t-\Phi(t)-u_s+\Phi(s)\ \leq\ u'_t-u_s \end{array}$$

and it follows that $R(t)-R(s)\leq t-s$, whereas $R(t)-R(s)=t-s$ is obvious when $(s,t)\subset A^c$. Hence R has the form (4.4) with ψ satisfying $1_{A^c}\leq\psi\leq 1$. Further $\{u_t\geq s\}=\{\Phi(s)=\Phi(t)\}\in\mathcal{F}_{\Phi(t)}$, hence u_t is $\mathcal{F}_{\Phi(t)}$–measurable. Since F and $F^\star$ are $(\mathcal{F}_t)$–adapted and right-continuous, $F(\Phi(t))$ and $F^\star(\Phi(t))$ are $\mathcal{F}_{\Phi(t)}$–measurable, and thus R and $R^\star$ are $(\mathcal{F}_{\Phi(t)})$–adapted. Therefore we can choose a version of ψ that is $(\mathcal{F}_{\Phi(t)})$– predictable.

iii) By definition of $R^\star$ we deduce from (4.1) and (4.6) that a.s.

$$\begin{array}{rcl} 0\ \leq\ R^\star(t)-R^\star(s) & \leq & \sqrt{\Phi(t)-\Phi(s)}\ \sqrt{F(\Phi(t)-)-F(\Phi(s))} \\ & \leq & \sqrt{\Phi(t)-\Phi(s)}\ \sqrt{R(t)-R(s)}. \end{array}$$

Exactly as in the proof of Lemma 3.1, we get (4.4) for $R^\star$ with $\psi^\star \leq \sqrt{\phi\psi}$, and $\psi^\star$ can be chosen $(\mathcal{F}_{\Phi(t)})$–predictable because $R^\star$ is $(\mathcal{F}_{\Phi(t)})$–adapted. $\square$

Lemma 4.2 *There exists an $(\mathcal{F}_t)$–predictable set B such that the processes θ and $\theta^\star$ in (3.1) have for λ–almost all t:*

$$\psi(t)1_B(\Phi(t)) = \phi(t)\theta_{\Phi(t)}, \qquad \psi^\star(t) = \phi(t)\theta^\star_{\Phi(t)}. \tag{4.7}$$

Proof. a) (4.3) and (4.4) give $\int_0^{\Phi(t)} \theta^\star_s ds = \int_0^t \psi^\star(s)ds$, and Lebesgue derivation Theorem yields the second property (4.7).

b) Observe that

$$\int_0^{u_t} h \circ \Phi(r)\ \psi(r)\ dr = \int_{[0,\Phi(t)]} h(r)\ \mu(dr) \tag{4.8}$$

is true for $h = 1_{[0,v]}$ (it then reduces to $R(u_t \bigwedge \Phi^{-1}(v)) = F(\Phi(t) \bigwedge v)$, which holds by (4.3) because $u_t \bigwedge \Phi^{-1}(v)$ belongs to A), hence for all bounded Borel function h.

Recall that $\mu^s = \mu - \mu^{ac}$. Since F is predictable, there is a predictable set B which supports μ^{ac} and is not charged by μ^s. In particular $1_B \cdot \mu = \mu^{ac} = \theta \cdot \lambda$. Further $\mu^s(B) = 0$ implies that $1_B \circ \Phi(r) = 1_B \circ \Phi(t) = 0$ if $t \leq r \leq u_t$ and $t < u_t$, because then $F[\Phi(t)-) < F(\Phi(t))$ by (4.6). Then applying (4.8) with $h = 1_B$ gives

$$\int_0^t 1_B(\Phi(s))\ \psi(s)\ ds = \int_{[0,\Phi(t)]} 1_B(r)\ \mu(dr) = \int_0^{\Phi(t)} \theta_s\ ds,$$

and Lebesgue derivation Theorem again implies the first part of (4.7). $\square$

4) Now we introduce a time-change. Set

$$S(t) = R^{-1}(t), \qquad \tau(t) = S \circ F(t). \tag{4.9}$$

Each $S(t)$ is a finite–valued $(\mathcal{F}_{\Phi(t)})$–stopping time, because $R(\infty) = \infty$ and R is adapted to the filtration $(\mathcal{F}_{\Phi(t)})$. Further,

Lemma 4.3 *Each $\tau(t)$ is a finite–valued $(\mathcal{F}_{\Phi(t)})$–stopping time given by the following formula, where $t_+ = \inf(v > t :\ F(v) > F(t))$.*

$$\tau(t) = \begin{cases} \Phi^{-1}(t_+) & \text{if } F(t) = F(t_+) \\ \Phi^{-1}(t_+-) & \text{if } F(t) < F(t_+). \end{cases} \tag{4.10}$$

Proof. Set $s = \tau(t)$. First $R(s) = F(t)$, hence $F(\Phi(s)-) \leq F(t)$ by (4.6), hence $\Phi(s) \leq t_+$. Second, for $\varepsilon > 0$ we have $R(s+\varepsilon) > F(t)$, hence $F(t) < F(\Phi(s+\varepsilon))$ by (4.6), hence $t_+ \leq \Phi(s+\varepsilon)$ and by continuity of Φ we get $t_+ \leq \Phi(s)$. Now, this and (4.6) imply $F(t) = R(s) = F(t_+) + s - u_s$; if $F(t) = F(t_+)$ this yields $s = u_s = \Phi^{-1}(t)$; otherwise $F(t) = F(t_+-)$, hence $s = u'_s = \Phi^{-1}(t_+-)$. Thus (4.10) is proved.

For every $(\mathcal{F}_t)$–stopping time T, we have $\{\Phi^{-1}(T) < r\} = \{T < \Phi(r)\} \in \mathcal{F}_{\Phi(r)}$ and $\{\Phi^{-1}(T-) \leq r\} = \{T \leq \Phi(r)\} \in \mathcal{F}_{\Phi(r)}$, hence both $\Phi^{-1}(T)$ and $\Phi^{-1}(T-)$ are $(\mathcal{F}_{\Phi(t)})$–stopping times. The stopping time property of $\tau(t)$ follows, because by (4.10) $\tau(t) = \Phi^{-1}(T) \bigwedge \Phi^{-1}(T'-)$ if $T = t_+$ (resp. ∞) and $T' = \infty$ (resp. t_+) if $F(t) = F(t_+)$ (resp. $F(t) < F(t_+)$). $\square$

Lemma 4.4 *Let k be a locally bounded $(\mathcal{F}_t)$–predictable process and $W'_t = W_{\Phi(t)}$. Then*

$$\int_0^{\tau(t)} k \circ \Phi(r)\ \psi(r)\ dr = \int_{[0,t]} k(r)\ \mu(dr), \tag{4.11}$$

$$\int_0^{\tau(t)} (k1_{\{\theta>0\}}) \circ \Phi(r)\ \psi(r)\ dW'_r = \int_{[0,t]} (k1_{\{\theta>0\}})(r)\ dW_r, \tag{4.12}$$

The process $(k1_{\{\theta>0\}}) \circ \Phi$ is $(\mathcal{F}_{\Phi(t)})$–predictable and $\tau(t)$ is an $(\mathcal{F}_{\Phi(t)})$–stopping time, hence the first stochastic integral in (4.12) is meaningful.

Proof. a) We use (4.10): if $F(t) = F(t_+)$ then $\tau(t) = u_{\tau(t)}$ and $\Phi(\tau(t)) = t_+$ hence (4.11) follows from (4.8) because $\mu((t,t_+]) = 0$. Suppose now $F(t) < F(t_+)$. Then $\tau(t) = u'_{\tau(t)}$ and $\Phi(\tau(t)) = t_+$ again, and $\psi = 1$ on $(u'_{\tau(t)}, u_{\tau(t)})$ by (4.5), so by (4.8):

$$\begin{aligned}\int_0^{\tau(t)} k \circ \Phi(r)\ \psi(r)\ dr &= \int_0^{u_{\tau(t)}} k \circ \Phi(r)\ \psi(r)\ dr - k \circ \Phi(\tau(t))\big(u_{\tau(t)} - u'_{\tau(t)}\big) \\ &= \int_{[0,t_+]} k(r)\ \mu(dr) - k(t_+)\mu(\{t_+\}) = \int_{[0,t]} k(r)\ \mu(dr).\end{aligned}$$

b) Set $M'_t = \int_0^t (k1_{\{\theta>0\}}) \circ \Phi(r)\ \psi(r)\ dW'_r$ and $M_t = \int_0^t (k1_{\{\theta>0\}})(r)\ dW_r$. The process Φ is a continuous time–change, hence $M'_t = M_{\Phi(t)}$ a.s. for all t (see e.g. Chapter 10 of [7]). In particular $M'_{\tau(t)} = M_{t_+}$ because $\Phi(\tau(t)) = t_+$. If $t_+ = t$ this gives (4.12). If $t < t_+$ we have $\theta = 0$ λ-a.s. on $[t, t_+]$, hence $M_{t_+} = M_t$ and (4.12) holds also in this case. □

5) In fact, Φ, R and $R^\star$ appear in the limiting behavior of some denser discretization schemes that are associated to the original ones as follows. We still assume (A1), (A2) and (A3).

First set $D_t^\varepsilon(\omega) = \{x \in [0,t] :\ d(x, D(\omega)) \geq \varepsilon\}$ (recall that D is the topological support of μ). Since $\mu(D_t^\varepsilon) = 0$ and D_t^ε is closed, A2 yields $\mu_n(D_t^\varepsilon) \to 0$ for all t. There is an increasing sequence $n_p \uparrow \infty$ with $n \geq n_p \Rightarrow P(\mu_n(D_p^{1/p}) > 1/p) \leq 1/p$, and thus $p_n = \sup(p : n_p \leq n)$ has:

$$p_n \uparrow \infty, \qquad P(\mu_n(D_{p_n}^{1/p_n}) > 1/p_n) \leq \frac{1}{p_n}. \tag{4.13}$$

Next we set $\alpha_n = (\delta_n \sqrt{p_n}) \bigwedge \sqrt{\delta_n}$, which is a sequence satisfying

$$\alpha_n \to 0, \qquad \delta_n/\alpha_n \to 0, \qquad \alpha_n/\delta_n p_n \to 0. \tag{4.14}$$

The idea of what follows is such: we first suppress the points $T(n,i)$ for which $\Delta(n,i) \geq \alpha_n$, and (4.14) ensures that we still keep (A2). Next we add subdivision points in the complement D^c of D, spaced by δ_n (so the corresponding "empirical" measure goes to Lebesgue measure on D^c) and distant from the initial subdivision points by α_n (which is small, yet "much bigger" than δ_n by (4.14)). Then we change time by substituting $T'(n,i)$ with $i\delta_n$ for the ith new subdivision point $T'(n,i)$. Since we must preserve some "stopping time" properties and keep track of the $S(n,i)$'s as well, things are a bit complicated. We do this step by step.

Step 1: Deleting points. We set

$$J_n = \{i \in I\!N :\ \Delta(n,i) < \alpha_n\}, \qquad J'_n = I\!N \backslash J_n, \qquad C(n) = \{T(n,i) :\ i \in J_n\}, \tag{4.15}$$

$$\nu_n = \delta_n \sum_{i \in J_n} \varepsilon_{T(n,i)}, \qquad \nu_n^\star = \sum_{i \in J_n} \sqrt{\Delta(n,i)\delta_n}\ \varepsilon_{T(n,i)}, \tag{4.16}$$

$$\Sigma(n,t) = \{i \in I\!N :\ S(n,i) \leq t\}. \tag{4.17}$$

Lemma 4.5 *We have*

$$\delta_n \ card(J'_n \cap \Sigma(n,t)) \leq t\delta_n/\alpha_n \to 0, \tag{4.18}$$

$$\nu_n \ \xrightarrow{P} \ \mu, \qquad \nu^\star_n \ \xrightarrow{P} \ \mu^\star. \tag{4.19}$$

Proof. Since $\sum_{i\in\Sigma(n,t)} \Delta(n,i) \leq t$ we have $\text{card}(J'_n \cap \Sigma(n,t)) \leq t/\alpha_n$ and (4.18) follows from (4.14). Next, set $\widehat{\nu}_n = \delta_n \sum_{i\in J_n} \varepsilon_{S(n,i)}$ and $\widehat{\nu}^\star_n = \sum_{i\in J_n} \sqrt{\Delta(n,i)\delta_n} \ \varepsilon_{S(n,i)}$. We have $\widehat{\nu}_n \leq \mu_n$ and $\widehat{\nu}^\star_n \leq \mu^\star_n$. Also $(\mu_n - \widehat{\nu}_n)([0,t]) = \delta_n \text{card}(J'_n \cap \Sigma(n,t))$ and $(\mu^\star_n - \widehat{\nu}_n)^\star([0,t]) = \sqrt{\delta_n \ \text{card}(J'_n \cap \Sigma(n,t))}$ by Cauchy–Schwarz inequality. Thus (A2) and (4.18) give us $\widehat{\nu}_n \xrightarrow{P} \mu$ and $\widehat{\nu}^\star_n \xrightarrow{P} \mu^\star$.

Now for all $i \in J_n$ we have $\Delta(n,i) < \alpha_n$, hence $0 \leq S(n,i) - T(n,i) \leq \alpha_n$, thus $\nu_n(f) - \widehat{\nu}_n(f)$ and $\nu^\star_n(f) - \widehat{\nu}^\star_n(f)$ tend to 0 in probability for every continuous function f with compact support, and (4.19) follows. □

Step 2: Adding points. Now we set

$$C(n,i) = \{T(n,i) + \alpha_n + j\delta_n : \ j \in I\!N\} \cap [0, T(n,i+1)) \cap D^c.$$

For n fixed, these sets are pairwise disjoint (some or even all may be empty), and also disjoint from $C(n)$. Set also

$$C''(n) = \bigcup_{i\in I\!N} C(n,i), \qquad C'(n) = C(n) \cup C''(n).$$

$C'(n)$ is an optional locally finite random set. We define a strictly increasing sequence of stopping times and a random measure by

$$\left.\begin{array}{l} T'(n,0) = 0, \qquad T'(n,i+1) = \inf(t \in C'(n) : \ t > T'(n,i)) \\ \mu'_n = \delta_n \sum_{i\geq 0} \ \varepsilon_{T'(n,i)}. \end{array}\right\} \tag{4.20}$$

Lemma 4.6 *We have $\mu'_n \xrightarrow{P} \mu'$, where the measure μ' is such that $\mu'([0,t]) = F'(t)$, as given by (4.2).*

Proof. Up to taking a subsequence we may assume that $\sum 1/p_n < \infty$ and that outside a P–null set (recall (4.13), (A2) and (4.19)):

$$\nu_n \to \mu, \qquad \mu_n \to \mu, \qquad \mu_n(D^{1/p_n}_{p_n}) \leq 1/p_n \quad \text{for } n \text{ large enough.} \tag{4.21}$$

We set $\overline{\mu}_n = \delta_n \sum_{s\in C'(n)} \varepsilon_s$ and $\overline{\mu} = 1_{D^c} \cdot \lambda$. Then $\mu' = \mu + \overline{\mu}$ and, since $C(n) \cap C''(n) = \emptyset$, we have $\mu'_n = \nu_n + \overline{\mu}_n$, so if we prove $\overline{\mu}_n \to \overline{\mu}$ for all ω having (4.21) then $\mu'_n \to \mu'$ for those ω, and the result will obtain. Hence below we fix an ω having (4.21).

Intervals between successive points in $C''(n)$ have length not smaller than δ_n, so $\overline{\mu}_n([s,t]) \leq t - s + \delta_n$. Since $\delta_n \to 0$ we deduce that the sequence $(\overline{\mu}_n)$ is relatively compact for the vague topology and all limit points are smaller than λ. Remembering that ω is fixed, it is then enough to show that if a subsequence still denoted by $(\overline{\mu}_n)$ converges to a limit $\overline{\mu}'$, then $\overline{\mu}' = \overline{\mu}$.

Let (U,V) be an interval contiguous to D and fix $t \in I\!R_+$ and $\varepsilon < (V-U)/2$. The set $C''(n) \cap (U,V) \cap [0,t]$ is a finite set whose points are equally spaced by δ_n, except for gaps of length smaller than $\delta_n + \alpha_n$ around all points $T(n,i)$ in $(U,V) \cap [0,t]$. Hence if N_n denotes the number of points $T(n,i)$ within $(U+\varepsilon, V-\varepsilon) \cap [0,t]$, the number of points in $C''(n) \cap (U+\varepsilon, V-\varepsilon) \cap [0,t]$ is bigger than $(V \bigwedge t - U \bigwedge t - 2\varepsilon - N_n(\delta_n + \alpha_n))/\delta_n$. Finally since $S(n,i) \leq T(n,i+1) < S(n,i+1)$, the last statement in (4.21) shows that for n large enough we have $p_n \geq t \bigvee (1/\varepsilon)$ and $N_n \leq 1 + 1/\delta_n p_n$, hence

$$\overline{\mu}_n([U+\varepsilon, V-\varepsilon] \cap [0,t]) \geq V \bigwedge t - U \bigwedge t - 2\varepsilon - (1 + 1/\delta_n p_n)(\delta_n + \alpha_n).$$

Since $\overline{\mu}'([U+\varepsilon, V-\varepsilon] \cap [0,t]) \geq \limsup_n \overline{\mu}_n([U+\varepsilon, V-\varepsilon] \cap [0,t])$ we deduce from (4.14) and the above that $\overline{\mu}'([U+\varepsilon, V-\varepsilon] \cap [0,t]) \geq V \bigwedge t - U \bigwedge t$, which equals $\overline{\mu}((U,V) \cap [0,t])$. Since $\overline{\mu}$ is supported by D^c, it follows that $\overline{\mu}' \geq \overline{\mu}$.

Finally $\overline{\mu}_n(D) = 0$ by construction, hence if D^0 is the (possibly empty) interior of D we have $\overline{\mu}'(D^0) = 0$ because $\overline{\mu}_n \to \overline{\mu}'$. Since the Lebesgue measure of a closed set with empty interior is null and $\overline{\mu}' \leq \lambda$, we deduce that $\overline{\mu}'(D \backslash D^0) = 0$, hence $\overline{\mu}'(D) = 0$, hence $\overline{\mu}' \leq \overline{\mu}$ because $\overline{\mu}' \leq \lambda$ and $\overline{\mu} = \lambda$ on the complement of D. Therefore $\overline{\mu}' = \overline{\mu}$ and the proof is finished. □

Step 3: Changing time. Set

$$\left.\begin{array}{l} A'_n = \{i \in I\!N : \ \exists j \in J_n \text{ such that } T'(n,i) = T(n,j)\} \\ \Delta'(n,i) = \Delta(n,j), \quad S'(n,i) = S(n,j) \quad \text{if } i \in A'_n \text{ and } T'(n,i) = T(n,j), \end{array}\right\} \tag{4.22}$$

$$T''(n,i) = T'(n,i) + i\delta_n, \qquad \text{and} \quad S''(n,i) = S'(n,i) + (i+1)\delta_n \quad \text{if } i \in A'_n. \tag{4.23}$$

If $j \in J_n$ we have $C'(n) \cap (T(n,j), S(n,j)] = \emptyset$. Therefore,

$$\left.\begin{array}{l} \text{if } i \in A'_n \text{ then } S'(n,i) \leq T'(n,i+1) \text{ and } S''(n,i) \leq T''(n,i+1), \\ \text{if further } S'(n,i) = T'(n,i+1) \text{ then } S''(n,i) = T''(n,i+1). \end{array}\right\} \tag{4.24}$$

The locally finite set $U(n) = \{T'(n,i) : i \in I\!N\} \cup \{S'(n,i) : i \in A'_n\}$ is re–ordered through the following strictly increasing sequence of stopping times:

$$R'(n,0) = 0, \qquad R'(n,i+1) = \inf(t > R'(n,i) : \ t \in U(n)). \tag{4.25}$$

Then we set

$$R''(n,i) = \left\{ \begin{array}{ll} T''(n,j) & \text{if } R'(n,i) = T'(n,j) \\ S''(n,j) & \text{if } R'(n,i) = S'(n,j) \text{ and } j \in A'_n. \end{array}\right. \tag{4.26}$$

(it is possible that $R'(n,i) = S'(n,j) = T'(n,j+1)$, but by (4.24) there is no ambiguity above), and

$$\left.\begin{array}{l} A_n = \{i \in I\!N : \ \text{there is a (unique) } j \in A'_n \text{ with } R'(n,i) = T'(n,j), \\ \nabla(n,i) = R'(n,i+1) - R'(n,i), \end{array}\right\} \tag{4.27}$$

$$\left.\begin{array}{l} \Sigma''(n,t) = \{i \in I\!N : \ R''(n,i+1) \leq t\} \\ \sigma(n,t) = \{i \in A'_n : \ R''(n,i+1) \leq t\} \\ \Phi_n(t) = R'(n,i+1) \quad \text{if} \quad R''(n,i) \leq t < R''(n,i+1). \end{array}\right\} \tag{4.28}$$

Step 4: Measurability properties. We have the following:

Lemma 4.7 *a) We have $\{i \in A_n\} \in \mathcal{F}_{R'(n,i)}$ and, in restriction to the set $\{i \in A_n\}$, the variables $R'(n,i+1)$ and $R''(n,i+1)$ are $\mathcal{F}_{R'(n,i)}$–measurable.*

b) Each $\Phi_n(t)$ is an $(\mathcal{F}_t)$–stopping time; we set $\mathcal{F}^n_t = \mathcal{F}_{\Phi_n(t)}$.

c) Each $R''(n,i)$ is an $(\mathcal{F}^n_t)$–stopping time, and $\mathcal{F}^n_{R''(n,i)} = \mathcal{F}_{R'(n,i+1)}$ and $\mathcal{F}^n_{R''(n,i)-} = \mathcal{F}_{R'(n,i)}$ ($\mathcal{F}^n_{0-}$ is the trivial σ–field, by convention).

Proof. a) It is enough to use (A1) and to observe that

$$\{i\in A_n\}\cap\{R'(n,i+1)\geq t\}=\cup_{j\in I\!N}\{R'(n,i)=T(n,j),\ t-T(n,j)\leq\Delta(n,j)<\alpha_n\},$$

$$\{i\in A_n\}\cap\{R''(n,i+1)\geq t\}=\cup_{j\in I\!N}\{R'(n,i)=T(n,j),\ t-T(n,j)-(j+1)\delta_n\leq\Delta(n,j)<\alpha_n\}.$$

b) By definition of $\Phi_n(t)$,

$$\{\Phi_n(t)\leq s\}=\cup_{i\in I\!N}D_i^n,\qquad D_i^n=\{R'(n,i+1)\leq s,\ R''(n,i)\leq t<R''(n,i+1)\}.$$

The sets $D_i^n\cap\{i\in A_n\}$ and $D_i^n\cap\{i+1\in A_n\}$ are in $\mathcal{F}_s$ by (a). The set $D_i^n\cap\{i\notin A_n\}\cap\{i+1\notin A_n\}$ is the union for all $k\in I\!N$ of the sets $\{R'(n,i+1)=S(n,k+1)\leq s,\ R'(n,i)=S(n,k),\ \Delta(n,k)<\alpha_n,\ \Delta(n,k)\leq t-T(n,k)-(k+1)\delta_n,\ t-T(n,k+1)-(k+2)\delta_n<\Delta(n,k+1)<\alpha_n\}$, also in $\mathcal{F}_s$ by (A1) and the fact that $R'(n,i+1)$ is a stopping time, hence the claim.

c) By definition of Φ_n again, $A\cap\{R''(n,i)\leq t\}=A\cap\{R'(n,i+1)\leq\Phi_n(t)\}$. Then if $A\in\mathcal{F}_{R'(n,i)}$ we get $A\cap\{R''(n,i)\leq t\}\in\mathcal{F}_{\Phi_n(t)}=\mathcal{F}^n_t$: hence $R''(n,i)$ is an $(\mathcal{F}^n_t)$–stopping time (take $A=\Omega$) and $\mathcal{F}_{R'(n,i+1)}\subset\mathcal{F}^n_{R''(n,i)}$. The opposite inclusion $\mathcal{F}^n_{R''(n,i)}\subset\mathcal{F}_{R'(n,i+1)}$ follows from $\Phi_n(R''(n,i))=R'(n,i+1)$ and from Lemma (10.5) of [7]. Hence $\mathcal{F}^n_{R''(n,i)}=\mathcal{F}_{R'(n,i+1)}$.

The last claim is obvious if $i=0$, so let $i\geq 1$. Since $R''(n,i-1)<R''(n,i)$ and $\mathcal{F}^n_{R''(n,i-1)}=\mathcal{F}_{R'(n,i)}$ we get $\mathcal{F}^n_{R'(n,i)}\subset\mathcal{F}_{R''(n,i)-}$. Conversely $\mathcal{F}^n_{R''(n,i)-}$ is generated by the sets $A\cap\{t<R''(n,i)\}$ for $t\geq 0$ and $A\in\mathcal{F}^n_t$; then $A\cap\{t<R''(n,i)\}=A\cap\{\Phi_n(t)\leq R'(n,i)\}$ is $\mathcal{F}_{R'(n,i)}$–measurable, hence $\mathcal{F}^n_{R''(n,i)-}\subset\mathcal{F}_{R'(n,i)}$. $\square$

Step 5: Limiting results. The following (with Φ, ψ, $\psi^\star$ as in (4.4)) will be crucial for the proof of the main theorems:

Lemma 4.8 *The following convergences, where f denotes a bounded continuous function, hold in probability uniformly on compact subsets of $I\!R_+$:*

$$\Phi_n(t)\ \to\ \Phi(t), \tag{4.29}$$

$$\delta_n\sum_{i\in\sigma(n,t)}f(R'(n,i))\ \to\ \int_0^t\psi(s)\ f\circ\Phi(s)\ ds, \tag{4.30}$$

$$\sum_{i\in\sigma(n,t)}\sqrt{V(n,i)\delta_n}\ f(R'(n,i))\ \to\ \int_0^t\psi^\star(s)\ f\circ\Phi(s)\ ds. \tag{4.31}$$

Proof. a) For (4.30) and (4.31) it suffices to consider nonnegative functions. Hence all processes above are increasing, and in addition the limiting processes are continuous: it is then enough to prove the convergence in probability for each $t\geq 0$. Up to taking subsequences, we can assume that in A2 and in Lemmas 4.5 and 4.6 the convergences hold a.s. So we fix ω such that $\nu_n\to\mu$, $\mu'_n\to\mu'$ and $\nu^\star_n\to\mu^\star$.

b) Consider the following measures on $I\!R_+$ (recall that $\Delta'(n,i)$ is well defined if $i\in A'_n$: see (4.22)):

$$\mu''_n=\delta_n\sum_{i\geq 0}\varepsilon_{T''(n,i)}\ ,$$

$$r_n=\delta_n\sum_{i\in A'_n}\varepsilon_{T''(n,i)}\ ,\qquad r^\star_n=\sum_{i\in A'_n}\sqrt{\Delta'(n,i)\delta_n}\ \varepsilon_{T''(n,i)}\ ,$$

and denote by F_n , $F_n^\star$, F_n', F_n'', R_n and $R_n^\star$ the repartition functions of ν_n, $\nu_n^\star$, μ_n', μ_n'', r_n, and $r_n^\star$ respectively.

c) $\mu_n' \to \mu'$ gives $F_n'(t) \to F'(t)$ for all t having $F'(t) = F'(t-)$. Since F'^{-1} is continuous it follows that

$$F_n'^{-1} \;\to\; F'^{-1} \quad \text{locally uniformly.} \tag{4.32}$$

Next, if t_n denotes the integer part of t/δ_n, we have

$$F_n''^{-1}(t) = T''(n, t_n + 1) = T'(n, t_n + 1) + (t_n + 1)\delta_n = F_n'^{-1}(t) + (t_n + 1)\delta_n \ ,$$

hence $F_n''^{-1}(t) \to F'^{-1}(t) + t$ by (4.32). Since F'' and F''^{-1} are continuous and strictly increasing, it follows that,

$$F_n'' \;\to\; F'' \quad \text{locally uniformly} \tag{4.33}$$

(i.e. $\mu_n'' \to \mu''$, with μ'' the measure having F'' for repartition function).

To obtain (4.29) it is enough to observe that $F_n'^{-1}[(F_n''(t) - \delta_n)^+] < \Phi_n(t) \le F_n'^{-1}[(F_n''(t)]$, and to apply (4.32) and (4.33) and the property $\Phi = F'^{-1} \circ F''$, which comes from the equivalence $F''^{-1}(v) = v + r \Leftrightarrow \Phi(v + r) = r$ in (ii) of the proof of Lemma 4.1.

d) Now we show that

$$R_n \;\to\; R \qquad \text{pointwise.} \tag{4.34}$$

Let $j \in A_n'$ and $i \in A_n$ be related by $R'(n, i) = T'(n, j)$ (or equivalently $R''(n, i) = T''(n, j)$: see (4.23)). We have the following sequence of equivalent properties: $T''(n, j) \le t \Leftrightarrow R''(n, i) \le t$ $\Leftrightarrow R'(n, i) < \Phi_n(t) \Leftrightarrow T'(n, j) < \Phi_n(t)$ (recall (4.28)). Further $j \in A_n'$ iff there is $k \in J_n$ with $T'(n, j) = T(n, k)$. Then in view of (4.16) we get $R_n(t) = F_n[\Phi_n(t)-]$. Then $\nu_n \to \mu$ and (4.29) yield

$$F[\Phi(t)-] \le \liminf_n R_n(t) \le \limsup_n R_n(t) \le F[\Phi(t)].$$

This and (4.6) imply $R_n(t) \to F[\Phi(t)]$ if $F[\Phi(t)] = F[\Phi(t)-]$, and otherwise,

$$\left.\begin{array}{ll} \limsup_n R_n(s) \le F[\Phi(t)-] & \text{if } \ s < u_t', \\[2mm] \liminf_n R_n(s) \ge F[\Phi(t)] & \text{if } \ s > u_t. \end{array}\right\} \tag{4.35}$$

On the other hand $r_n \le \mu_n''$, hence $R_n(\beta) - R_n(\alpha) \le F_n''(\beta) - F_n''(\alpha)$ if $\alpha \le \beta$. Then (4.33) and the fact that $F''(\beta) - F''(\alpha) \le \beta - \alpha$ yield

$$\limsup_n [R_n(\beta) - R_n(\alpha)] \le \beta - \alpha. \tag{4.36}$$

Putting together (4.35), (4.36) and $F[\Phi(t)] - F[\Phi(t)-] = u_n - u_t'$ readily yields $R_n(t) \to F[\Phi(t)] - u_t + t = R(t)$: hence (4.34) holds.

Now we can prove (4.30). Denote by $\Psi_n(t)$ the left–hand side of (4.30), and by $\overline{\Psi}_n(t)$ the same quantity with $R'(n, i+1)$ instead of $R'(n, i)$. If $i \in A_n$ we have $R'(n, i+1) - R'(n, i) \le \alpha_n$ (combine (4.15), (4.22) and (4.25)), while $\delta_n \mathrm{card}(\sigma(n, t)) \le R_n(t) \to R(t)$ by (4.34): since f is uniformly continuous on $[0, t]$, we deduce that $\overline{\Psi}_n(t) - \Psi_n(t) \to 0$. Now $R'(n, i + 1) = \Phi_n(R''(n, i))$, and $i \in A_n$ iff there is a (unique) $J \in A_n'$ such that $R''(n, i) = T''(n, j)$, hence

$$\overline{\Psi}_n(f) = \int_0^t f \circ \Phi_n(s) r_n(ds) - \delta_n \sum_{i \in A_n, R''(n,i) \le t < R''(n,i+1)} f(R''(n, i + 1))$$

and the sum above is in fact bounded by $\delta_n \sup |f|$. By (4.29) $f \circ \Phi_n$ converges uniformly to the bounded continuous function $f \circ \Phi$ on $[0, t]$, and (4.34) means that r_n weakly converges to the measure $\psi(s)ds$, hence $\overline{\Psi}_n$ converges to the right-hand side of (4.30), and (4.30) is proved.

e) Exactly as before, $R_n^\star(t) = F_n^\star[\Phi_n(t)-]$. Then $\nu_n^\star \to \mu^\star$ and (4.29) and the continuity of $F^\star$ give $R_n^\star \to R^\star$ pointwise, and (4.31) is deduced from this as (4.30) is from (4.34) in (c) above. $\square$

5 Proof of Theorem 3.3

1) Let g satisfy (K). Since the process $(\gamma_t)_{t\geq 0}$ is $I\!R_+$-valued predictable increasing and γ_0 is a constant, there is a sequence τ_p of stopping times increasing to ∞, with $\gamma_t \leq p \vee \gamma_0$ for all $t \leq \tau_p$. Letting $g_p(\omega,t,x) = g(\omega, t\wedge\tau_p(\omega), x)$, we see that g_p satisfies (K) with a process γ which is the constant $p\vee\gamma_0$, and obviously $U^n(g)_t = U^n(g_p)_t$ and $g * B_t = g_p * B_t$ for all $t \leq \tau_p$. Since $\tau_p \to \infty$, it is obvious that if the sequence $U^n(g_p)$ enjoys the limiting behavior described in Theorem 3.3 for any fixed p, the same is true of the sequence $U^n(g)$.

In other words, it is enough to consider test functions g having (K) with $\gamma_t(\omega)$ being a constant. We assume this below, as well as (A1), (A2) and (A3) (as seen before, assuming (A3) is not a restriction). We use all notation of Section 4, and add some more. First, for any process Z we set (recall (4.27) for $\nabla(n,i)$):

$$\nabla_i'^n Z = \nabla(n,i)^{-1/2}\ (Z_{R'(n,i+1)} - Z_{R'(n,i)}).$$

Then, define the following processes (I_d is the $d\times d$ identity matrix):

$$f_t = \rho(g_t g_t^T) - \rho(g_t)\rho(g_t^T), \qquad h_t = \rho(g_t x^T),$$

$$F_t^n = \delta_n \sum_{i\in\sigma(n,t)} f_{R'(n,i)}, \qquad F_t = \int_0^t f_{\Phi(s)}\psi(s)\ ds, \tag{5.1}$$

$$H_t^n = \sum_{i\in\sigma(n,t)} \sqrt{\nabla(n,i)\delta_n}\ h_{R'(n,i)}, \qquad H_t = \int_0^t h_{\Phi(s)}\psi^\star(s)\ ds,$$

$$\begin{aligned} K_t^n &= \Phi_n(t)\ I_d, \qquad K_t = \Phi(t)\ I_d, \\ W_t'^n &= W_{\Phi_n(t)}, \qquad W_t' = W_{\Phi(t)}, \end{aligned} \tag{5.2}$$

$$\left.\begin{aligned} &U_t'^n = \textstyle\sum_{i\in\Sigma''(n,t)}\ \chi_i^n, \quad \text{where} \\ &\chi_i^n = \sqrt{\delta_n}\ 1_{A_n}(i)\ (g(R'(n,i),\nabla_i^n W) - \rho(g_{R'(n,i)})). \end{aligned}\right\} \tag{5.3}$$

2) Now we proceed to study the limiting behavior of U'^n. Note that $t\mapsto f_t$ and $t\mapsto h_t$ are continuous. Then Lemma 4.8 yields the following convergences in probability, locally uniform in time:

$$W'^n \to W, \quad F^n \to F, \quad H^n \to H, \quad K^n \to K. \tag{5.4}$$

Recalling that $\{i\in A_n\}\in\mathcal{F}_{R'(n,i)}$ and that the restriction to $\{i\in A_n\}$ of the variable $\nabla(n,i)$ is $\mathcal{F}_{R'(n,i)}$–measurable (Lemma 4.7-a), we easily deduce from (5.3) that, for some constant K,

$$\left.\begin{aligned} &E(\chi_i^n|\mathcal{F}_{R'(n,i)}) = 0 \\ &E(\chi_i^n\ \chi_i^{n,T}|\mathcal{F}_{R'(n,i)}) = 1_{A_n}(i)\ \delta_n\ f_{R'(n,i)} \\ &E(\chi_i^n\ (\nabla_i^n W)^T|\mathcal{F}_{R'(n,i)}) = 1_{A_n}(i)\ \sqrt{\delta_n}\ h_{R'(n,i)} \\ &E(|\chi_i^n|^4|\mathcal{F}_{R'(n,i)}) \leq K\delta_n^2. \end{aligned}\right\} \tag{5.5}$$

Lemma 5.1 *The processes U'^n, W'^n, $U'^n U'^{n,T} - F^n$, $W'^n W'^{n,T} - K^n$, $U'^n W'^{n,T} - H^n$ are $(\mathcal{F}_t^n)$–local martingales* (recall that $\mathcal{F}_t^n = \mathcal{F}_{\Phi_n(t)}$: see Lemma 4.7).

Proof. In view of Lemma 4.7-b, of the fact that $\Phi_n(t) \to \infty$ as $t \to \infty$ and of Theorems (10.9) and (10.10) of [7], the process W'^n and $W'^n W'^{n,T} - K^n$ are $(\mathcal{F}^n_t)$–local martingale.

Now consider a process $V^n_t = \sum_{i\in\Sigma''(n,t)} \eta^n_i = \sum_{i\geq 0} \eta^n_i \; 1_{\{R''(n,i+1)\leq t\}}$ with $\mathcal{F}^n_{R''(n,i+1)}$–measurable η^n_i satisfying $\eta^n_i = 0$ when $i \notin A_n$. By virtue of Lemma 4.7-a,c V^n is an $(\mathcal{F}^n_t)$–local martingale iff $E(\eta^n_i|\mathcal{F}_{R'(n,i)}) = 0$. By (5.5) this applies to $V^n = U'^n$ with $\eta^n_i = \chi^n_i$, and to $V^n = U'^n U'^{n,T} - F^n$ with

$$\eta^n_i = \chi^n_i \chi^{n,T}_i + U'^n_{R''(n,i)} \chi^{n,T}_i + \chi^{n,T}_i U'^{n,T}_{R'(n,i)} - 1_{A_n}(i)\; \delta_n \; f_{R'(n,i)}.$$

Set $\alpha^n_i = \sqrt{\nabla(n,i)}\; \nabla^n_i W$. If $Y^n_t = \sum_{i\in\sigma(n,t)} \alpha^n_i$, and again due to (5.5), the previous result also applies to $V^n = U'^n Y^{n,T} - H^n$, with

$$\eta^n_i = \chi^n_i \alpha^{n,T}_i + U'^n_{R''(n,i)} \alpha^{n,T}_i + \chi^{n,T}_i Y^{n,T}_{R'(n,i)} - 1_{A_n}(i)\; \sqrt{\nabla(n,i)\delta_n}\; h_{R'(n,i)}.$$

Finally $U'^n W'^{n,T} - H^n = U'^n Y^{n,T} - H^n + U'^n(W'^{n,T} - Y^{n,T})$. Now U'^n and $W'^{n,T} - Y^{n,T}$ are two $(\mathcal{F}^n_t)$–local martingale, purely discontinuous and with no common jump, hence their product is again a local martingale. □

An application of Aldous' criterion (apply (5.4) and Lemma 4.8, and combine Theorem 4.18 and Lemma 4.22 of Chapter VI of [9]), shows that the sequence U'^n is tight, and even C-tight (the last inequality in (5.5) implies Lindeberg's condition). Applying again (5.4) yields that the sequence $\zeta^n = (W'^n, F^n, H^n, K^n, U'^n, U'^n U'^{n,T} - F^n)$ is C-tight and that if $\zeta = (\overline{W}', \overline{F}, \overline{H}, \overline{K}, \overline{U}', \overline{M})$ is a limiting process for this sequence, (W', F, H, K) and $(\overline{W}', \overline{F}, \overline{H}, \overline{K})$ have the same distribution and $\overline{M} = \overline{U}'\overline{U}'^T - \overline{F}$ a.s.

In other words, if $C^q = C(\mathbb{R}_+, \mathbb{R}^q)$ is endowed with the canonical process U' and with the canonical filtration $(\mathcal{C}^q_t)$, we can realize any limit ζ on the product space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, (\widetilde{\mathcal{F}}_t)) = (\Omega, \mathcal{F}, (\mathcal{F}_t)) \otimes (C^q, \mathcal{C}^q_1, (\mathcal{C}^q_t)_t)$, so that

$$\left.\begin{array}{l}\text{If we consider a converging subsequence, still denoted by } \zeta^n\text{, there is a probability}\\ \text{measure } \widetilde{P} \text{ on } (\widetilde{\Omega}, \widetilde{\mathcal{F}}) \text{ whose } \Omega\text{–marginal is } P\text{, and such that the laws of } \zeta^n\\ \text{converge to the law of } \zeta = (W', F, H, K, U', U'U'^T - F) \text{ under } \widetilde{P}.\end{array}\right\} \qquad (5.6)$$

Lemma 5.2 *Under $\widetilde{P}$ the processes U', W', $U'U'^T - F$, $W'W'^T - K$, $U'W'^T - H$ are $(\widetilde{\mathcal{F}}_t)$–local martingales, continuous and null at* 0.

Proof. That the processes are continuous and null at 0 is obvious. We show the martingale property for $U'U'^T - F$ only; it is the same (or simpler) for the other processes.

Set $M = U'^j U'^k - F^{jk}$ and $M^n = U'^{n,j} U'^{n,k} - F^{n,jk}$, and also

$$L(n,y) = \inf(t:\; |M^n_t| + |F^n_t| + |U'^n_t| > y),$$

$$L(y) = \inf(t:\; |M_t| + |F_t| + |U'_t| > y),$$

Observe that $|M^n_t| \leq y$ if $t < L(n,y)$ and $|M^n_{L(n,y)}| \leq y + 2y|\chi^n_i| + |\chi^n_i|^2 + K'$ for some constant K', if $L(n,y) = R''(n,i+1)$. Thus $E(|M^n_{t\wedge L(n,y)}|^2) \leq (y+1)K''$ for another constant K'' by (5.5), from which we deduce the uniform integrability of the sequences $(M^n_{t\wedge L(n,y)})_{n\geq 1}$.

On the other hand (5.4) and (5.6) imply the convergence in law of (ζ^n, M^n, G^n) to $(\zeta, M, 0)$. Then (see e.g. Proposition VI-2.11 of [9]) for all y in a dense subset of $\mathbb{R}_+$, $(\zeta^n, M^n_{\cdot\wedge L(n,y)})_{n\geq 1}$ converge in law to $(\zeta, M_{\cdot\wedge L(y)})$. From the uniform integrability above and from Lemma 5.1 we deduce that $M_{\cdot\wedge L(y)}$ is a $\widetilde{P}$–martingale for the filtration generated by $(\zeta, M_{\cdot\wedge L(y)})$, i.e. for $(\widetilde{\mathcal{F}}_t)$. Since $L(y) \to \infty$ as $y \to \infty$, it follows that M is a local martingale. $\square$

Recalling that $0 \leq \psi^\star \leq \sqrt{\phi\psi}$ and that the process h_Φ (time-changed of h by Φ) is $(\mathcal{F}_{\Phi(t)})$–predictable, and setting $0/0 = 0$, we can define the following continuous local martingales on the extension $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, (\widetilde{\mathcal{F}}_t)), \widetilde{P})$:

$$M' = \alpha \cdot W' \quad \text{with} \quad \alpha_s = \frac{\psi^\star(s)}{\psi(s)}\, h_{\Phi(s)}, \qquad M'' = U' - M'. \tag{5.7}$$

Next, due to the structure of $(C^q, \mathcal{C}^q)$, there is a regular disintegration $\widetilde{P}(d\omega, dx) = P(d\omega)\widetilde{Q}_\omega(dx)$.

Lemma 5.3 *a) The space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, (\widetilde{\mathcal{F}}_t)), \widetilde{P})$ is a very good extension of the space $(\Omega, \mathcal{F}, (\mathcal{F}_{\Phi(t)}), P)$.*

b) M'' is an $(\mathcal{F}_t)$–conditional centered Gaussian martingale, $(\mathcal{F}_t)$–locally square-integrable, with bracket

$$\left.\begin{array}{l} F''_t = \int_0^t f''_s ds, \qquad \text{where} \\ f''_s = \psi(s) f_{\Phi(s)} - \phi(s)\alpha_s\alpha_s^T = \psi(s)(f_{\Phi(s)} - \frac{\phi^{\star 2}}{\phi\psi}(s) h_{\Phi(s)} h^T_{\Phi(s)}). \end{array}\right\} \tag{5.8}$$

Proof. a) Let Z be a bounded martingale on $(\Omega, \mathcal{F}, (\mathcal{F}_{\Phi(t)}), P)$, and set $N_t = E(Z_\infty|\mathcal{F}_t)$. We know that $N = l \cdot W$ for some $(\mathcal{F}_t)$–predictable process l. Like in the proof of Lemma 4.4, we then have

$$Z_t = E(Z_\infty|\mathcal{F}_{\Phi(t)}) = N_{\Phi(t)} = \int_0^t l_{\Phi(s)}\, dW'_s.$$

Now W' is a martingale on the extension $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, (\widetilde{\mathcal{F}}_t)), \widetilde{P})$ and $l \circ \Phi$ is predictable w.r.t. $(\widetilde{\mathcal{F}}_t)$: then Z is a martingale on the extension, which is thus very good.

b) Lemma 5.2 implies that the continuous local martingale U' has $\langle U', U'^T\rangle = F$ and $\langle U', W'^T\rangle = H$, and simple calculations show that $\langle M'', M''^T\rangle = F''$ given by (5.8) and $\langle M'', W'^T\rangle = 0$.

We deduce first that $\langle M'', M''^T\rangle$ is $(\mathcal{F}_{\Phi(t)})$–adapted. Next, since all bounded $(\mathcal{F}_{\Phi(t)})$–martingales are stochastic integrals w.r.t. W' (see (a) above) we deduce that M'' is orthogonal to all bounded $(\mathcal{F}_{\Phi(t)})$–martingales. Finally $M''_0 = 0$, and M'' is continuous. It remains to apply Lemma 2.3. $\square$

Corollary 5.4 *a) The measure $\widetilde{P}$ is unique, and (5.6) holds for the initial sequence ζ^n.*

b) We can even strengthen the convergence (5.6) as follows: for all bounded continuous functions k on the Skorokhod space $\mathbb{D}(\mathbb{R}_+, \mathbb{R}^q)$ and all bounded random variables Z on $(\Omega, \mathcal{F})$, we have

$$E(Z\, k(U'^n)) \;\to\; \widetilde{E}(Z\, k(U')). \tag{5.9}$$

Proof. a) By Lemmas 2.2 and 5.3 the $\mathcal{F}$–conditional law of M'' is determined by F'', so the $\mathcal{F}$–conditional law of $U' = M' + M''$, that is $\widetilde{Q}_\omega$, is P–a.s. unique, so $\widetilde{P}$ is unique and thus (5.6) holds for the original sequence ζ^n;

b) Clearly (5.4) and (5.6) imply (5.9) when $Z = l(W')$, where l is a continuous bounded function on $\mathbb{D}(\mathbb{R}_+, \mathbb{R}^q)$.

Next, let $\mathcal{F}'$ be the σ–field generated by all variables W'_t, $t \geq 0$. W' is a continuous $(\mathcal{F}_{\Phi(t)})$–local martingale with bracket $K_t = \Phi(t) I_d$, the process Φ is $\mathcal{F}'$–measurable, as well as its inverse Φ^{-1}. We have $W_t = W'_{\Phi^{-1}(t)}$ because $\Phi \circ \Phi^{-1}(t) = t$, hence W_t is $\mathcal{F}'$–measurable: thus $\mathcal{F}' = \mathcal{F}$.

Now let Z be bounded and $\mathcal{F}$–measurable. Since $\mathcal{F}' = \mathcal{F}$ there are $Z_p = l_p(W')$ with l_p continuous bounded and $Z_p \to Z$ in $L^1(P)$. (5.9) holds for each Z_p, and if $C = \sup |k|$ we obtain:

$$|E(Z_p\ k(U'^n)) - E(Z\ k(U'^n))| \leq C\ E(|Z - Z_p|),$$

$$|E(Z_p\ k(U')) - E(Z\ k(U'))| \leq C\ E(|Z - Z_p|),$$

so (5.9) follows. □

3) Now we state the relations between the process U' above and the process $g \star B$ of Theorem 3.3, defined on the extension $(\overline{\Omega}, \overline{\mathcal{F}}, (\overline{\mathcal{F}}_t)), \overline{P})$ of $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$. For this, we set

$$U_t = U'_{\tau(t)} \qquad (\tau(t) \text{ is given by (4.9)}. \tag{5.10}$$

Lemma 5.5 *Both processes U on the (non-filtered) extension $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{P})$ of $(\Omega, \mathcal{F}, P)$ and $g \star B$ on the (non-filtered) extension $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{P})$ of the same space have the same $\mathcal{F}$–conditional law.*

Proof. a) First we show that $g \star B'_t = M'_{\tau(t)}$ (see (3.9) and (5.7)). By definition the process W' is constant on the intervals contiguous to A, hence $1_{\{\phi=0\}} \cdot W' = 0$ by (4.5). Further the bracket of W' is absolutely continuous w.r.t. Lebesgue measure by (4.4), hence $1_C \cdot W' = 0$ if $\lambda(C) = 0$. Therefore $M' = [(\theta^\star h\ 1_{\{\theta>0\}}) \circ \Phi] \cdot W'$ by (4.7) and (3.2), hence $M'_{\tau(t)} = g \star B'_t$ follows from (4.12).

b) Since $g \star B'$ is $\mathcal{F}$–measurable, it remains at this point to show that both processes $g \star B''$ and $\widetilde{M}''_t = M''_{\tau(t)}$ have the same $\mathcal{F}$–conditional law. Now the time-change $\tau(t)$ is $\mathcal{F}$–measurable, so it follows from Lemma 5.3-b that M'' is an $\mathcal{F}$–conditional centered Gaussian martingale with bracket $\widetilde{F}''_t = F''_{\tau(t)}$, while $g \star B''$ is an $\mathcal{F}$–conditional centered Gaussian martingale with bracket given by (3.10). By Lemma 2.2-b it remains to show that $\widetilde{F}''$ is given by (3.10).

Using (4.7) and $\psi = 0 \Rightarrow \psi^\star = 0$ and $\theta = 0 \Rightarrow \theta^\star = 0$, we deduce from (5.8):

$$F''_t = \int_0^t \left(f_{\Phi(r)} - \frac{\theta^{\star 2}}{\theta} \circ \Phi(r)\ h_{\Phi(r)} h^T_{\Phi(r)} \right) \psi(r)\ dr,$$

and (4.11) gives

$$F''_{\tau(t)} = \int_{[0,t]} \left(f_r - \frac{\theta^{\star 2}}{\theta}(r)\ h_r h^T_r \right) \mu(dr).$$

Thus $F''_{\tau(t)}$ is equal to (3.10), since $\frac{\theta^{\star 2}}{\theta}(r)\ \mu(dr) = \theta^{\star 2}(r) dr$ by Lemma 3.1. □

Proof of Theorem 3.3. a) In a first step, we prove that if

$$\overline{\chi}^n_i = \sqrt{\delta_n}\ \left(g(T(n,i), \xi^n_i) - \rho(g_{T(n,i)})\right), \qquad \overline{U}^n_t = \sum_{i \in \Sigma(n,t) \cap J_n} \overline{\chi}^n_i, \tag{5.11}$$

(recall (1.3) for ξ^n_i and (4.15) for J_n and J'_n below), then

$$\sup_{s \leq t}\ |U^n_s(g) - \overline{U}^n_s|\ \xrightarrow{P}\ 0. \tag{5.12}$$

Set $\zeta^n_i = \overline{\chi}^n_i\ 1_{J'_n}(i)$, $X^n_i = \sum_{j \leq i} \zeta^n_j$, $L^n_i = \sum_{j \leq i} E(\zeta^n_j \zeta^{n,T}_j | \mathcal{F}_{T(n,j)})$, Then L^n is the predictable bracket of the (discrete-time) locally square-integrable martingale X^n w.r.t. the filtration $(\mathcal{F}_{T(n,i+1)})_{i \geq 0}$, for which $\theta(n,t)$ =card$(\Sigma(n,t))$ is a stopping time. Since $L^n_{\theta(n,t)} = \delta_n \sum_{i \in \Sigma(n,t)} f_{T(n,i)}\ 1_{J'_n}(i)\ \xrightarrow{P}\ 0.$

by (4.18), it follows from Lenglart's inequality that $\sup_{i\leq\theta(n,t)}|X_i^n| \stackrel{P}{\longrightarrow} 0$. It remains to observe that $U_t^n(g)-\overline{U}_t^n = X_{\theta(n,t)}^n$, hence (5.12). Therefore it is enough to prove the claims of Theorem 3.3 for $\overline{U}^n$ instead of $U^n(g)$.

b) Next we observe that i belongs to A iff there is a $j\in J_n$ such that $R'(n,i)=T(n,j)$, in which case $\nabla(n,i)=\Delta(n,j)$ (see (4.22) and (4.27)) and $\xi_i^n=\overline{\chi}_j^n$. Hence comparing (5.3) and (5.11) gives that $\overline{U}_t^n = U_s'^n$ iff there are as many points in $\Sigma(n,t)\cap J_n$ and in $\sigma(n,s)$. With the notation of the proof of Lemma 4.8, these numbers are $F_n(t)/\delta_n$ or $1+F_n(t)/\delta_n$ (resp. $R_n(s)/\delta_n$ or $1+R_n(s)/\delta_n$). Then there is $\tau_n(t)$ with

$$\overline{U}_t^n = U_{\tau_n(t)}'^n, \qquad R_n^{-1}(F_n(t)-\delta_n)\leq\tau_n(t)\leq R_n^{-1}(F_n(t)). \tag{5.13}$$

c) Set $\mathbb{D}=\mathbb{D}(\mathbb{R}_+,\mathbb{R}^q)$, with its Borel σ–field $\mathcal{D}$. Set $Y=\Omega\times\mathbb{D}$, with the σ–field $\mathcal{Y}=\mathcal{F}\otimes\mathcal{D}$. We endow $(Y,\mathcal{Y})$ with the probability measures χ_n and χ defined by

$$\chi_n(A\times B)=E(1_A\,1_B(U'^n)), \qquad \chi(A\times B)=\widetilde{E}(1_A\,1_B(U')). \tag{5.14}$$

By (5.9), $\chi_n(Z\otimes k)\to\chi(Z\otimes k)$ for all bounded measurable Z on $(\Omega,\mathcal{F})$ and all bounded continuous k on the Polish space $(\mathbb{D},\mathcal{D})$. By [7], Theorem (3.4), we deduce that $\chi_n(l)\to\chi(l)$ for every bounded measurable l on $(Y,\mathcal{Y})$ such that $x\mapsto l(\omega,x)$ is continuous at χ–almost all points (ω,x).

Applying this to $l(\omega,x)=Z(\omega)k((x_{\tau(\omega,t)})_{t\geq0})$, where Z is bounded measurable on $(\Omega,\mathcal{F})$ and k is bounded continuous on $(\mathbb{D},\mathcal{D})$, we get (see Lemma 4.2)

$$\chi_n(l)=E(Z\ k(U_{\tau(.)}'^n))\to\chi(l)=\overline{E}(Z\ k(U'_{\tau(.)}))=\overline{E}(Z\ k(g\star B)).$$

Applying this to $l(\omega,x)=Z(\omega)k((x_{\tau(\omega,t_1)},\ldots,x_{\tau(\omega,t_r)}))$ with k bounded continuous on $(\mathbb{R}^q)^r$ and using the fact that U' is continuous in time (hence $x\mapsto l(\omega,x)$ is again χ–a.s. continuous), we get similarly

$$E(Z\ k(U_{\tau(t_1)}'^n),\ldots,(U_{\tau(t_r)}'^n))\to\overline{E}(Z\ k(g\star B_{t_1},\ldots,g\star B_{t_r})).$$

Therefore, in view of (5.13), the result will follow if we prove the following two properties:

$$U_{\tau_n(t)}'^n-U_{\tau(t)}'^n \stackrel{P}{\longrightarrow} 0 \quad \text{for all } t\in I \text{ (recall (3.12) for } I), \tag{5.15}$$

$$\sup_{t\leq s}|U_{\tau_n(t)}'^n-U_{\tau(t)}'^n| \stackrel{P}{\longrightarrow} 0 \quad \text{for all } s \text{ if } \mu \text{ has a.s. no atom.} \tag{5.16}$$

Up to taking subsequences, we may assume that the convergences (4.19) and (4.34) hold a.s.

d) Let us prove two auxiliary facts. First, if $t\in I$ then (4.19) gives that outside a null set $F_n(t_n)\to F(t)$ whenever $t_n\to t$, and if μ has a.s. no atom we have $F_n\to F$ a.s., locally uniformly. Then we have a.s.:

$$\left.\begin{array}{l} F_n(t)-\delta_n\to F(t), \quad F_n(t)\to F(t) \qquad \text{if } t\in I \\ \sup_{t\leq s}|F_n(t)-\delta_n-F(t)|\to 0, \qquad \sup_{t\leq s}|F_n(t)-F(t)|\to 0 \\ \qquad\qquad\qquad \text{for all } s \text{ if } \mu \text{ has no atom.} \end{array}\right\} \tag{5.17}$$

Second, because of Lemma 5.2, U' is a martingale with bracket F given by (5.1). Hence U' is a.s. constant over the intervals where F is constant, hence over those on which R is constant, and we have a.s.:

$$U'_s=U'_{S(t)} \quad \text{if} \quad S(t-)\leq s\leq S(t). \tag{5.18}$$

e) Now we prove (5.15). Let $t \in I$. Then (5.17) and (4.34) imply that a.s.:

$$S(F(t)-) \leq \liminf_n \tau_n(t) \leq \limsup_n \tau_n(t) \leq S(F(t)) = \tau(t). \tag{5.19}$$

Since U'^n converges in law to the continuous process U' satisfying (5.18), these inequalities imply (5.15).

f) Finally, assume that μ has a.s. no atom. Suppose that (5.16) does not hold. There is $\varepsilon > 0$, $s \in I\!\!R_+$ and a subsequence still denoted by n, and a (random) sequence t_n in $[0,s]$, such that

$$P\left(|U'^n_{\tau_n(t_n)} - U'^n_{\tau(t_n)}| > \varepsilon\right) \geq \varepsilon \qquad \text{for all} \;\; n. \tag{5.20}$$

Up to taking a further subsequence, we can even assume that $t_n \to t \in [0,s]$ a.s. Since F is continuous, we then have a.s. by (5.17) and (4.34):

$$S(F(t)-) \leq \liminf_n \tau_n(t_n) \leq \limsup_n \tau_n(t_n)$$

as well as (5.19). Then once more because U'^n converges in law to the continuous process U' satisfying (5.18), these relations imply $|U'^n_{\tau_n(t_n)} - U'^n_{\tau(t)}| \stackrel{P}{\longrightarrow} 0$, which contradicts (5.20). Thus (5.15) holds, and we are finished. $\square$

PART II: BROWNIAN SEMIMARTINGALES

6 The results

In this section the setting is the same as in Section 3, but in addition we have an $I\!\!R^m$–valued Brownian semimartingale X of the form (1.10), satisfying (H). We set

$$\Delta_i^n X = \Delta(n,i)^{-1/2}(X_{S(n,i)} - X_{T(n,i)}). \tag{6.1}$$

We also set $c = aa^T$, and call $\rho_t^X = \rho_t^X(\omega, dx)$ the centered Gaussian distribution on $I\!\!R^m$ with covariance matrix $c_t(\omega)$. Then we write $\rho_t^X(f) = \int \rho_t^X(\omega, dx) f(\omega, t, x)$ for any function f on $\Omega \times I\!\!R_+ \times I\!\!R^m$.

We are interested in the limiting behavior of processes like $U^n(g)$ of (1.4), with ξ_i^n replaced by $\Delta_i^n X$. Of course we should also modify the centering term in (1.4), and there are several possibilities for this. The most natural one is the following:

$$U_t^{1,n}(g) = \sqrt{\delta_n} \sum_{i \in \Sigma(n,t)} \left(g(T(n,i), \Delta_i^n X) - E(g(T(n,i), \Delta_i^n X) | \mathcal{F}_{T(n,i)})\right) \tag{6.2}$$

(see (4.17) for $\Sigma(n,t)$), provided the conditional expectations above make sense. However, these conditional expectations are difficult to compute, and it may be more useful to consider

$$U_t^{2,n}(g) = \sqrt{\delta_n} \sum_{i \in \Sigma(n,t)} \left(g(T(n,i), \Delta_i^n X) - \rho^X_{T(n,i)}(g)\right), \tag{6.3}$$

which is well-defined if g satisfies (K). Finally, the following has also some interest:

$$U_t^{3,n}(g) = \sqrt{\delta_n} \sum_{i \in \Sigma(n,t)} \left(g(T(n,i), a_{T(n,i)}\xi_i^n) - \rho^X_{T(n,i)}(g)\right). \tag{6.4}$$

Observe that under (H) and (K), $t \mapsto \rho_t^X(g)$ is continuous, and Lemma 4.5 yields for $t \in I$ (recall (3.12) for I):

$$\delta_n \sum_{i \in \Sigma(n,t)} \rho^X_{T(n,i)}(g) \stackrel{P}{\longrightarrow} \int_{[0,t]} \rho_s^X(g)\ \mu(ds), \tag{6.5}$$

and this convergence in probability holds locally uniformly in t if μ has a.s. no atom.

The behavior of $U^{3,n}(g)$ is very simple. Indeed if g: $\Omega \times I\!R_+ \times I\!R^m \to I\!R^q$ satisfies (K), and if (H) holds (hence a is locally bounded), the function g': $\Omega \times I\!R_+ \times I\!R^d \to I\!R^q$ defined by $g'(\omega,t,x) = g(\omega,t,a_t(\omega)x)$ also satisfies (K) and $\rho_t^X(g) = \rho(g'_t)$. Hence Theorem 3.3 yields:

Theorem 6.1 *Assume (A1), (A2), (H) and let B be a tangent measure to W along $(\mathcal{T}_n)$. Let g satisfy (K).*

a) If μ has a.s. no atom, the processes $U^{3,n}(g)$ converge stably in law to $U(g)$ given by

$$U(g) = g' \star B, \qquad \textit{with} \quad g'(\omega,t,x) = g(\omega,t,a_t(\omega)x). \tag{6.6}$$

b) For all $(t_1,\ldots,t_k)$ in I, the variables $(U^{3,n}_{t_1}(g),\ldots,U^{3,n}_{t_k}(g))$ converge stably in law to the variable $(U_{t_1}(g),\ldots,U_{t_k}(g))$.

In view of (6.5), we have the

Corollary 6.2 *Assume (A1), (A2), (H), and let g satisfy (K). Then the following convergence*

$$\delta_n \sum_{i \in \Sigma(n,t)} g(T(n,i), a_{T(n,i)}\Delta_i^n X) \ \to\ \int_{[0,t]} \rho_s^X(g)\ \mu(ds) \tag{6.7}$$

holds in probability, for all $t \in I$, and also locally uniformly in time if μ has a.s. no atom.

Now let us consider the following processes, for $A \in \mathcal{R}^m$:

$$B^X(A)_t = f \star B_t, \qquad \text{where} \quad f(\omega,t,x) = 1_A(a_t(\omega)x). \tag{6.8}$$

It is obvious that $B^X = (B^X(A)_t : t \geq 0, A \in \mathcal{R}^m)$ is a worthy martingale measure on $I\!R^m$, and that $U(g)$ in (6.6) is $U(g) = g \star B^X$. Further if B'^X and B''^X are defined by (6.8) with B' and B'' instead of B (recall Proposition 3.2), then B'^X is an L^2–valued martingale measure on the Wiener space and B''^X is an $\mathcal{F}$–conditional centered Gaussian measure. Therefore $B^X = B'^X + B''^X$ is an $\mathcal{F}$–conditional Gaussian measure. An easy computation using (3.8) and (3.9) shows that, with the notation

$$\beta_t^X(g) = \int x\ g(t,a_t x)\ \rho(dx), \tag{6.9}$$

B^X satisfies all conditions of the following:

Definition 2: A tangent measure to X along the sequence $(\mathcal{T}_n)$ is an $\mathcal{F}$–conditional Gaussian measure B^X on $I\!R^m$, defined on a very good extension $(\overline{\Omega},\overline{\mathcal{F}},(\overline{\mathcal{F}}_t),\overline{P})$ of $(\Omega,\mathcal{F},(\mathcal{F}_t),P)$, such that $\overline{E}[B^X(A)_0] = 0$ and

$$\langle W, B^X(A)\rangle_t = \int_0^t \beta_s^X(1_A)\ \mu^\star(ds) \tag{6.10}$$

for all $A \in \mathcal{R}^m$, and having the covariance measure

$$\nu^X([0,t] \times A \times A') = \int_{[0,t]} (\rho_s^X(A \cap A') - \rho_s^X(A)\rho_s^X(A'))\mu(ds). \tag{6.11}$$

Again B^X is "essentially unique" (its $\mathcal{F}$–conditional law is completely determined). In fact we can construct the tangent measures to all Brownian semimartingales having (H) on the same extension $(\overline{\Omega},\overline{\mathcal{F}},(\overline{\mathcal{F}}_t)),\overline{P})$, via (6.8). A result similar to Proposition 3.2, and formulas similar to (3.8), (3.9) and 3.10) hold for B^X: we leave this to the reader.

3) In the rest of the section, B^X is a tangent measure to X, and all results below are proved in Section 8. For studying $U^{1,n}(g)$ we need additional assumptions:

Assumption H-r ($r \in I\!R_+$): $E(\sup_{t\leq s}(|a_t|^r + |b_t|^r)) < \infty$ for all $s < \infty$.

Assumption K1: The function g: $\Omega \times I\!R_+ \times I\!R^m \to I\!R^q$ satisfies (K), and for all ω, s the family of functions $x \mapsto g(\omega,t,x)$ indexed by $t \in [0,s]$ is uniformly equicontinuous on each compact subset of $I\!R^m$.

Assumption K2-r ($r \in I\!R_+$): We have (K1) and, for some nondecreasing adapted finite–valued process $\gamma = (\gamma_t)$,

$$|g(\omega,t,x)| \leq \gamma_t(\omega)(1+|x|^r). \tag{6.12}$$

Observe that (H-0) is empty, and that if $p < r$ then (K2-p)implies (K2-r), while (H-r) implies (H-p).

Theorem 6.3 *Assume (A1), A2), H) and one of the following:*

(i) (H-r) for all $r < \infty$, and (K1),

(ii) (H-r) and (K2-r) for some $r \in [1,\infty)$,

(iii) (K2-0) (i.e. (K1) and $|g(t,x)| \leq \gamma_t$).

Then: a) The processes $U^{1,n}(g)$ are well–defined (i.e. the conditional expectations in (6.2) make sense), and satisfy for all $s < \infty$:

$$\sup_{t\leq s}|U^{1,n}_t(g) - U^{3,n}_t(g)| \stackrel{P}{\longrightarrow} 0. \tag{6.13}$$

b) If μ has a.s. no atom, the processes $U^{1,n}(g)$ converge stably in law to $g \star B^X$.

c) For all $t_1,\ldots,t_)$ in I, the variables $(U^{1,n}_{t_1}(g),\ldots,U^{1,n}_{t_k}(g))$ converge stably in law to $(g\star B^X_{t_1},\ldots,g\star B^X_{t_k})$.

Corollary 6.4 *Assume (A1), (A2), (H) and (K1). Then the following convergence*

$$\delta_n \sum_{i\in\Sigma(n,t)} g(T(n,i),\Delta^n_i X) \;\to\; \int_{[0,t]} \rho^X_s(g)\;\mu(ds) \tag{6.14}$$

holds in probability, for all $t \in I$, and also locally uniformly in time if μ has a.s. no atom.

4) Let us turn to the processes $U^{2,n}(g)$. Again, we need new assumptions:

Assumption H' : a) $t \mapsto b_t$ is adapted continuous.

b) The process a is a Brownian semimartingale of the form

$$a_t = a_0 + \int_0^t a'_s dW_s + \int_0^t b'_s ds, \tag{6.15}$$

with a' and b' predictable locally bounded and $t \mapsto a'_t$ continuous. □

Observe that (H') implies (H). On the other hand, the following implies (K1):

Assumption K' : The function g satisfy (K1), and $x \mapsto g(\omega, t, x)$ is differentiable, and the function ∇g (gradient in x) also satisfies (K1). □

In order to define the limiting process, we also need some more notation. First, we consider the process,

$$\overline{\rho}^X_t(\nabla g) = \frac{1}{2} \int \rho(dx) \sum_{1\le i\le d, 1\le j,k\le m} \frac{\partial g}{\partial x_i}(t, a_t x) a'^{ijk}_t (x^j x^k - \delta^{jk}). \tag{6.16}$$

In the above formula δ^{jk} is the Kronecker symbol; recall $a = (a^{ij})_{i\le m, j\le d}$, so $a' = (a'^{ijk})_{i\le m; j,k\le d}$ and (6.15) reads componentwise as

$$a^{ij}_t = a^{ij}_0 + \sum_{1\le k\le d} \int_0^t a'^{ijk}_s dW^k_s + \int_0^t b'^{ij}_s ds.$$

Under the above assumptions, $\overline{\rho}^X_t(\nabla g)$ is continuous in t. Finally, we define the q-dimensional process:

$$\overline{U}(g)_t = g \star B^X_t + \int_0^t (\rho^X_s(\nabla g) b_s + \overline{\rho}^X_s(\nabla g))\ \mu^\star(ds). \tag{6.17}$$

Theorem 6.5 *Assume (A1), (A2), (H') and (K'). Then*

a) If μ has a.s. no atom, the processes $U^{2,n}(g)$ converge stably in law to $\overline{U}(g)$.

b) For all $t_1, \ldots, t_k$ in I, the variables $(U^{2,n}_{t_1}(g), \ldots, U^{2,n}_{t_k}(g))$ converge stably in law to $(\overline{U}(g)_{t_1}, \ldots, \overline{U}(g)_{t_k})$.

5) Finally, we could hope for a central limit theorem associated with the convergence (6.14). For this we need rather strong regularity of g as a function of time. To remain simple, we consider the very special case where $g(\omega, t, x) = g(x)$ depends on x only. For such g, (K') amounts to saying that g is continuously differentiable, with ∇g having polynomial growth.

Further, this desired central limit theorem is not true in general (see Remark 4 below), and we consider only the regular case $T(n,i) = i/n$ and $\Delta(n,i) = 1/n$. Then we are led to consider the processes

$$V^n_t(g) = \frac{1}{n} \sum_{1\le i\le [nt]} g(\sqrt{n}\ (X_{i/n} - X_{(i-1)/n})) - \int_0^t \rho^X_s(g) ds. \tag{6.18}$$

Corollary 6.6 *Let g be a continuously differentiable function on $\mathbb{R}^m$ with ∇g having polynomial growth. Assume (H'). Then*

a) $\sup_{t\le s} |\sqrt{n}\ V^n_t(g) - U^{2,n}_t(g)| \xrightarrow{P} 0$ for all s.

b) The processes $\sqrt{n}\ V^n(g)$ converge stably in law to the process $\overline{U}(g)$ of (6.17) (with $\mu =$ Lebesgue measure).

Remark 4: In contrast with the regular case we do not have in general a rate of convergence $\sqrt{\delta_n}$ in (6.14), even when $\delta_n = 1/n$ and even when the $T(n,i)$'s are deterministic.

Here is a counter-example: take $m=d=q=1$, and $a_t=t$ and $b=0$, and $g(x)=x^2$: we have (H') and (K'). Take $T(n,i)=i/n^\alpha$ for some $\alpha>1$ if $i\leq n$ and $T(n,i)=\infty$ otherwise, and $\Delta(n,i)=1/n^\alpha$.Then (A1) and (A2) are satisfied with $\delta_n=1/n$ and $\mu=\varepsilon_0$ and $\mu^\star=0$.

We have $\rho^X_t(g)=t$, hence if $t\leq 1$ the limit in (6.14) is 0. Denote by V^n_t the left hand side of (6.14). Then $\sqrt{n}\,V^n_1-U^{2,n}_1(g)=n^{-1/2}\sum_{1\leq i\leq n}\rho^X_{T(n,i-1)}(g)=\sum_{1\leq i\leq n}(i-1)n^{-\alpha-1/2}=\frac{1}{2}(n-1)n^{1.2-\alpha}$, which is equivalent to $n^{3/2-\alpha}/2$. By Theorem 6.5 we have non–degenerate convergence of $\sqrt{n}\;V^n_1$ if $\alpha\geq 3/2$ (with a non–centered limit if $\alpha=3/2$), and if $1<\alpha<3/2$ we have convergence of $n^{\alpha-1}V^n_1$ to 1/2 in probability. □

6) *The case of stochastic differential equations.* Here we explain how the above assumptions on a,b read when the process X of (1.10) is the solution of the following stochastic differential equation:

$$dX_t=A(t,X_t)dW_t+B(t,X_t)dt,\qquad X_0=x_0\ \text{ given in }\ I\!R^m. \tag{6.19}$$

Assume that A and B are locally Lipschitz in space (locally uniformly in time) and with at most linear growth (locally uniformly in time). Then (6.19) has a unique strong non-exploding solution X, and $\sup_{s\leq t}|X_s|^p$ is integrable for all $p<\infty$, $t<\infty$, and X is of the form (1.10) with $a_t=A(t,X_t)$, $b_t=B(t,X_t)$. If further A is continuous in time, clearly (H) and (H-r) hold for all r: hence Theorem 6.3 applies, provided g satisfies (K1).

For (H') to hold, we need further assumptions: for instance, that A is of class $C^{1,2}$ on $I\!R_+\times I\!R^m$ and B is continuous in time.

7 Some estimates

Below, K_r denotes a constant depending on r and which may change from line to line, but which does not depend on a,b,g. If $s>0$ and $t\geq 0$, set

$$\delta(t,s)=s^{-1/2}(X_{t+s}-X_t),\qquad \delta'(t,s)=s^{-1/2}a_t(W_{t+s}-W_t). \tag{7.1}$$

Below, *increasing process on* $I\!R^j_+$ means a process, say G, indexed by $I\!R^j_+$, whose paths $(t_1,\dots,t_j)\mapsto G(t_1,\dots,t_j)(\omega)$ are a.s. with values in $I\!R_+$ and non–decreasing and right–continuous separately in each variable t_i. We also denote by $\mathcal{S}$ the family of all pairs (T,Δ) where T is a finite stopping time and Δ an $\mathcal{F}_T$–measurable $(0,\infty)$–valued random variable.

Lemma 7.1 *Assume (H) and (H-r) for some $r\geq 2$. There exist two increasing processes χ_r and χ'_r on $I\!R^2_+$, with $\chi'_r(u,0)=0$ and such that for all $(T,\Delta)\in\mathcal{S}$:*

$$E(|\delta(T,\Delta)|^r|\mathcal{F}_T)\leq\chi_r(T,\Delta),\qquad E(|\delta'(T,\Delta)|^r|\mathcal{F}_T)\leq\chi_r(T,\Delta), \tag{7.2}$$

$$E(|\delta(T,\Delta)-\delta'(T,\Delta)|^r|\mathcal{F}_T)\leq\chi'_r(T,\Delta). \tag{7.3}$$

Proof. a) Since $E(|\delta'(T,\Delta)|^r|\mathcal{F}_T)\leq|a_T|^r\;E(\Delta^{-r/2}|W_{t+\Delta}-W_t|^r|\mathcal{F}_T)$ and Δ is $\mathcal{F}_T$–measurable, the second inequality in (7.2) holds with $\chi_r(u,v)=\sup_{t\leq u}|a_t|^r$. By Cauchy-Schwarz and Burkholder-Davis-Gundy inequalities and again the $\mathcal{F}_T$–measurability of Δ,

$$\begin{aligned}E(|\delta(T,\Delta)|^r|\mathcal{F}_T)\;&\leq\;K_r\Delta^{-r/2}\;E\Big(\Big(\int_T^{T+\Delta}|b_s|ds\Big)^r+\Big(\int_T^{T+\Delta}|a_s|^2ds\Big)^{r/2}|\mathcal{F}_T\Big)\\&\leq\;K_T\;\frac{1}{\Delta}\int_T^{T+\Delta}E\left(|b_s|^r\Delta^{r/2}+|a_s|^r\;|\mathcal{F}_T\right)\;ds.\end{aligned}$$

The first inequality in (7.2) holds if we take

$$\chi_r(u,v) = K_r \lim_{v'\downarrow v} \sup_{t\le u} E(\sup_{s\le u+v'} (|b_s|^r v^{r/2} + |a_s|^r)|\mathcal{F}_T),$$

which is finite–valued by (H-r) and Doob's inequality for martingales.

b) Observing that $\delta(t,s) - \delta'(t,s) = s^{-1/2}\left(\int_t^{t+s}(a_u - a_t)dW_u + \int_t^{t+s} b_u du\right)$, the same argument as above shows that

$$E(|\delta(T,\Delta) - \delta'(T,\Delta)|^r|\mathcal{F}_T) \le K_T\ \frac{1}{\Delta}\int_T^{T+\Delta} E\left(|b_s|^r\Delta^{r/2} + |a_s - a_T|^r\ |\mathcal{F}_T\right)\ ds. \qquad (7.4)$$

Then if $\beta(u,v) = \sup(|a_{t+s} - a_t| :\ 0 \le t \le u, 0 \le s \le v)$, (7.3) holds with

$$\chi'_r(u,v) = K_r \lim_{v'\downarrow v} \sup_{t\le u} E(\sup_{s\le u+v'} (|b_s|^r v^{r/2} + \beta(u,v')^r)|\mathcal{F}_T), \qquad (7.5)$$

Further $\beta(u,v) \to 0$ as $v \to 0$ by (H), and this convergence also takes place in L^r if (H-r) holds. Then Doob's inequality again gives $\chi'_r(u,0) = 0$. □

Lemma 7.2 *Assume (H), (H-r) for all $r<\infty$, and (K1). Then with γ_t as in (K), for all $r<\infty$ there is an increasing process χ''_r on $I\!R^3_+$, with $\chi''_r(u,0,w) = 0$ a.s. and such that for all $(T,\Delta)\in\mathcal{S}$:*

$$E(|g(T,\delta(T,\Delta)) - g(T,\delta'(T,\Delta))|^r|\mathcal{F}_T) \le \chi''_r(T,\Delta,\gamma_T). \qquad (7.6)$$

Proof. Let $(T,\Delta)\in\mathcal{S}$ and $q<\infty$. Set $\delta = \delta(T,\Delta)$ and $\delta' = \delta'(T,\Delta)$ and $\gamma = \gamma_T$. By (K1), for all $p<\infty$, $\varepsilon>0$ there is a strictly positive random variable $\nu(\varepsilon,p)$ such that $|x|\le p$, $|y|\le p$ and $|x-y|\le\nu(\omega,\varepsilon,p)$ imply $|g(\omega,t,x) - g(\omega,t,y)|\le\varepsilon$. Then by (K):

$$\beta := |g(T,\delta) - g(T,\delta')|^r \le \begin{cases} K_r\gamma^r(1 + |\delta|^{r\gamma} + |\delta'|^{r\gamma}) \\ \varepsilon^r \quad \text{if} \quad |\delta|, |\delta'|\le p, \quad |\delta-\delta'|\le\nu(\varepsilon,p). \end{cases}$$

Then for some constant K_r, for all $\varepsilon,\theta,u,v,w>0$ we have on $\{T\le u, \Delta\le v, \gamma\le w\}$:

$$E(\beta|\mathcal{F}_T) \le \varepsilon^r + K_r w^r E\left((1 + |\delta|^{rw} + |\delta'|^{rw})\ 1_{\{|\delta|>p\}\cup\{|\delta'|>p\}\cup\{|\delta-\delta'|>\nu(\varepsilon,p)\}}\ |\mathcal{F}_T\right)$$

$$\le \varepsilon^r + K_r w^r\big(1 + \chi_{2rw}(u,v)\big)^{1/2}\ \left(\frac{2}{p^2}\ \chi_2(u,v) + \sqrt{\chi'_2(u,v)}/\theta\ + Z(\varepsilon,p,\theta)\right)^{1/2}, \qquad (7.7)$$

where $Z(\varepsilon,p,\theta) = \sup_t\ P(\nu(\varepsilon,p)\le\theta|\mathcal{F}_T)$ (use (7.2), (7.3) and the inequalities of Cauchy–Schwarz and Bienaymé–Tchebicheff). If $Y(\varepsilon,p,\theta,u,v,w)$ is the right–hand side of (7.7), then (7.6) holds with $\chi''_r(u,v,w) = \lim_{v'\downarrow v}\ \inf_{\varepsilon,p,\theta>0} Y(\varepsilon,p,\theta,u,v',w)$. Further, there exist finite variables $Z'(u,w)$ such that for all $\varepsilon,p,\theta>0$ and $v\in[0,1]$, we have

$$\chi''_r(u,v,w) \le \varepsilon^r + Z'(u,w)\left(p^{-2}\sqrt{\chi'_2(u,2v)}/\theta\ + Z(\varepsilon,p,\theta)\right)^{1/2}.$$

Since $P(\nu(\varepsilon,p)\le\theta)\to 0$ as $\theta\to 0$ we clearly have $Z(\varepsilon,p,\theta)\ \stackrel{P}{\longrightarrow}\ 0$ as $\theta\to 0$ for all $\varepsilon,p>0$, while $\chi'_2(u,2v)\to 0$ as $v\to 0$. Then by choosing first p, then θ, then v, it is clear that $\chi''_r(u,v,w)\to 0$ as $v\to 0$. □

Next, we will assume (H') and the following (implying (H-r) for all $r<\infty$:

Assumption H'-∞: The processes b and a', b' of (6.15) are bounded by a constant C, and $|a_0|$ belongs to L^r for all r. □

By definition a' takes its values in $I\!R^d \otimes I\!R^m \otimes I\!R^m$, and we define the $I\!R^d$–valued variables $Y(t,s) = (Y(t,s)^i)_{1\le i\le d}$ by

$$Y(t,s)^i = b_t^i + \frac{1}{s}\sum_{1\le j,k\le d} a_t'^{ijk}\int_t^{t+s}(W_u^j - W_t^j)dW_u^k. \tag{7.8}$$

Lemma 7.3 *Assume (H') and (H'-∞). For all $r<\infty$ there is an increasing process $\overline{\chi}_r'$ on $I\!R_+^2$, with $\overline{\chi}_r'(u,0)=0$ a.s. and such that for all $(T,\Delta)\in\mathcal{S}$,*

$$E(|\Delta^{-1/2}(\delta(T,\Delta)-\delta'(T,\Delta))-Y(T,\Delta)|^r\ |\mathcal{F}_T)\le\overline{\chi}_r'(T,\Delta). \tag{7.9}$$

Proof. It is enough to prove the result for $r\ge 2$. Observe first that $\Delta^{-1/2}(\delta(T,\Delta)-\delta'(T,\Delta)) - Y(T,\Delta) = A(T,\Delta)+B(T,\Delta)$, where (see (7.1), (6.15) and (7.9)):

$$A(T,\Delta)=\frac{1}{\Delta}\int_T^{T+\Delta}D_s(T)dW_s,\quad \text{where }\ D_t(T)=\int_T^{T+t}(a_s'-a_T')dW_s+\int_T^{T+t}b_s'ds,$$

$$B(T,\Delta)=\frac{1}{\Delta}\int_T^{T+\Delta}(b_s-b_T)ds.$$

Then it is enough to prove the result separately for $E(|B(T,\Delta)|^r|\mathcal{F}_T)$ and for $E(|A(T,\Delta)|^r|\mathcal{F}_T)$.

In the first case it holds with

$$\overline{\chi}_r'(u,v)=\lim_{v'\downarrow v}\ \sup_{t\le u}\ E(\sup_{s\le u,s'\le v'}|b_{s+s'}-b_s|^r|\mathcal{F}_t),$$

which has $\overline{\chi}_r'(u,0)=0$ because here $t\mapsto b_t$ is continuous and uniformly bounded (same argument as in (b) of Lemma 7.1). Next, as in Lemma 7.1:

$$E(|A(T,\Delta)|^r|\mathcal{F}_T)\le K_r\Delta^{-r/2-1}\int_T^{T+\delta}E(|D_t(T)|^r|\mathcal{F}_T)\ dt.$$

Since a' and b' are uniformly bounded and a' is continuous, exactly as in the proof of Lemma 7.1 again we obtain an increasing process ζ_r on $I\!R_+^2$ with $\zeta_r(u,0)=0$, such that $E(|D_t(T)/\sqrt{t}\,|^r|\mathcal{F}_T)\le\zeta_r(T,t)$. Then if $(T,\Delta)\in\mathcal{S}$,

$$E(|A(T,\Delta)|^r|\mathcal{F}_T)\le K_r\ \frac{1}{\Delta}\int_T^{T+\delta}E(|D_t(T)/\sqrt{t}\,|^r|\mathcal{F}_T)\ dt\le K_r\zeta_r(T,\Delta)$$

and the result follows. □

Lemma 7.4 *Assume (H'), (H'-∞) and (K'1). Then with γ_t satisfying (1.5) for both g and ∇g, for all $r<\infty$ there is an increasing process $\overline{\chi}_r''$ on $I\!R_+^3$ with $\overline{\chi}_r''(u,0,w)=0$ a.s. and such that for all $(T,\Delta)\in\mathcal{S}$:*

$$E(|\Delta^{-1/2}(g(T,\delta(T,\Delta))-g(T,\delta'(T,\Delta)))-\nabla g(T,\delta'(T,\Delta))Y(T,\Delta)|^r\ |\mathcal{F}_T)\le\overline{\chi}_r''(T,\Delta,\gamma_T). \tag{7.10}$$

Proof. a) Here again it is enough to prove the result for $r\ge 2$. Due to our assumptions, we can apply Lemma 7.1 to the process a instead of X, hence with the same notation χ_T we get any finite stopping time T:

$$E\left(|t^{-1/2}(a_{T+t}-a_T)|^r|\mathcal{F}_T\right)\le\chi_r(T,t). \tag{7.11}$$

Plugging this into (7.4) gives, instead of (7.5): $\chi'_r(u,v) = v^{r/2}\zeta_r(u,v)$, where ζ_r is the following increasing process on $I\!R_+^2$:

$$\zeta_r(u,v) = K_r \lim_{v'\downarrow v} \sup_{t\le u} E(\sup_{s\le u+v'} |b_s|^r|\mathcal{F}_t) + \chi_r(u,v)).$$

b) Let $(T,\Delta)\in\mathcal{S}$. Set $\delta=\delta(T,\Delta)$, $\delta'=\delta'(T,\Delta)$, $Y=Y(T,\Delta)$, $Z=\delta-\delta'-\sqrt{\Delta}\ Y$. Taylor's formula yields $\Delta^{-1/2}(g(T,\delta)-g(T,\delta'))-\nabla g(T,\delta')Y = A(T,\Delta)+B(T,\Delta)$, with $A(T,\Delta)=\Delta^{-1/2}\nabla g(T,\delta')Z$, and $B(T,\Delta)=\Delta^{-1/2}(\nabla g(T,\delta'')-\nabla g(T,\delta'))(\delta-\delta')$ and $\delta''=\delta'+\theta(\delta-\delta')$ for a random variable θ taking values in $[0,1]$.

Our assumptions imply (H-r) for all r, hence we can reproduce the proof of Lemma 7.2 with ∇g instead of g and δ'' instead of δ, after observing that $|\delta''-\delta'|\le|\delta-\delta'|$. We obtain

$$E(|\nabla g(T,\delta'')-\nabla g(T,\delta')|^r|\mathcal{F}_T)\le\chi''_r(T,\Delta,\gamma_T).$$

Combining this and (7.3) and (a) above, Cauchy–Schwarz inequality gives

$$E(|B(T,\Delta)|^r|\mathcal{F}_T)\le\left(\chi''_{2r}(T,\Delta,\gamma_T)\ \zeta_{2r}(T,\Delta)\right)^{1/2}. \tag{7.12}$$

c) Finally 7.6) for ∇g and (7.2) yield $E(|\nabla g(T,\delta')|^r|\mathcal{F}_T)\le\zeta'_r(T,\Delta,\gamma_T)$ for some other increasing process ζ'_r. This and (7.9) give us

$$E(|A(T,\Delta)|^r|\mathcal{F}_T)\le\left(\overline{\chi}'_{2r}(T,\Delta)\ \zeta'_{2r}(T,\Delta,\gamma_T)\right)^{1/2}. \tag{7.13}$$

Then adding (7.12) and (7.13) gives (7.14) with the required properties for $\overline{\chi}''_r$. □

We end this section with an estimate for functions g: $I\!R^d\to I\!R^q$ that are continuously differentiable and have for some r:

$$|\nabla g(x)|\le r(1+|x|^r). \tag{7.14}$$

Set also $U(t,s)=\rho_{t+s}(g)-\rho_t(g)$. Then

Lemma 7.5 *Assume (H'), (H'-∞) and (7.14). There are increasing processes ζ and ζ' on $I\!R_+^2$ with $\zeta(u,0)=0$ a.s. and such that for all $(T,\Delta)\in\mathcal{S}$:*

$$|E(U(T,\Delta)|\mathcal{F}_T)|\le\sqrt{\Delta}\ \zeta(T,\Delta), \tag{7.15}$$

$$E|(U(T,\Delta)|^2|\mathcal{F}_T)|\le\Delta\ \zeta'(T,\Delta). \tag{7.16}$$

Proof. Below the constant K changes from line to line. We fix $u<\infty$ and set $\theta=1+\sup_t|a_t|$ and $\overline{\theta}_p=\sup_t E(\theta^p|\mathcal{F}_t)$, which is integrable for all $p<\infty$. We always take below (T,Δ) in $\mathcal{T}(u)$.

a) (7.14) implies $|g(x)-g(y)|\le K(1+|x|^r+|y|^r)|x-y|$, so $|g(a_{T+\Delta}x)-g(a_Tx)|\le K(1+|x|^r)\theta^r|a_{T+\Delta}-a_T|$ and integrating w.r.t. the normal measure G gives $|U(T,\Delta)|\le K\theta^r|a_{T+\Delta}-a_T|$. Then (7.11) readily gives (7.16) with $\zeta'(u,v)=K(\overline{\theta}_{4r}\ \chi_4(u,v))^{1/2}$ for a suitable constant K.

b) Taylor's formula gives $g(y)-g(x)=(\nabla g(x)+\alpha(x,y))(y-x)$ with $|\alpha(x,y)|\le K(1+|x|^r+|y|^r)$ and $\alpha(x,y)\to0$ as $y\to x$, uniformly in x on each compact subset of $I\!R^d$. Therefore there are reals $\nu(\varepsilon,p)>0$ such that $|x|\le p$ and $|y-x|\le\nu(\varepsilon,p)$ imply $|\alpha(x,y)|\le\varepsilon$.

By definition of $U(T,\Delta)$ we have

$$U(T,\Delta)=U_1+U_2,\qquad \text{where}\quad U_i=\int u_i(x)\,\rho(dx)$$

and

$$u_1(x) = \nabla g(a_T x)(a_{T+\Delta} - a_T)x, \qquad u_2(x) = \alpha(a_T x, a_{T+\Delta} x)(a_{T+\Delta} - a_T)x.$$

It is enough to prove (7.15) separately for U_1 and U_2.

c) We have $|u_2(x)| \leq K\theta^r(1+|x|^{r+1})|a_{T+\Delta}-a_T|$ and, as soon as $\theta|x| \leq p$ and $|a_{T+\Delta}-a_T|\,|x| \leq \nu(\varepsilon,p)$, then $|u_2(x)| \leq c|a_{T+\Delta}-a_T|\,|x|$. Integrating w.r.t. ρ, we obtain for all $\varepsilon, p > 0$, as for (7.7) (recall that K changes from line to line):

$$|U_2| \leq K\left(\varepsilon + \theta^{r+1}\left(\frac{1}{p} + \frac{|a_{T+\Delta}-a_T|}{\nu(\varepsilon,p)}\right)\right)|a_{T+\Delta}-a_T|.$$

We deduce from (7.11) that $|E(U_2|\mathcal{F}_T)| \leq \sqrt{\Delta}\; Y(\varepsilon,p,T,\Delta)$, where

$$Y(\varepsilon,p,u,v) = K\left((\varepsilon + \overline{\theta}_{2r+2}^{1/2}/p)\;\sqrt{\chi_2(u,v)} + \overline{\theta}_{2r+2}^{1/2}\sqrt{v\;\chi_4(u,v)}/\nu(\varepsilon,p)\right).$$

This is true for all $\varepsilon, p > 0$. Then (7.15) is satisfied by U_2 with $\zeta(u,v) = \lim_{v'\downarrow v}\;\inf_{\varepsilon,p>0}\;Y(\varepsilon,p,u,v')$, and that $\zeta(u,0) = 0$ is easily checked by choosing first p, then ε, the v.

d) Finally, (5.15) allows us to write (recall that a' and b' are bounded):

$$\begin{aligned}|E(U_1|\mathcal{F}_T)| \;&=\; \Big|\int \nabla g(a_T x)\Big(\int_T^{T+\Delta} E(b'_s|\mathcal{F}_T)\;ds\Big)x\,\rho(dx)\Big| \\ &\leq\; K\Delta\int|\nabla g(a_T x)|\,|x|\,\rho(dx) \leq\; K\theta^r\Delta\end{aligned}$$

use (7.14)). Then (7.15) holds for U_1, with $\zeta(u,v) = k\theta^r\sqrt{v}$. □

8 Proof of the results of Section 6

Proof of Theorem 6.3. In view of Theorem 6.1 it is enough to prove the claim (a) of Theorem 6.3. We do that in several steps.

Step 1. First we prove that under the assumptions of Theorem 6.3, $U^{1,n}(g)$ is well–defined. First assume (i), and let γ_t be as in (K). Set $T = T(n,i)$ and $\Delta = \Delta(n,i)$, so that on the $\mathcal{F}_T$–measurable set $\{\gamma_T \leq p\}$ we have $|g(T,\Delta_i^n X)| \leq p(1+|\Delta_i^n X|^p)$. Then $E(|g(T,\Delta_i^n X)| \mid \mathcal{F}_T) \leq p(1+\chi_p(T,\Delta)) < \infty$ by (7.2), and since $\{\gamma_T \leq p\} \uparrow \Omega$ as $p \to \infty$, the conditional expectations in (6.2) are well defined.

In cases (ii) and (iii) the same argument works, with γ_t as in (K2-r) (with $r = 0$ in case (iii)), so that $|g(T,\Delta_i^n X)| \leq p(1+|\Delta_i^n X|^r)$.

Step 2. Now we prove (6.13) under (i). Set

$$\chi_i^n = g(T(n,i),\Delta_i^n X) - g(T(n,i), a_{T(n,i)}\xi_i^n) \tag{8.1}$$

$$G_t^n = \delta_n \sum_{i\in\Sigma(n,t)} E(|\chi_i^n|^2|\mathcal{F}_{T(n,i)}). \tag{8.2}$$

Then since $\Delta(n,i)$ and $S(n,i)$ are $\mathcal{F}_{T(n,i)}$–measurable,

$$Y_t^n := U_t^{1,n}(g) - U_t^{3,n}(g) = \sqrt{\delta_n}\sum_{i\in\Sigma(n,t)}\left(\chi_i^n - E(\chi_i^n|\mathcal{F}_{T(n,i)})\right).$$

As in part (a) of the proof of Theorem 3.3, we get (6.13) if $\sum_{i\in\Sigma(n,t)} E(\xi_i^n\xi_i^{n,T}|\mathcal{F}_{T(n,i)}) \stackrel{P}{\longrightarrow} 0$ with $\xi_i^n = \sqrt{\delta_n}\ (\chi_i^n - E(\chi_i^n|\mathcal{F}_{T(n,i)}))$. In view of (8.2) it is then enough to prove that

$$G_t^n \stackrel{P}{\longrightarrow} 0. \tag{8.3}$$

Then with γ_t as in (K), we deduce from (7.6) that (recall that χ_2'' is increasing in each of its arguments, and that $\delta_n\mathrm{card}(\Sigma(n,t)) = \mu_n([0,t])$):

$$G_t^n \leq \delta_n \sum_{i\in\Sigma(n,t)} \chi_2''(T(n,i),\Delta(n,i)),\gamma_{T(n,i)})$$

$$\leq \mu_n([0,t] = \ \chi_2''(t,\sqrt{\delta_n},\gamma_t) + \chi_2''(t,t,\gamma_t) \sum_{i\in\Sigma(n,t)} 1_{\{\Delta(n,i)>\sqrt{\delta_n}\}}.$$

We have $\sum_{i\in\Sigma(n,t)} \Delta(n,i) \leq t$: hence the last sum above is smaller than $t/\sqrt{\delta_n}$. That is, $G_t^n \leq \mu_n([0,t] = \ \chi_2''(t,\sqrt{\delta_n},\gamma_t) + t\sqrt{\delta_n}\ \chi_2''(t,t,\gamma_t)$. Since $\delta_n \to 0$ and $\chi_2''(t,v,\gamma_t) \to 0$ a.s. as $v \to 0$ and since the sequence $\mu_n([0,t])$ is bounded in probability by (A2), we deduce (8.3) and (6.13).

Step 3. Here we assume (ii) or (iii) of Theorem 6.3. In order to apply Step 2, although (H-r) does not hold for all r, we "localize" the coefficients: since a and b are locally bounded, there exists an increasing sequence (τ_l) of stopping times satisfying $\tau_l = 0$ if $|a_0| + |b_0| > l$ and $|a_t| + |b_t| \leq l$ if $t \leq \tau_l$ and $\tau_l > 0$, and

$$\tau_l \uparrow +\infty \quad \text{a.s. as} \quad l \to \infty. \tag{8.4}$$

Set $a(l) = a_{t\wedge\tau_l}$ and $b(l) = b_{t\wedge\tau_l}$ if $\tau_l > 0$, and $a(l)_t = b(l)_t = 0$ if $\tau_l = 0$, and

$$X(l)_t = x_0 + \int_0^t a(l)_s dW_s + \int_0^t b(l)_s ds. \tag{8.5}$$

We denote by $U^{i,n}(l,g)$ the processes defined by (6.2), (6.3) and (6.4), with $(a(l),X(l))$ instead of (a,X). Now, $a(l)$ and $b(l)$ satisfy (H) and (H-r) for all $r<\infty$, hence Step 2 implies

$$\sup_{s\leq t} |U_s^{1,n}(l,g) - U_s^{3,n}(l,g)| \ \stackrel{P}{\longrightarrow}\ 0 \quad \text{as } n\to\infty, \text{ for all } \ l<\infty. \tag{8.6}$$

Further, on $\{\tau_l \geq t\}$, $U_s^{i,n}(g) = U_s^{i,n}(l,g)$ for all $s\in[0,t]$, $i=1,2,3$ (this is obvious for $i=2$ and $i=3$; for $i=1$ it comes from the fact that $S(n,j)$ is $\mathcal{F}_{T(n,i)}$–measurable). Then (6.13) readily follows from (8.4) and (8.6).

Proof of Corollary 6.4. Assume (H) and (K1). In view of (6.7) it is enough to prove that, for each $t<\infty$ and with χ_i^n defined by (8.1), $G_t^n = \delta_n \sum_{i\in\Sigma(n,t)} |\xi_i^n| \ \stackrel{P}{\longrightarrow}\ 0$. Because $(\chi_i^n : i\in\Sigma(n,t))$ are the same for X and for $X(l)$ on $\{R_n \geq t\}$ and because of (8.4), we can in fact work with each process $X(l)$, or equivalently assume (H-r) for all $r<\infty$.

Further, with $\theta(n,t)$ as in part (a) of the proof of Theorem 3.3 and $X_i^n = \sum_{j\leq i} \delta_n |\chi_j^n|$, we have $G_t^n = X_{\theta(n,t)}^n$ and the predictable compensator of X^n for the filtration $(\mathcal{F}_{T(n,i+1)})_{i\geq 0}$ is $\widetilde{X}_i^n = \sum_{j\leq i} \delta_n\ E(|\chi_j^n|\ |\mathcal{F}_{T(n,j)})$. Then by Lenglart's inequality, $\widetilde{X}_{\theta(n,t)}^n \ \stackrel{P}{\longrightarrow}\ 0$ implies $X_{\theta(n,t)}^n \ \stackrel{P}{\longrightarrow}\ 0$ (because $\theta(n,t)$ is a stopping time). Now, we can reproduce the proof of Step 2 in the previous proof to obtain $\widetilde{X}_{\theta(n,t)}^n = \delta_n \sum_{i\in\Sigma(n,t)} E(|\chi_j^n|\ |\mathcal{F}_{T(n,j)}) \ \stackrel{P}{\longrightarrow}\ 0$ (substituting $|\chi_i^n|^2$ with $|\chi_i^n|$, and thus χ_2'' with χ_1''). □

Proof of Theorem 6.5. Note that if U^n, Y^n, U, Y are $\mathbb{R}^k$–valued random variables, with Y^n going to Y in probability and U^n going to U stably in law, then $U^n + Y^n$ converge stably in law to $U+Y$. The same holds for the Skorokhod topology if U^n, Y^n, U, Y are càdlàg processes and further Y is continuous in time. Therefore if we set

$$Y_t^n = U_t^{2,n}(g) - U_t^{3,n}(g), \tag{8.7}$$

$$Y_t = \int_0^t (\rho_s^X(\nabla g) b_s + \overline{\rho}_s^X(\nabla g))\ \mu^\star(ds), \tag{8.8}$$

in order to deduce Theorem 6.5 from Theorem 6.1, it is enough to prove that

$$\sup_{s\leq t} |Y_s^n - Y_s| \ \stackrel{P}{\longrightarrow}\ 0 \tag{8.9}$$

under (A1), (A2), (H') and (K'). The proof goes through several steps.

Step 1. We wish to show that for every (small enough) function f on $I\!R^d$ and every pair (T,Δ) in $\mathcal{S}$ (see Section 7, recall also that δ^{jk} is the Kronecker symbol), we have

$$\begin{aligned} &E\Big((f(W_{T+\Delta}) - f(W_T))\ \int_T^{T+\Delta} (W_s^j - W_T^j) dW_s^k\ |\mathcal{F}_T\Big) \\ &\quad = \frac{1}{2}\ E\left((f(W_{T+\Delta}) - f(W_T))\left((W_{T+\Delta}^j - W_T^j)(W_{T+\Delta}^k - W_T^k) - \Delta\delta^{jk}\right) |\mathcal{F}_T\right). \end{aligned} \tag{8.10}$$

When $j=k$ this is just Itô's formula applied to $s\mapsto (W_{T+s}^j - W_T^j)^2$ and the equality holds even before taking conditional expectations. If $j\neq k$, and since W has stationary independent increments and independent components, it is enough to prove (8.10) when $T=0$ and Δ is deterministic and $f(x)=\exp(iux^j+ivx^k)$ for some $u,v\in I\!R$. In other words, we need to prove that if B, B' are two independent one-dimensional Brownian motion, and $Z_t=\int_0^t B_s\,dB'_s$,

$$E\left(e^{iuB_s+ivB'_s}\ Z_s\right) = \frac{1}{2}\ E\left(e^{iuB_s+ivB'_s}\ B_sB'_s\right). \tag{8.11}$$

Set $V=e^{iuB+ivB'}$. Itô's formula yields that the process YZ equals a martingale plus the following process:

$$\frac{1}{2}\int_0^s \left(-(u^2+v^2)V_tZ_t + 2ivV_tB_t)\right)\ dt.$$

Hence if $h(s)$ denotes the left-hand side of (8.11), we have,

$$h(s) = \frac{1}{2}\int_0^s \left(-(u^2+v^2)h(t) + 2iv\ E(V_tB_t)\right)\ dt$$

and, since $E(V_tB_t) = iut\,e^{-(u^2+v^2)t/2}$, we easily deduce that $h(s) = -\frac{uvs^2}{2}\ e^{-(u^2+v^2)s/2}$, which is equal to the right-hand side of (8.11).

Step 2. Here we assume in addition (H'-∞). Recalling (7.8), we set

$$\eta_i^n = \nabla g(T(n,i), a_{T(n,i)}\xi_i^n)\ Y(T(n,i),\Delta(n,i)).$$

Then (8.10) and (6.16) yield,

$$E(\eta_i^n|\mathcal{F}_{T(n,i)}) = \rho_{T(n,i)}^X(\nabla g) b_{T(n,i)} + \overline{\rho}_{T(n,i)}^X(\nabla g).$$

Since $t\mapsto \rho_t^X(\nabla g)b_t + \overline{\rho}_t^X(\nabla g)$ is continuous, one proves exactly as in lemma 4.5 the following convergence in probability, locally uniform in time:

$$\sum_{i\in\Sigma(n,t)} \sqrt{\delta_n\Delta(n,i)}\ E(\eta_i^n|\mathcal{F}_{T(n,i)}) \to Y_t.$$

Recalling (8.1), we have $Y_t^n = \sqrt{\delta_n}\,\sum_{i\in\Sigma(n,t)} E(\chi_i^n|\mathcal{F}_{T(n,i)})$. Therefore, the same argument as in the proof of Corollary 6.4 shows that (8.9) holds, provided we have for all $t<\infty$

$$G_t^n \;:=\; \sum_{i\in\Sigma(n,t)} \sqrt{\delta_n\Delta(n,i)}\; E(|\Delta(n,i)^{-1/2}\chi_i^n - \eta_i^n|\mathcal{F}_{T(n,i)}) \;\stackrel{P}{\longrightarrow}\; 0.$$

We reproduce Step 2 of the proof of Theorem 6.3, for $|\Delta(n,i)^{-1/2}\chi_i^n - \eta_i^n|$ instead of $|\chi_i^n|^2$: use (7.10) with $r=1$ and $\overline{\chi}''_1$ instead of (7.6) and χ''_2, and truncate at $\Delta(n,i)>\delta_n^{1/4}$, so $G_t^n \le \mu_n^\star([0,t])\;\overline{\chi}''_1(t,\delta_n^{1/4},\gamma_t) + t^{3/2}\delta_n^{1/4}\overline{\chi}''_1(t,t,\gamma_t)$.

Step 3. We no longer assume (H'-∞), but we localize as in Step 3 of the proof of Theorem 6.3: we have an increasing sequence (τ_l) of stopping times satisfying (8.4), and $\tau_l=0$ if $|a_0|+|b_0|+|a'_0|+|b'_0|>l$, and $|a_t|+|a'_t|+|b'_t|\le l$ if $t\le\tau_l$ and $\tau_l>0$.

Set $a(l)'_t = a'_{t\wedge\tau_l}$, $b(l)_t = b_{t\wedge\tau_l}$, $b(l)'_t = b'_{t\wedge\tau_l}$ and

$$a(l)_t = a_0 + \int_0^t a(l)'_s dW_s + \int_0^t b(l)'_s ds$$

if $\tau_l>0$, and $a(l)_t=0$, $b(l)_t=0$, $a(l)'_t=0$, $b(l)'_t=0$ if $\tau_l=0$. Finally, let $X(l)$ be defined by (8.5), and denote by $Y(l)^n$, $Y(l)$ the quantities associated with these processes indexed by l via (8.7), (8.8). For each l the term $(a(l),b(l),a(l)',b(l)')$ satisfies (H') and (H'-∞). Hence Step 1 implies (8.9) for $(Y(l)^n,Y(l))$ for each l, while on $\{R_l\ge t\}$ we have $Y_s=Y_s(l)$ and $Y_s^n=Y_s^n(l)$ for all $s\le t$. Then (8.9) for (Y^n,Y) follows from (8.4). □

Proof of Corollary 6.6. We only need to prove the claim (a). Recall that now $T(n,i)=i/n$ and $\Delta(n,i)=1/n$. Observe first that,

$$Y_t^n := U_t^{2,n}(g) - \sqrt{n}\;V_t^n(g) = \sqrt{n}\sum_{0\le i\le[nt]-1}\eta_i^n,$$

where $\eta_i^n = \int_{i/n}^{(i+1)/n}(\rho_s^X(g)-\rho_{i/n}^X(g))ds$.

Next, let us localize as in Step 3 of the proof of Theorem 6.5, and call $Y_t^n(l)$ the above quantity associated with the localized processes. Since $Y_s^n = Y_s^n(l)$ for all $s\le t$ on $\{\tau_l\ge t\}$, we see by (8.4) that it is enough to prove $\sup_{s\le t}|Y_s^n(l)| \stackrel{P}{\longrightarrow} 0$ for each l, or in other words we can and will assume (H'-∞).

Now we can apply Lemma 7.5 with $T=i/n$ and $\Delta=1/n$. Integrating (7.15) and (7.16) against Lebesgue measure on $[i/n,(i+1)/n]$, we get for $i\le[nt]-1$:

$$|E(\eta_i^n|\mathcal{F}_{i/n})| \le n^{-3/2}\zeta(t,1/n), \qquad E(|\eta_i^n|^2|\mathcal{F}_{i/n})| \le n^{-3}\zeta'(t,1/n).$$

Therefore if $A_t^n = \sqrt{n}\sum_{0\le i\le[nt]-1}E(\eta_i^n|\mathcal{F}_{i/n})$ and $B_t^n = V_t^n - A_t^n$, we deduce $\sup_{s\le t}|A_s^n| \stackrel{P}{\longrightarrow} 0$ (because $\zeta(t,v)\to 0$ a.s. as $v\to 0$), and the bracket of the $(\mathcal{F}_{[nt]})$–local martingale B^n is $|\langle B^n,B^{n,T}\rangle_t| \le \zeta'(t,1/n)/n$. Then Lenglart's inequality implies that $\sup_{s\le t}|B_s^n| \stackrel{P}{\longrightarrow} 0$, hence $\sup_{s\le t}|Y_s^n| \stackrel{P}{\longrightarrow} 0$ as well. □

9 Applications and examples

We will consider below a Brownian semimartingale X satisfying (H). Our first remark is that the measure ρ_t^X is symmetric about 0. Hence (see (6.9)):

$$\left.\begin{array}{l}\text{If } x\mapsto g(\omega,t,x) \text{ is an even function, } \rho_t^X(g)=0 \text{ and } \rho_t^X(\nabla g)=0,\\ \text{and also } \overline{\rho}_t^X(\nabla g)=0 \text{ and } \overline{U}(g)=g\star B^X \text{ in (6.17) if further (K') holds.}\end{array}\right\} \tag{9.1}$$

Let us for example consider the even function $g(\omega,t,x)=xx^T$ (taking values in $I\!R^d\otimes I\!R^d$, hence $q=d^2$). (6.14) yields the following well-known approximation of the quadratic variation:

$$\sup_t \Big| \sum_{1\le i\le [nt]} (X_{i/n}-X_{(i-1)/n})(X_{i/n}-X_{(i-1)/n})^T - \int_0^t c_s ds\Big| \xrightarrow{P} 0. \tag{9.2}$$

Further, Corollary 6.6 gives a rate of convergence in (9.2), which is easily proved directly but is not so well-known (apply the easily proved fact that $\rho_s(g_{jk}g_{il})=c_s^{jk}c_s^{il}+c_s^{ji}c_s^{kl}+c_s^{jl}c_s^{ki}$.

Proposition 9.1 *Assume (H'). The d^2–dimensional processes*

$$Y_t^n=\sqrt{n}\left(\sum_{1\le i\le [nt]} (X_{i/n}-X_{(i-1)/n})(X_{i/n}-X_{(i-1)/n})^T - \int_0^t c_s ds\right) \tag{9.3}$$

converge stably to a process Y defined on a very good extension of the space $(\Omega,\mathcal{F},(\mathcal{F}_t),P)$, and which is $\mathcal{F}$–conditionally a continuous Gaussian martingale with "deterministic" bracket given by

$$\langle Y^{jk},Y^{il}\rangle_t=\int_0^t (c_s^{jk}c_s^{il}+c_s^{ji}c_s^{kl}+c_s^{jl}c_s^{ki})ds. \tag{9.4}$$

Now we assume for simplicity that $d=m=1$. Consider $g(\omega,t,x)=x^p$ for some $p\in I\!N$. Then if α_p denotes the pth moment of the distribution $\mathcal{N}(0,1)$, Corollary 6.6 gives:

Proposition 9.2 *Assume (H'). The processes*

$$\sqrt{n}\left(n^{p/2-1}\sum_{1\le i\le [nt]} (X_{i/n}-X_{(i-1)/n})^p - \alpha_p\int_0^t (c_s)^{p/2}\, ds\right) \tag{9.5}$$

converge stably in law to a process Y defined on a very good extension of the space $(\Omega,\mathcal{F},(\mathcal{F}_t),P)$ which is as follows:

a) If p is even, Y is $\mathcal{F}$–conditionally a continuous Gaussian martingale with "deterministic" bracket given by

$$\langle Y,Y\rangle_t=(\alpha_{2p}-(\alpha_p)^2)\int_0^t (c_s)^p\, ds. \tag{9.6}$$

b) If p is odd and $p\ge 3$, $Y=Y'+Y''$ where

$$Y'_t=\alpha_{p+1}\int_0^t (c_s)^{(p-1)/2}dX_s^c+p\int_0^t (\alpha_{p-1}(b_s-a'_s/2)+\alpha_{p+1}a'_s/2)(c_s)^{(p-1)/2}\, ds, \tag{9.7}$$

and Y'' is $\mathcal{F}$–conditionally a continuous Gaussian martingale with deterministic bracket given by (9.6).

The first summand in (9.7) is a local martingale, but the second one is not: this is a good example of the "drift" introduced in the error term of the approximation (6.14) when the function g is not even.

We also deduce results on the approximations of the β–variation of X ($\beta > 0$), defined by

$$\mathrm{Var}(X,\beta)^n_t = \sum_{1\le i\le [nt]} |X_{i/n} - X_{(i-1)/n}|^\beta.$$

This is done by applying the previous results to $g(\omega,t,x) = |x|^\beta$. If $\alpha'_r = \int G(dx)|x|^r$ (hence $\alpha'_r = \alpha_r$ if α is an even integer), we have under (H):

$$n^{\beta/2-1}\ \mathrm{Var}(X,\beta)^n_t \ \to\ \alpha'_\beta \int_0^t (c_s)^{\beta/2}\, ds$$

uniformly in time, in probability. Further if $\beta > 1$, (K') holds and the processes

$$\sqrt{n}\,\big(n^{\beta/2-1}\ \mathrm{Var}(X,\beta)^n_t - \alpha'_\beta \int_0^t (c_s)^{\beta/2}\, ds\big)$$

converge stably to a process which, conditionally on $\mathcal{F}$, is a continuous Gaussian martingale with bracket equal to $(\alpha'_{2\beta} - (\alpha'_\beta)^2)\int_0^t (c_s)^\beta\, ds$.

Another interesting type of results, closely related to the previous ones, goes as follows. We consider only the situation of the β–variations (which include the quadratic variation of Proposition 9.1 for $\beta = 2$). Assume that a does not vanish and take $g(\omega,t,x) = |x/a_t(\omega)|^\beta$. Set

$$\mathrm{Var}'(X,\beta)^n_t = \sum_{1\le i\le [nt]} |(X_{i/n} - X_{(i-1)/n})/a_{(i-1)/n}|^\beta.$$

Then

$$n^{\beta/2-1}\ \mathrm{Var}'(X,\beta)^n_t \ \to\ \alpha'_\beta t$$

uniformly in time, in probability. Further if $\beta > 1$, the processes

$$\sqrt{n}\ \left(n^{\beta/2-1}\ \mathrm{Var}'(X,\beta)^n_t - \alpha'_\beta t\right)$$

converge stably to a process which, conditionally on $\mathcal{F}$, is a continuous Gaussian martingale with bracket given by $|\alpha'_{2\beta} - (\alpha'_\beta)^2|t$.

2) The previous examples were concerned with regular schemes. Now consider, again in the case $m = d = 1$, an example of random schemes. Set

$$T(n,0) = 0, \quad T(n,i+1) = \inf(t > T(n,i):\ nt \in I\!N, |X_t| \le h_n), \quad \Delta(n,i) = 1/n, \tag{9.8}$$

where h_n is a sequence of positive numbers tending to 0 and such that $\delta_n = 1/2nh_n$ tends to 0. Clearly (A1) holds, and we have

$$L^n_t := \mu_n([0,t]) = \frac{1}{2nh_n} \sum_{1\le i\le [nt]} 1_{\{|X-(i-1)/n|\le h_n\}} \tag{9.9}$$

and $\mu^\star_n = \sqrt{2h_n}\ \mu_n$. Then, as is well known, (A2) is met with $\mu(dt) = dL_t$ and $\mu^\star = 0$, where L is the local time of X at 0.

We cannot use Corollary 6.6 here. However, Theorem 6.1 gives the following result, when $g(\omega,t,x) = x^p$ for some $p \in I\!N$:

Proposition 9.3 *Assume (H). The processes,*

$$\frac{1}{\sqrt{2nh_n}} \sum_{1\leq i\leq [nt]} \left(n^{p/2}(X_{i/n}-X_{(i-1)/n})^p - \alpha_p (c_{(i-1)/n})^{p/2}\right) 1_{\{|X_{(i-1)/n}|\leq h_n\}}$$

converge stably in law to a process Y defined on a very good extension of the space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$, which is $\mathcal{F}$–conditionally a continuous Gaussian martingale with "deterministic" bracket given by

$$\langle Y, Y\rangle_t = (\alpha_{2p} - (\alpha_p)^2) \int_0^t (c_s)^p \; dL_s.$$

Although we cannot deduce a rate of convergence of L^n in (9.9) to L, it is interesting to re–state Corollary 6.6 here: take g satisfying (K1), and assume (H). Then the following convergence holds in probability, locally uniformly in time:

$$\frac{1}{2nh_n} \sum_{1\leq i\leq [nt]} g\left(\frac{i-1}{n}, \sqrt{n}\; (X_{i/n}-X_{(i-1)/n})\right) 1_{\{|X_{(i-1)/n}|\leq h_n\}} \;\to\; \int_0^t \rho_s(g) dL_s.$$

Let us mention that results similar to Proposition 9.3 have already been used in statistics: see Florens-Zmirou [5]. Analogous results when $d \geq 2$ have also been proved by Brugière [2] via a method of moments, but are not consequences of this paper since (A2) is violated in this case by the sequence (9.8) (there is no local time when $d \geq 2$, and the processes L^n of (9.9) converge in law, but not in probability; note that the normalization in (9.9) should be changed, and it depends on the dimension d.

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