CHICAGO BOOTH
The University of Chicago Booth School of Business

Working Paper No. 13-77

Measuring the Macroeconomic Impact of Monetary Policy at the Zero Lower Bound

Jing Cynthia Wu
University of Chicago Booth School of Business

Fan Dora Xia
University of California, San Diego

Measuring the Macroeconomic Impact of Monetary Policy at the Zero Lower Bound *

Jing Cynthia Wu
Chicago Booth and NBER

Fan Dora Xia
Merrill Lynch

First draft: August 19, 2013
This draft: July 20, 2014

Abstract

This paper employs an approximation that makes a nonlinear term structure model extremely tractable for analysis of an economy operating near the zero lower bound for interest rates. We show that such a model offers an excellent description of the data and can be used to summarize the macroeconomic effects of unconventional monetary policy at the zero lower bound. Our estimates imply that the efforts by the Federal Reserve to stimulate the economy since July 2009 succeeded in making the unemployment rate in December 2013 0.13% lower than it otherwise would have been.

Keywords: monetary policy, zero lower bound, unemployment, shadow rate, dynamic term structure model

JEL classification codes: E43, E44, E52, E58

*We have benefited from extensive discussions with Jim Hamilton and Drew Creal. We also thank Jim Bullard, John Cochrane, Greg Duffee, Benjamin Friedman, Refet Gurkaynak, Kinda Hachem, Anil Kashyap, Leo Krippner, Randy Kroszner, Jun Ma, David Romer, Glenn Rudebusch, Jeff Russell, Frank Schorfheide, Matthew Shapiro, Eric Swanson, Ruey Tsay, Johannes Wieland, John Williams and seminar and conference participants at Chicago Booth, UCSD, NBER Summer Institute Monetary Economics Workshop, St. Louis Fed Applied Time Series Econometrics Workshop, FRBSF ZLB workshop, Atlanta Fed, Boston Fed, Chicago Fed, Dallas Fed, and Kansas City Fed for helpful suggestions. Cynthia Wu gratefully acknowledges financial support from the IBM Faculty Research Fund at the University of Chicago Booth School of Business.
Correspondence: cynthia.wu@chicagobooth.edu, dora.xia@baml.com.

1 Introduction

Historically the Federal Reserve has used the federal funds rate as the primary instrument of monetary policy, lowering the rate to provide more stimulus and raising it to slow economic activity and control inflation. But since December 2008, the fed funds rate has been near zero, so that lowering it further to produce more stimulus has not been an option. Consequently, the Fed has relied on unconventional policy tools such as large-scale asset purchases and forward guidance to try to affect long-term interest rates and influence the economy. Assessing the impact of these measures or summarizing the overall stance of monetary policy in the new environment has proven to be a big challenge. Pevious efforts include Gagnon, Raskin, Remache, and Sack(2011), Hamilton and Wu(2012), Krishnamurthy and Vissing-Jorgensen(2011), D'Amico and King(2013), Wright(2012), Bauer and Rudebusch(forthcoming), and Swanson and Williams(forthcoming). However, these papers only focused on measuring the effects on the yield curve. In contrast, the goal of this paper is to assess the overall effects on the economy.

A related challenge has been to describe the relations between the yields on assets of different maturities in the new environment. The workhorse model in the term structure literature has been the Gaussian affine term structure model (GATSM); for surveys, see Piazzesi(2010), Duffee(forthcoming), Gürkaynak and Wright(2012), and Diebold and Rudebusch(2013). However, because this model is linear in Gaussian factors, it potentially allows nominal interest rates to go negative and faces real difficulties in the zero lower bound (ZLB) environment. One approach that could potentially prove helpful for both measuring the effects of policy and describing the relations between different yields is the shadow rate term structure model (SRTSM) first proposed by Black(1995). This model posits the existence of a shadow interest rate that is linear in Gaussian factors, with the actual short-term interest rate the maximum of the shadow rate and zero. However, the fact that an analytical solution to this model is known only in the case of a one-factor model makes using it more challenging.

In this paper we propose a simple analytical representation for bond prices in the multi-factor SRTSM that provides an excellent approximation and is extremely tractable for analysis and empirical implementation. It can be applied directly to discrete-time data to gain immediate insights into the nature of the SRTSM predictions. We demonstrate that this model offers an excellent empirical description of the recent behavior of interest rates.

More importantly, we show using a simple factor-augmented vector autoregression (FAVAR) that the shadow rate calculated by our model exhibits similar dynamic correlations with macro variables of interest in the period since July 2009 as the fed funds rate did in data prior to the Great Recession. This result gives us a tool for measuring the effects of monetary policy at the ZLB, and offers an important insight to the empirical macro literature where people use the effective federal funds rate in vector autoregressive (VAR) models to study the relationship between monetary policy and the macroeconomy. Examples of this literature include Christiano, Eichenbaum, and Evans(1999), Stock and Watson(2001), and Bernanke, Boivin, and Eliasz(2005). The evident structural break in the effective fed funds rate prevents researchers from getting meaningful information out of a VAR during and even post the ZLB. In contrast, the continuation of our series allows researchers to update their favorite VAR using the shadow rate for the ZLB period.[1]

Using the series combining the historical effective fed funds rate with the shadow rate at the ZLB, we show that the Fed has used unconventional policy measures to successfully lower the shadow rate. Our estimates also imply that the Fed's efforts to stimulate the economy since July 2009 have succeeded in lowering the unemployment rate by 0.13% relative to where it would have been in the absence of these measures.

The SRTSM has been used to describe the recent behavior of interest rates and monetary policy by Kim and Singleton(2012) and Bauer and Rudebusch(2013), but these authors relied on simulation methods to estimate and study the model. Krippner(2013) proposed a continuous-time analog to our solution, where he added a call option feature to derive the so-

[1]Our shadow rate data with monthly update is available at http://faculty.chicagobooth.edu/jing.wu/research/data/WX.html.

lution. Ichiue and Ueno(2013) derived similar approximate bond prices by ignoring Jensen's inequality. Both derivations are in continuous time, which requires numerical integration when applied to discrete-time data.

Our paper also contributes to the recent discussion on the usefulness of the shadow rate as a measure for the monetary policy stance. Christensen and Rudebusch(2014) and Bauer and Rudebusch(2013) pointed out that the estimated shadow rate varied across different models. Bullard(2012) and Krippner(2012) advocated the potential of the shadow rate to describe the monetary policy stance. Our results provide further empirical evidence to support the latter view, and demonstrate that the shadow rate is a powerful tool to summarize information at the ZLB.

The rest of the paper proceeds as follows. Section 2 describes the SRTSM. Section 3 proposes a new measure for monetary policy at the ZLB and demonstrates its advantage over the effective federal funds rate. Section 4 summarizes the implication of unconventional monetary policy on the macroeconomy using historical data from 1960 to 2013, and Section 5 zooms in on the ZLB period. Section 6 concludes.

2 Shadow rate term structure model

2.1 Shadow rate

Similar to Black(1995), we assume that the short term interest rate is the maximum of the shadow rate s_t and a lower bound $\underline{r}$:

$$r_t = max(\underline{r}, s_t). \tag{1}$$

If the shadow rate s_t is greater than the lower bound, then s_t is the short rate. Note that when the lower bound is binding, the shadow rate contains more information about the current state of the economy than does the short rate itself. Since the end of 2008, the

Federal Reserve has paid interest on reserves at an annual interest rate of 0.25%, proposing the choice of $\underline{r} = 0.25$.[2]

2.2 Factor dynamics

We assume that the shadow rate s_t is an affine function of some state variables X_t,

$$s_t = \delta_0 + \delta_1' X_t. \tag{2}$$

The state variables follow a first order vector autoregressive process (VAR(1)) under the physical measure ($\mathbb{P}$):

$$X_{t+1} \quad = \quad \mu + \rho X_t + \Sigma \varepsilon_{t+1}, \quad \varepsilon_{t+1} \sim N(0, I). \tag{3}$$

The log stochastic discount factor is essentially affine as in Duffee(2002)

$$M_{t+1} \quad = \quad \exp\left(-r_t - \frac{1}{2}\lambda_t' \lambda_t - \lambda_t' \varepsilon_{t+1}\right), \tag{4}$$

where the price of risk λ_t is linear in the factors

$$\lambda_t = \lambda_0 + \lambda_1 X_t.$$

This implies that the risk neutral measure ($\mathbb{Q}$) dynamics for the factors are also a VAR(1):

$$X_{t+1} \quad = \quad \mu^{\mathbb{Q}} + \rho^{\mathbb{Q}} X_t + \Sigma \varepsilon_{t+1}^{\mathbb{Q}}, \quad \varepsilon_{t+1}^{\mathbb{Q}} \overset{\mathbb{Q}}{\sim} N(0, I). \tag{5}$$

[2] Our main results are robust if we estimate $\underline{r}$ as a free parameter, see Section 3.2 for example.

The parameters under the $\mathbb{P}$ and $\mathbb{Q}$ measures are related as follows:

$$\begin{aligned} \mu - \mu^{\mathbb{Q}} &= \Sigma\lambda_0, \\ \rho - \rho^{\mathbb{Q}} &= \Sigma\lambda_1. \end{aligned}$$

2.3 Forward rates

Equation (1) introduces non-linearity into an otherwise linear system. A closed-form pricing formula for the SRTSM described in Sections 2.1 - 2.2 is not available beyond one factor. In this section, we propose an analytical approximation for the forward rate in the SRTSM, making the otherwise complicated model extremely tractable. Our formula is simple and intuitive, and we will compare it to the solution in a Gaussian model in Section 2.4 to gain some intuition. A simulation study in Section 2.6 demonstrates that the error associated with our approximation is only a few basis points.

Define $f_{n,n+1,t}$ as the forward rate at time t for a loan starting at $t+n$ and maturing at $t+n+1$. The forward rate in the SRTSM described in equations (1) to (5) can be approximated by

$$f^{SRTSM}_{n,n+1,t} = \underline{r} + \sigma^{\mathbb{Q}}_n g\left(\frac{a_n + b'_n X_t - \underline{r}}{\sigma^{\mathbb{Q}}_n}\right), \tag{6}$$

where $(\sigma^{\mathbb{Q}}_n)^2 \equiv \mathbb{V}ar^{\mathbb{Q}}_t(s_{t+n})$. The function $g(z) \equiv z\Phi(z)+\phi(z)$ consists of a normal cumulative distribution function $\Phi(.)$ and a normal probability density function $\phi(.)$. Its non-linearity comes from moments of the truncated normal distribution. The expressions for a_n and b_n as well as the derivation are in Appendix A.

To our knowledge, we are the first in the literature to propose an analytical approximation for the forward rate in the SRTSM that can be applied to discrete-time data directly. For example, Bauer and Rudebusch(2013) used a simulation-based method. Krippner(2013) proposed an approximation for the instantaneous forward rate in continuous-time. To apply his formula to the one-month ahead forward rate in the data, a researcher needs to numerically integrate the instantaneous forward rate over that month, see Christensen and Rudebusch(2014) for example. Conversely, our discrete-time formula can be applied directly to the data. In summary, our analytical approximation is free of any numerical error associated with simulation methods and numerical integration.

2.4 Relation to Gaussian Affine Term Structure Models

If we replace equation (1) with

$$r_t = s_t,$$

the SRTSM becomes a GATSM, the benchmark model in the term structure literature. The forward rate in the GATSM is an affine function of the factors:

$$f^{GATSM}_{n,n+1,t} = a_n + b_n' X_t, \tag{7}$$

where a_n and b_n are the same as in equation (6), and the detailed expressions are in Appendix A.

The difference between (6) and (7) is the function $g(.)$. We plot it in Figure 1 together with the 45 degree line. It is a non-linear and increasing function. The function value is indistinguishable from the 45 degree line for inputs greater than 2, and is practically zero for z less than -2. The limiting behavior demonstrates that the GATSM is a simple and close approximation for the SRTSM, when the economy is away from the ZLB.

2.5 Estimation

State space representation for the SRTSM We write the SRTSM as a nonlinear state space model. The transition equation for the state variables is equation (3). From equation (6), the measurement equation relates the observed forward rate $f^o_{n,n+1,t}$ to the factors as follows:

$$f^o_{n,n+1,t} \;=\; \underline{r} + \sigma^{\mathbb{Q}}_n g\left(\frac{a_n + b_n' X_t - \underline{r}}{\sigma^{\mathbb{Q}}_n}\right) + \eta_{nt}, \tag{8}$$

where the measurement error η_{nt} is i.i.d. normal, $\eta_{nt} \sim N(0, \omega)$. The observation equation is not linear in the factors. We use the extended Kalman filter for estimation, which applies the Kalman filter by linearizing the nonlinear function $g(.)$ around the current estimates. See Appendix B for details.

The extended Kalman filter is extremely easy to apply due to the closed-form formula in equation (6). We take the observation equation (8) directly to data without any further numerical approximation, which is necessary for pricing formulas derived in continuous time. The likelihood surface behaves similarly to a GATSM, because the function $g(.)$ is monotonically increasing. These features together make our formula appealing.

State space representation for the GATSM For the GATSM described in Section 2.4, equation (3) is still the transition equation. Equation (7) implies the measurement equation:

$$f^o_{n,n+1,t} \;=\; a_n + b_n' X_t + \eta_{nt}, \tag{9}$$

with $\eta_{nt} \sim N(0, \omega)$. We apply the Kalman filter for the GATSM, because it is a linear Gaussian state space model. See Appendix B for details.

Data We construct one-month forward rates for maturities of 3 and 6 months, 1, 2, 5, 7 and 10 years from the Gürkaynak, Sack, and Wright(2007) dataset, using observations at the end of the month.[3] Our sample spans from January 1990 to December 2013.[4] We plot the time series of these forward rates in Figure 2. In December 2008, the Federal Open

[3]As a robustness check, we also estimate the SRTSM and extract the shadow rate with Fama and Bliss(1987) zero coupon bond data from CRSP, and we get similar results. See Section 3.2 for example.

[4]Starting the sample from 1990 is standard in the GATSM literature, see Wright(2011) and Bauer, Rudebusch, and Wu(2012) for examples.

Market Committee (FOMC) lowered the target range for the federal funds rate to 0 to 25 basis points. We refer to the period from January 2009 to the end of the sample as the ZLB period, and highlight with shaded area. For this period, forward rates of shorter maturities are essentially stuck at zero, and do not display meaningful variation. Those with longer maturities are still far away from the lower bound, and display significant variation.

Normalization The consensus in the term structure literature is that three factors are sufficient to account for almost all of the cross-sectional variation in yields. Therefore, we focus our discussion on three factor models.[5] The collection of parameters we estimate include $(\mu, \mu^{\mathbb{Q}}, \rho, \rho^{\mathbb{Q}}, \Sigma, \delta_0, \delta_1)$. For identification, we impose normalizing restrictions on the $\mathbb{Q}$ parameters similar to Joslin, Singleton, and Zhu(2011) and Hamilton and Wu(2014): (i) $\delta_1 = [1, 1, 0]'$; (ii) $\mu^{\mathbb{Q}} = 0$; (iii) $\rho^{\mathbb{Q}}$ is in real Jordan form with eigenvalues in descending order; and (iv) Σ is lower triangular. Note that these restrictions are for statistical identification only, i.e. they prevent the latent factors from rotating. Imposing this or other sets of restrictions does not change economic implications of the model.

Repeated eigenvalues Estimation assuming that $\rho^{\mathbb{Q}}$ has three distinct eigenvalues produces two smaller eigenvalues almost identical to each other, with the difference in the order of 10^{-3}. This evidence points to repeated eigenvalues. Creal and Wu(2014) have documented a similar observation using a different dataset and a different model. With repeated eigenvalues, the real Jordan form becomes

$$\rho^{\mathbb{Q}} = \begin{bmatrix} \rho_1^{\mathbb{Q}} & 0 & 0 \\ 0 & \rho_2^{\mathbb{Q}} & 1 \\ 0 & 0 & \rho_2^{\mathbb{Q}} \end{bmatrix}.$$

[5]All of our main results relating to the macroeconomy, from Section 3 onward, are robust to two-factor models, see Section 3.2 for example. But for the term structure models themselves, two-factor models perform worse than three-factor models in terms of fitting the data.

Model comparison Maximum likelihood estimates, and robust standard errors (See Hamilton(1994) p. 145) are reported in Table 1. The log likelihood value is 755.46 for the GATSM, and 855.57 for the SRTSM. The superior performance of the SRTSM comes from its ability to fit the short end of the forward curve when the lower bound binds. In Figure 3, we plot average observed (red dots) and fitted (blue curves) forward curves in 2012. The left panel illustrates that the SRTSM fitted forward curve flattens at the short end, because the $g(.)$ function is very close to zero when the input is sufficiently negative. This is consistent with the feature of the data. In contrast, the GATSM in the right panel has trouble fitting the short end. Instead of having a flat short end as the data suggest, the GATSM generates too much curvature. That is the only way it can approximate the yield curve at the ZLB.

As demonstrated in Section 2.4, the GATSM is a good approximation for the SRTSM when forward rates are sufficiently higher than the lower bound. We illustrate this property using the following numerical example. When both models are estimated over the period of January 1990 to December 1999, the maximum log likelihood is 475.71 for the SRTSM, and 476.69 for the GATSM. The slight difference in the likelihood comes from the linear approximation of the extended Kalman filter.

2.6 Approximation error

An alternative to equation (6) to compute forward rates or yields is simulation. In Table 2, we compare forward rates and yields implied by equation (6) and by an average of 10 million simulated paths to measure the size of the approximation error of equation (6). The details of our simulation exercise are explained in Table 2. The approximation errors grow with the time to maturity for both forward rates and yields. We focus on the longest end to report the worst case scenario. The average absolute approximation error of the 24 Januaries between 1990 and 2013 for the 10-year ahead forward rate is 2.3 basis points, about 0.36% of the average forward rate for this period (6.37%). The number is 0.78 basis points for the 10 year yield with an average level of 5.29%, yielding a ratio of 0.14%. The approximation errors for long term forward rates are larger than those for yields, because yields factor in the smaller approximation errors of short term and medium term forward rates. Regardless, the approximation errors are at most a few basis points, orders of magnitude smaller than the level of interest rates. The approximation errors in Table 2 contain simulation errors. With the large number of draws (10 million), the simulation errors are negligible. To show that, we compare the analytical solution in equation (7) for the GATSM with simulation. The average absolute simulation errors are 0.1 basis points for the 10 year ahead forward rate and 0.04 for the 10 year yield.

3 Policy rate

The effective federal funds rate has been the primary measure for the Fed's monetary policy stance in the literature, and it provided the basis for most empirical studies of the interaction between monetary policy and the economy. However, since 2009, the effective federal funds rate has been stuck at the lower bound, and no longer conveys any information due to its lack of variability. How do we summarize the effects of monetary policy in this situation? Most research singles out the ZLB period. The issue with this approach is that it throws out half a century of historical data even when the economy exits the ZLB and the short rate regains its role as the summary for monetary policy. Is there a way economists can keep using the long historical data, with the presence of the ZLB period? The shadow rate from the SRTSM is a potential solution. Section 3.2 demonstrates that the shadow rate interacts with macro variables similarly as the fed funds rate did historically. Section 5.1 reinforces this key result.

We construct the new policy rate s_t^o by splicing together the effective federal funds rate before 2009 and the estimated shadow rate since 2009. This combination makes the most use out of both series. We plot the model implied shadow rate (in blue) and the effective federal funds rate (in green) in Figure 4. Before 2009, the ZLB was not binding, the model implied short rate was equal to the shadow rate. The difference between the two lines in Figure 4 reflects measurement error, in units of basis points. The two rates have diverged since 2009. The effective federal funds rate has been stuck at the ZLB. In contrast, the shadow rate has become negative and still displays meaningful variation. We update our shadow rate monthly at http://faculty.chicagobooth.edu/jing.wu/research/data/WX.html.

3.1 Factor augmented vector autoregression

We use the FAVAR model proposed by Bernanke, Boivin, and Eliasz(2005) to study the effects of monetary policy. The basic idea of the FAVAR is to compactly summarize the rich information contained in a large set of economic variables Y_t^m using a low-dimensional vector of factors x_t^m. This model allows us to study monetary policy's impact on any macroeconomic variable of interest. The factor structure also ensures that the number of parameters does not explode.

Model Following Bernanke, Boivin, and Eliasz(2005), we use 3 factors, and assume that the factors x_t^m and the policy rate s_t^o jointly follow a VAR(13):[6]

$$\begin{bmatrix} x_t^m \\ s_t^o \end{bmatrix} = \begin{bmatrix} \mu^x \\ \mu^s \end{bmatrix} + \rho^m \begin{bmatrix} X_{t-1}^m \\ S_{t-1}^o \end{bmatrix} + \Sigma^m \begin{bmatrix} \varepsilon_t^m \\ \varepsilon_t^{\mathrm{MP}} \end{bmatrix}, \qquad \begin{bmatrix} \varepsilon_t^m \\ \varepsilon_t^{\mathrm{MP}} \end{bmatrix} \sim N(0, I), \tag{10}$$

where we summarize the current value of x_t^m (and s_t^o) and its 12 lags using a capital letter to capture the state of the economy, $X_t^m = [x_t^{m\prime}, x_{t-1}^{m\prime}, ..., x_{t-12}^{m\prime}]'$ (and $S_t^o = [s_t^o, s_{t-1}^o, ..., s_{t-12}^o]'$). Constants μ^x and μ^s are the intercepts, and ρ^m is the autoregressive coefficient. The matrix Σ^m is the cholesky decomposition of the covariance matrix. The monetary policy shock is $\varepsilon_t^{\mathrm{MP}}$. We identify the monetary policy shock through the recursiveness assumption as in Bernanke, Boivin, and Eliasz(2005); for details see Appendix C. Observed macroeconomic

[6] Our results hold with different numbers of factors (3 or 5) and with different lag lengths (6, 7, 12 or 13).

variables load on the macroeconomic factors and policy rate as follows:

$$Y_t^m = a_m + b_x x_t^m + b_s s_t^o + \eta_t^m, \quad \eta_t^m \sim N(0, \Omega), \tag{11}$$

where a_m is the intercept, and b_x and b_s are factor loadings.

Data Similar to Bernanke, Boivin, and Eliasz(2005), Y_t^m consists of a balanced panel of 97 macroeconomic time series from the Global Insight Basic Economics, and our data spans from January 1960 to December 2013.[7] We have a total of $T = 635$ observations. We apply the same data transformations as in the original paper to ensure stationarity. See Table 3 for detailed data description.

Estimation First, we extract the first three principal components of the observed macroeconomic variables over the period of January 1960 to December 2013, and take the part that is orthogonal to the policy rate as the macroeconomic factors. Then, we estimate equation (11) by ordinary least squares (OLS). See Appendix C for details. Next, we estimate equation (10) by OLS.

Macroeconomic variables and factors The loadings of the 97 macro variables on the factors are plotted in Figure 5. Real activity measures load heavily on factor 1, price level indexes load more on factor 2, and factor 3 contributes primarily to employment and prices. For the contemporaneous regression in equation (11), more than one third of the variables have an R^2 above 60%, which confirms the three-factor structure. Besides the policy rate, we focus on the following five macroeconomic variables: industrial production, consumer price index, capacity utilization, unemployment rate and housing starts. They represent the three factors, and cover both real activity and price levels. The R^2s for these macroeconomic

[7] Global Insight Basic Economics does not maintain all 120 series used in Bernanke, Boivin, and Eliasz(2005). Only 97 series are available from January 1960 to December 2013. The main results from Bernanke, Boivin, and Eliasz(2005) can be replicated by using the 97 series in our paper for the same sample period.

variables are 73%, 89%, 64%, 64% and 67% respectively.

3.2 Measures of monetary policy

The natural question is whether the shadow rate could be used in place of the fed funds rate to describe the stance and effects of monetary policy under the ZLB. We first approach this using a formal hypothesis test - can we reject the hypothesis that the parameters relating the shadow rate to macroeconomic variables of interest under the ZLB are the same as those that related the fed funds rate to those variables in normal times?

We begin this exercise by acknowledging that we do not attempt to model the Great Recession in our paper, because it was associated with some extreme financial events and monetary policy responses. For example, Ng and Wright(2013) provided some empirical evidence that the Great Recession is different in nature from other post-war recessions. Instead, we are interested in the behavior of monetary policy and the economy in the period following the Great Recession, when policy returned to a new normal that ended up being implemented through the traditional 6-week FOMC calendar but using the unconventional tools of large scale asset purchases and forward guidance. We investigate whether a summary of this new normal based on our derived shadow rate shows similar dynamic correlations as did the fed funds rate in the period prior to the Great Recession.

We rewrite the first equation in (10)

$$\begin{aligned}
x_t^m &= \mu^x + \rho^{xx} X_{t-1}^m \\
&+ \mathbf{1}_{(t<\text{December 2007})} \rho_1^{xs} S_{t-1}^o \\
&+ \mathbf{1}_{(\text{December 2007} \leq t \leq \text{June 2009})} \rho_2^{xs} S_{t-1}^o \\
&+ \mathbf{1}_{(t>\text{June 2009})} \rho_3^{xs} S_{t-1}^o \\
&+ \Sigma^{xx} \varepsilon_t^m, \qquad (12)
\end{aligned}$$

The null hypothesis is that the coefficient ρ^{xs} is the same before and after the Great Recession:

$$H_0 : \rho_1^{xs} = \rho_3^{xs}.$$

We construct the likelihood ratio statistic as follows (see Hamilton(1994) p. 297):

$$(T-k)(\log|\widehat{\Sigma_R^{xx}\Sigma_R^{xx\prime}}| - \log|\widehat{\Sigma_U^{xx}\Sigma_U^{xx\prime}}|,$$

where T is the sample size, k is the number of regressors on the right hand side of equation (12), $\widehat{\Sigma_U^{xx}\Sigma_U^{xx\prime}}$ is the estimated covariance matrix, and $\widehat{\Sigma_R^{xx}\Sigma_R^{xx\prime}}$ is the estimated covariance matrix with the restriction imposed by the null hypothesis.

The likelihood ratio statistic has an asymptotic χ^2 distribution with 39 degrees of freedom. The p-value is 0.29 for our policy rate s_t^o. We fail to reject the null hypothesis at any conventional significance level. This is consistent with the claim that our proposed policy rate impacts the macroeconomy the same way at the ZLB as before. If we use the effective federal funds rate instead, the p-value is 0.0007, and we would reject the null hypothesis at any conventional significance level. Our results show that there is a structural break if one tries to use the conventional monetary policy rate. Using a similar procedure for the coefficients relating lagged macro factors to the policy rate, the p-values are 1 for both our policy rate and the effective fed funds rate. In summary, our policy rate exhibits similar dynamic relations to key macro variables before and after the Great Recession, and captures meaningful information missing from the effective federal funds rate after the economy reached the ZLB. The immediate implication of this result is that researchers can use the shadow rate to update earlier studies that had been based on the historical fed funds rate.

Robustness We check the robustness of our main result with the following alternatives. A1: We estimate $\underline{r}$ in (1) as a free parameter. A2: We use two factors instead of three for the SRTSM in Section 2. A3: We use Fama and Bliss(1987) zero coupon bond yields from

CRSP, with maturities of 3 months and 1 through 5 years. A4: We include 5 factors in the FAVAR. A5: For the FAVAR, we use 6, 7 and 12 lags. The structural break test results are summarized in Table 4. The first column shows the p values for the test $H_0 : \rho_1^{xs} = \rho_3^{xs}$. We cannot reject the null hypothesis at 5% level for all the specifications, with all but one p value greater than 0.1. The second column illustrates that we cannot reject the null hypothesis $H_0 : \rho_1^{sx} = \rho_3^{sx}$ for any alternative at any level. Overall, our result is robust to different model choices.

4 Macroeconomic implications

After the Great Recession the Federal Reserve implemented a sequence of unconventional monetary policy measures including large-scale asset purchases and forward guidance. The literature has thus far focused on large-scale asset purchases, and its effects on the yield curve. In contrast to previous studies, here we attempt to answer some more fundamental questions: what is the overall impact of these new unconventional policy tools on the real economy? Is the Fed able to achieve its stated goal of lowering the unemployment rate?

4.1 Historical decomposition

In this section, we attempt to assess the effect of the various unconventional policy measures adopted by the Federal Reserve after the Great Recession with a historical decomposition. The basic idea is that we can write each variable in equation (10) as a sum of past shocks and its initial condition. Specifically, the contribution of monetary policy shocks after the Great Recession (between $[t_1 = \text{July 2009}, t_2 = \text{December 2013}]$) to an individual economic variable $Y_t^{m,i}$ can be summarized by

$$\sum_{\tau=t_1}^{max(t,t_2)} \Psi_{t-\tau}^{\text{MP},i} \varepsilon_\tau^{\text{MP}}, \tag{13}$$

where $\Psi_j^{\mathrm{MP},i}$ is the impulse response

$$\Psi_j^{\mathrm{MP},i} = \frac{\partial Y_{t+j}^{m,i}}{\partial \varepsilon_t^{\mathrm{MP}}} = b_{x,i}\frac{\partial x_{t+j}^m}{\partial \varepsilon_t^{\mathrm{MP}}} + b_{s,i}\frac{\partial s_{t+j}^o}{\partial \varepsilon_t^{\mathrm{MP}}}, \tag{14}$$

for variable i after j periods in response to a one unit shock in $\varepsilon_t^{\mathrm{MP}}$, and the derivatives on the right hand side are the impulse responses from a standard VAR.

In Figure 6, we plot the observed time series for the six variables in blue, and counterfactual paths in red dashed lines for an alternative world where all the monetary policy shocks at the ZLB were zero. In the top left panel, we show the difference between the realized and counterfactual policy rates. Without any deviation from the traditional monetary policy rule, the shadow rate would have been about -1% in December 2013, whereas the actual shadow rate then was about -2%. On average, the shadow rate would have been 0.4% higher between 2011 and 2013 if the monetary policy shocks were set to zero. These results indicate that unconventional monetary policy has been actively lowering the policy rate, and the Federal Reserve has employed an expansionary monetary policy since 2011.

Next consider implications for the real economy. In the absence of expansionary monetary policy, in December 2013, the unemployment rate would be 0.13% higher at the 6.83% level rather than 6.7% in the data. The industrial production index would have been 101.0 rather than 101.8, and capacity utilization would be 0.3% lower than what we observe. Housing starts would be 11,000 lower (988,000 vs. 999,000). These numbers suggest that unconventional monetary policy achieved its goal of stimulating the economy. Interestingly, the accommodative monetary policy during this period has not boosted real activity at the cost of high inflation. Instead, monetary policy shocks have contributed to decreasing the consumer price index by 1. Our result exhibits the same price puzzle that has been discussed in earlier macro studies.[8]

The historical decomposition exercise calculates the contribution of monetary policy shocks defined as deviations of the realized shadow rate from the policy rate implied by

[8] Examples include Sims(1992) and Eichenbaum(1992).

the historical monetary policy rule. Another question of interest is what would happen if the Fed had adopted no unconventional monetary policy at all. This question is more difficult to answer, because it is not clear what the counterfactual shadow rate would be. One possible counterfactual to consider would be what would have happened if the shadow rate had never fallen below the lower bound $\underline{r}$. Specifically, we replace the realized monetary policy shock ($\varepsilon_\tau^{\mathrm{MP}}$) in equation (13) with the counterfactual shocks, $\varepsilon_\tau^{\mathrm{MP},II}$, such that these shocks would have kept the shadow rate at the lower bound. One might view the difference between the actual shadow rate and this counterfactual as an upper bound on the contribution of unconventional monetary policy measures. If instead of the realized shadow rate, monetary policy had been such that the shadow rate never fell below 0.25%, the result would have been an unemployment rate 1% higher than observed.

Our estimated effect of unconventional monetary policy on the unemployment rate is smaller than the ones found in Chung, Laforte, Reifschneider, and Williams(2012) and Baumeister and Benati(2013). This is primarily because they assumed that unconventional monetary policy had a big impact on the yield curve. For example, Chung, Laforte, Reifschneider, and Williams(2012) assumed that the large-scale asset purchases reduced the long term interest rates by 50 basis points, and then translated this number into a 1.5% decrease in the unemployment rate. If we were to use Hamilton and Wu(2012)'s estimate of 13 basis-point decrease in the 10 year rate, a simple linear calculation would translate this number into a 0.39% reduction in the unemployment rate. This is comparable to our estimate.

4.2 Impulse responses

What would happen to the unemployment rate one year later if the Fed decreases the policy rate by 25 basis points now? An impulse response function offers a way to think about questions like this by describing monetary policy's dynamic impact on the economy.

We compute the impulse responses using equation (14) and plot them in Figure 7 for six economic variables (the policy rate, industrial production, consumer price index, capacity utilization, unemployment rate and housing starts) to a loosening monetary policy shock with a size of 25 basis points ($\Sigma^{ss}\varepsilon_t^{\mathrm{MP}} = -25$ bps). The 90% confidence intervals are in the shaded areas.[9] With an expansionary monetary policy shock, real activity increases as expected: industrial production, capacity utilization and housing starts increase while the unemployment rate decreases. The impacts peak after about a year. Specifically, one year after a -25 basis-point shock to the policy rate, industrial production is 0.5% higher than its steady state level, capacity utilization increases by 0.2% , the unemployment rate decreases by 0.06% , and housing starts is 1.3% above its steady state level. After the peak, the effects die off slowly, and they are eventually gone in about 8 years.

5 Macroeconomic impact at the ZLB

Our main results in Section 3 and 4 are based on a constant structure before and after the Great Recession. Despite a much smaller sample, the ZLB period provides an alternative angle, complementing the results we have so far. Section 5.1 serves as a robustness check – we compare the full sample impulse responses with those from the ZLB period, demonstrating the usefulness of the shadow rate. Section 5.2 studies forward guidance. With a sample size of 53 months, we replace the 13-lag FAVAR with a 1-lag FAVAR.

5.1 New vs. conventional policy rates

Consider first an attempt to estimate a first-order FAVAR for data at the ZLB period in which the effective fed funds rate is used as the policy rate. We plot impulse responses to an expansionary policy shock of 25 basis points in Figure 8. The turquoise lines are median responses, and 90% confidence intervals are in the turquoise areas. For comparison, we also plot the impulse responses for the full sample with our policy rate in blue. These are identical to the impulse responses presented in Figure 7. For the ZLB subsample, the

[9] Confidence intervals are constructed by bootstrapping.

impulse responses to a shock to the effective federal funds rate are associated with huge uncertainty, with the confidence intervals orders of magnitude bigger than those for the full sample. This indicates that the effective federal funds rate does not carry much information at the ZLB. The reason is simple: it is bounded by the lower bound, and does not display any meaningful variation. We can also see this from Figure 4.

By contrast, Figure 9 plots the ZLB impulse-response functions in turquoise with our policy rate introduced in Section 3. Again, we compare them with full sample impulse responses in blue. Overall, the subsample impulse responses are qualitatively the same as those for the full sample. Specifically, an expansionary monetary policy shock boosts real economic activity. The impulse responses for the subsample and full sample also look quantitatively similar. The point estimates and confidence intervals have the same orders of magnitude. Therefore, at the ZLB, our new policy rate conveys important and economically meaningful information; while the conventional policy rate gets stuck around zero.

5.2 Forward guidance

Since December 2008, the federal funds rate has been restricted by the ZLB. The conventional monetary policy is no longer effective, because the Federal Reserve cannot further decrease the federal funds rate below zero to boost the economy. Consequently, the central bank has resorted to a sequence of unconventional monetary policy tools. One prominent example is forward guidance, or central bank communications with the public about the future federal funds rate. In particular, forward guidance aims to lower the market's expectation regarding the future short rate. Market expectations about future short rates feed back through the financial market to affect the current yield curve, especially at the longer end. Lower long term interest rates in turn stimulate aggregate demand. The Federal Reserve has made considerable use of forward guidance since the federal funds rate first hit the ZLB. In Table 5, we summarize a list of forward guidance quotes, when the Fed expected a different lift-off date or condition for the ZLB. Some of these dates overlap with Woodford(2012).

The wording focuses either on (i) the length of the ZLB, or (ii) the target unemployment rate. Section 5.2.1 compares the length of the ZLB prescribed by forward guidance and the market's expectation from our model. Section 5.2.2 studies the impact of forward guidance on the unemployment rate.

5.2.1 ZLB duration

One focus of forward guidance is for the Federal Reserve to implicitly or explicitly communicate with the general public about how long it intends to keep the federal funds rate near zero, as demonstrated in Table 5. For example, in the earlier FOMC statements in late 2008 and early 2009, they used phrases such as "some time" and "an extended period". Later on, starting from late 2011, the Federal Reserve decided to be more transparent and specific about forward guidance. In each statement, they unambiguously revised the date, on which they expected the ZLB to end, according to the development of the overall economy.

Our model implies a closely related concept: the ZLB duration. It measures the market's perception of when the economy will finally escape from the ZLB. This is a random variable defined as

$$\tau_t \equiv \inf\{\tau_t \geq 0 | s_{t+\tau} \geq \underline{r}\}.$$

Thus τ_t represents how much time passes before the shadow rate first crosses the lower bound from below. At time t, $s_{t+\tau}$ is unknown. We simulate out $N = 10000$ paths of the future shadow rate given the information at time t.[10] Every simulated path generates an estimate of τ_t. Therefore, we have a distribution of τ_t, and we take the median across N simulations as our measure of the market's expected ZLB duration.

We summarize the time series of the market's expected ZLB duration in Figure 10 as the difference between the blue dots and dashed 45 degree line. The duration increased since early 2009 and kept above the two-and-a-half-year level from late-2011 to mid-2013, when it

[10] Similar to (Bauer and Rudebusch(2013)), we use the $\mathbb{Q}$ parameters for simulation, because (i) $\mathbb{Q}$ is the probability measure reflected in assets price, and (ii) $\mathbb{Q}$ parameters are estimated with much more precision than $\mathbb{P}$ parameters (see discussion in Creal and Wu(2014) for example).

plummeted to around one year and a half. Since then, it has been between one and a half to two years. We highlight four different months: August 2011, January 2012, September 2012 and June 2013. They correspond to those dates when the Fed explicitly spelled out the ZLB lift-off dates (see Table 5). On August 9, 2011, the Federal Reserve promised to keep the rate low "at least through mid-2013". The market anticipated this development one month ahead. When the lift-off date was postponed to "at least through late 2014" on January 25, 2012, the market expected the ZLB to last another three years. The two expectations overlap each other. On September 13, 2012, the forward guidance further extended the lift-off date to "at least through mid-2015", the market's expected duration increased to three and a half years. On June 19, 2013, Federal Reserve Board Chairman Ben Bernanke expressed in a press conference the Federal Reserve's plan to maintain accommodative monetary policy until 2015 based on the economic outlook at that time. Following his remarks, the market's expected lift-off date jump right on top of Bernanke's expectation.[11]

Overall, evidence suggests that forward guidance and the market's expectation align well. The market seems to adjust towards the Fed's announcements ahead of time. For multiple occasions, the two expectations overlapped each other. In the next section, we will use the expected ZLB duration as a proxy for forward guidance, and study its impact on the real economy, especially the unemployment rate.

5.2.2 Impact on unemployment

We have demonstrated that forward guidance is consistent with the market's expectation. The ultimate question central bankers and economists care about is whether forward guidance is as successful in terms of its impact on the real economy, especially unemployment. We phrase this question in a FAVAR(1) framework with the expected ZLB duration measuring the monetary policy, and use this tool to study the transmission mechanism of forward guidance. For the macroeconomic factors, we keep them as they were. Figure 11 shows the

[11]The results look very similar if we use real time duration instead, i.e., compute the ZLB duration at time t using only data up to t.

impulse responses to a shock to the expected ZLB duration of one year for the same set of variables. Overall, in response to an easing of monetary policy, the economy starts to expand. Most interestingly, a one year increase in the expected ZLB duration translates into a 0.25% decrease in the unemployment rate, although the impulse response is not statistically significant at 10% level.

A simple calculation suggests that a one year increase in the expected ZLB duration has roughly the same effect on the macroeconomy as a 35 basis-point decrease in the policy rate. The visual comparison is in Figure 12, where the blue part is identical to Figure 11, and the turquoise portion is 35/25 times the turquoise in figure 9. Figure 12 suggests that in response to a one year shock to the expected ZLB duration, or a negative 35 basis-point shock to the policy rate, capacity utilization goes up by 0.6%, unemployment rate decreases by 0.25% and housing starts is about 5% over its steady state.

6 Conclusion

We have developed an analytical approximation for the forward rate in the SRTSM, making the otherwise complicated model extremely tractable. The SRTSM is an excellent description of the data especially when the economy is at the ZLB, with the approximation error being only a couple of basis points. We used the shadow rate from the SRTSM to construct a new measure for the monetary policy stance when the effective federal funds rate is bounded below by zero, and employed this measure to study unconventional monetary policy's impact on the real economy. We have found that our policy rate impacts the real economy since July 2009 in a similar fashion as the effective federal funds rate did before the Great Recession. An expansionary monetary policy shock boosts the real economy. More specifically, at the ZLB, in response to a negative 35 basis-point shock to the policy rate, the unemployment rate decreases by 0.25%. This quantity is equivalent to a one year extension of the expected ZLB period, prescribed by forward guidance. Our historical decomposition has found that the efforts by the Federal Reserve to stimulate the economy since July 2009 succeeded in making the unemployment rate in December 2013 0.13% lower than it otherwise would have been.

The continuation in our policy rate series provides empirical researchers – who used the effective federal funds rate in a VAR to study monetary policy in the macroeconomy – a tool to update their historical analysis. It also has potential applications in other areas in macroeconomics, such as dynamic stochastic general equilibrium models.

References

Bauer, Michael D., and Glenn D. Rudebusch (2013) "Monetary Policy Expectations at the Zero Lower Bound" Federal Reserve Bank of San Francisco Working Paper.

Bauer, Michael D., and Glenn D. Rudebusch (forthcoming) "The Signaling Channel for Federal Reserve Bond Purchases" *International Journal of Central Banking*.

Bauer, Michael D., Glenn D. Rudebusch, and Jing Cynthia Wu (2012) "Correcting Estimation Bias in Dynamic Term Structure Models" *Journal of Business & Economic Statistics* 30, 454–467.

Baumeister, Christiane, and Luca Benati (2013) "Unconventional Monetary Policy and the Great Recession: Estimating the Macroeconomic Effects of a Spread Compression at the Zero Lower Bound" *International Journal of Central Banking* 9, 165–212.

Bernanke, Ben S., Jean Boivin, and Piotr Eliasz (2005) "Measuring the Effects of Monetary Policy: a Factor-Augmented Vector Autoregressive (FAVAR) Approach" *Quarterly Journal of Economics* 120, 387–422.

Black, Fischer (1995) "Interest Rates as Options" *Journal of Finance* 50, 1371–1376.

Bullard, James. (2012) "Shadow Interest Rates and the Stance of US Monetary Policy" Presentation at the Center for Finance and Accounting Research Annual Corporate Finance Conference, Washington University in St. Louis.

Christensen, J. H. E., and Glenn D. Rudebusch (2014) "Estimating shadow-rate term structure models with near-zero yields" *Journal of Financial Econometrics* 0, 1–34.

Christiano, Lawrence J., Martin Eichenbaum, and Charles L. Evans (1999) "Monetary policy shocks: What have we learned and to what end?" in *Handbook of macroeconomics* Elsevier.

Chung, Hess, Jean-philippe Laforte, David Reifschneider, and John C. Williams (2012) "Have We Underestimated the Likelihood and Severity of Zero Lower Bound

Events?" *Journal of Money, Credit and Banking* 44, 47–82.

Creal, Drew D., and Jing Cynthia Wu (2014) "Estimation of Affine Term Structure Models with Spanned or Unspanned Stochastic Volatility" NBER Working Paper No. 20115.

D'Amico, Stefania, and Thomas King (2013) "Flow and Stock Effects of Large-Scale Treasury Purchases: Evidence on the Importance of Local Supply" *Journal of Financial Economics* 108, 425–448.

Diebold, Francis X., and Glenn D. Rudebusch (2013) *Yield Curve Modeling and Forecasting.* Princeton University Press, Princeton, NJ.

Duffee, Gregory R. (2002) "Term Premia and Interest Rate Forecasts in Affine Models" *Journal of Finance* 57, 405–443.

Duffee, Gregory R. (forthcoming) "Bond Pricing and the Macroeconomy" in *Handbook of the Economics of Finance.*

Eichenbaum, Martin (1992) "Comment on Interpreting the Macroeconomic Time Series Facts: the Effects of Monetary Policy" *European Economic Review* 36, 1001–1011.

Fama, Eugene F., and Robert R. Bliss (1987) "The Information in Long-Maturity Forward Rates" *American Economic Review*, 680–692.

Gagnon, Joseph, Mattew Raskin, Julie Remache, and Brian Sack (2011) "The Financial Market Effects of the Federal Reserve's Large-Scale Asseet Purchase" *International Journal of Central Banking* 7, 3–43.

Gürkaynak, Refet S., Brian Sack, and Jonathan H. Wright (2007) "The U.S. Treasury Yield Curve: 1961 to the Present" *Journal of Monetary Economics* 54, 2291–2304.

Gürkaynak, Refet S., and Jonathan H. Wright (2012) "Macroeconomics and the Term Structure" *Journal of Economic Literature* 50, 331–367.

Hamilton, James D. (1994) *Time Series Analysis* Princeton University Press, Princeton, New Jersey.

Hamilton, James D., and Jing Cynthia Wu (2012) “The Effectiveness of Alternative Monetary Policy Tools in a Zero Lower Bound Environment” *Journal of Money, Credit, and Banking* 44 (s1), 3–46.

Hamilton, James D., and Jing Cynthia Wu (2014) “Testable Implications of Affine Term Structure Models” *Journal of Econometrics* 178, 231–242.

Ichiue, Hibiki, and Yoichi Ueno (2013) “Estimating Term Premia at the Zero Bound : an Analysis of Japanese, US, and UK Yields” Bank of Japan Working Paper.

Joslin, Scott, Kenneth J. Singleton, and Haoxiang Zhu (2011) “A New Perspective On Gaussian Dynamic Term Structure Models” *Review of Financial Studies* 27, 926–970.

Kim, Don H., and Kenneth J. Singleton (2012) “Term Structure Models and the Zero Bound: an Empirical Investigation of Japanese Yields” *Journal of Econometrics* 170, 32–49.

Krippner, Leo (2012) “Modifying Gaussian Term Structure Models When Interest Rates are Near the Zero Lower Bound” Reserve Bank of New Zealand Discussion Paper 2012/02.

Krippner, Leo (2013) “A Tractable Framework for Zero Lower Bound Gaussian Term Structure Models” Australian National University CAMA Working Paper 49/2013.

Krishnamurthy, Arvind, and Annette Vissing-Jorgensen (2011) “The Effects of Quantitative Easing on Interest Rates” *Brooking Papers on Economic Activity* 43, 215–287.

Ng, Serena, and Jonathan H. Wright (2013) “Facts and Challenges from the Great Recession for Forecasting and Macroeconomic Modeling” NBER Working Paper 19469.

Piazzesi, Monika (2010) “Affine Term Structure Models” in *Handbook of Financial Econometrics*, edited by Y. Ait-Sahalia and L. P. Hansen Elsevier, New York pages 691–766.

Sims, Christopher A (1992) “Interpreting the Macroeconomic Time Series Facts: the Effects of Monetary Policy” *European Economic Review* 36, 975–1000.

Stock, James H., and Mark W. Watson (2001) “Vector Autoregression” *Journal of Economic Perspectives* 15, 101–115.

Swanson, Eric T., and John C. Williams (forthcoming) “Measuring the Effect of the Zero Lower Bound on Medium- and Longer-Term Interest Rates” *American Economic Review.*

Woodford, Michael (2012) “Methods of Policy Accommodation at the Interest-rate Lower Bound” Jackson Hole symposium, August, Federal Reserve Bank of Kansas City.

Wright, J. H. (2011) “Term Premia and Inflation Uncertainty: Empirical Evidence from an International Panel Dataset” *American Economic Review* 101, 1514–1534.

Wright, Jonathan H. (2012) “What does Monetary Policy do to Long-term Interest Rates at the Zero Lower Bound?” *Economic Journal* 122, F447–F466.

Appendix A Approximation to Forward rates

Define

$$\begin{aligned}
\bar{a}_n &\equiv \delta_0 + \delta_1' \left(\sum_{j=0}^{n-1} \left(\rho^{\mathbb{Q}}\right)^j \right) \mu^{\mathbb{Q}}, \\
a_n &\equiv \bar{a}_n - \frac{1}{2}\delta_1' \left(\sum_{j=0}^{n-1} \left(\rho^{\mathbb{Q}}\right)^j \right) \Sigma\Sigma' \left(\sum_{j=0}^{n-1} \left(\rho^{\mathbb{Q}}\right)^j \right)' \delta_1, \\
b_n' &\equiv \delta_1' \left(\rho^{\mathbb{Q}}\right)^n.
\end{aligned}$$

Shadow rate The shadow rate is affine in the state variables. Under the risk neutral measure, it is conditionally normally distributed. The conditional mean is

$$\mathbb{E}_t^{\mathbb{Q}}\left[s_{t+n}\right] = \bar{a}_n + b_n' X_t,$$

the conditional variance is

$$\mathbb{V}\mathrm{ar}_t^{\mathbb{Q}}\left[s_{t+n}\right] \equiv \left(\sigma_n^{\mathbb{Q}}\right)^2 = \sum_{j=0}^{n-1} \delta_1' \left(\rho^{\mathbb{Q}}\right)^j \Sigma\Sigma' \left(\rho^{\mathbb{Q}\prime}\right)^j \delta_1,$$

and

$$\frac{1}{2}\left(\mathbb{V}\mathrm{ar}_t^{\mathbb{Q}}\left[\sum_{j=1}^{n} s_{t+j}\right] - \mathbb{V}\mathrm{ar}_t^{\mathbb{Q}}\left[\sum_{j=1}^{n-1} s_{t+j}\right] \right) = \bar{a}_n - a_n.$$

SRTSM We start the derivation of equation (6) with the following approximation: $\log\left(\mathbb{E}\left[e^Z\right]\right) \approx \mathbb{E}\left[Z\right] + \frac{1}{2}\mathbb{V}\mathrm{ar}\left[Z\right]$ for any random variable Z. This approximation uses Taylor series expansions for the exponential and natural logarithm functions. For the special case of a Gaussian random variable Z, this approximation is exact. Then the forward rate between $t+n$ and $t+n+1$ can be approximated as follows:

$$\begin{aligned}
f_{n,n+1,t}^{SRTSM} &= (n+1)y_{n+1,t} - n y_{nt} \\
&= -\log\left(e^{-r_t}\mathbb{E}_t^{\mathbb{Q}}\left[e^{-\sum_{j=1}^{n} r_{t+j}}\right]\right) + \log\left(e^{-r_t}\mathbb{E}_t^{\mathbb{Q}}\left[e^{-\sum_{j=1}^{n-1} r_{t+j}}\right]\right) \\
&\approx \mathbb{E}_t^{\mathbb{Q}}\left[\sum_{j=1}^{n} r_{t+j}\right] - \frac{1}{2}\mathbb{V}\mathrm{ar}_t^{\mathbb{Q}}\left[\sum_{j=1}^{n} r_{t+j}\right] - \mathbb{E}_t^{\mathbb{Q}}\left[\sum_{j=1}^{n-1} r_{t+j}\right] + \frac{1}{2}\mathbb{V}\mathrm{ar}_t^{\mathbb{Q}}\left[\sum_{j=1}^{n-1} r_{t+j}\right] \\
&= \mathbb{E}_t^{\mathbb{Q}}\left[r_{t+n}\right] - \frac{1}{2}\left(\mathbb{V}\mathrm{ar}_t^{\mathbb{Q}}\left[\sum_{j=1}^{n} r_{t+j}\right] - \mathbb{V}\mathrm{ar}_t^{\mathbb{Q}}\left[\sum_{j=1}^{n-1} r_{t+j}\right] \right). \qquad \text{(A.1)}
\end{aligned}$$

We calculate the first term $\mathbb{E}_t^{\mathbb{Q}}[r_{t+n}]$ analytically:

$$\begin{aligned}
\mathbb{E}_t^{\mathbb{Q}}[r_{t+n}] &= \mathbb{E}_t^{\mathbb{Q}}[\max(\underline{r}, s_{t+n})] \\
&= \Pr_t^{\mathbb{Q}}[s_{t+n} < \underline{r}] \times \underline{r} + \Pr_t^{\mathbb{Q}}[s_{t+n} \geq \underline{r}] \times \mathbb{E}_t^{\mathbb{Q}}[s_{t+n}|s_{t+n} \geq \underline{r}] \\
&= \underline{r} + \sigma_n^{\mathbb{Q}}\left(\left(\frac{\bar{a}_n + b_n' X_t - \underline{r}}{\sigma_n^{\mathbb{Q}}}\right)\Phi\left(\frac{\bar{a}_n + b_n' X_t - \underline{r}}{\sigma_n^{\mathbb{Q}}}\right) + \phi\left(\frac{\bar{a}_n + b_n' X_t - \underline{r}}{\sigma_n^{\mathbb{Q}}}\right)\right) \\
&= \underline{r} + \sigma_n^{\mathbb{Q}} g\left(\frac{\bar{a}_n + b_n' X_t - \underline{r}}{\sigma_n^{\mathbb{Q}}}\right). \qquad \text{(A.2)}
\end{aligned}$$

Using the second moments for the truncated normal distribution, we have the following approximations for the conditional variance and covariance (see details in Appendix A.1):

$$\begin{aligned}
\mathbb{V}\mathrm{ar}_t^{\mathbb{Q}}[r_{t+n}] &\approx \Pr_t^{\mathbb{Q}}[s_{t+n} \geq \underline{r}]\, \mathbb{V}\mathrm{ar}_t^{\mathbb{Q}}[s_{t+n}], \qquad \text{(A.3)} \\
\mathbb{C}\mathrm{ov}_t^{\mathbb{Q}}[r_{t+n-j}, r_{t+n}] &\approx \Pr_t^{\mathbb{Q}}[s_{t+n-j} \geq \underline{r}, s_{t+n} \geq \underline{r}]\, \mathbb{C}\mathrm{ov}_t^{\mathbb{Q}}[s_{t+n-j}, s_{t+n}], \ \forall j = 1, ..., n-1. \qquad \text{(A.4)}
\end{aligned}$$

Next, we take the approximation

$$\Pr_t^{\mathbb{Q}}[s_{t+n-j} \geq \underline{r} | s_{t+n} \geq \underline{r}] \approx 1,$$

using the fact that the shadow rate is very persistent. Equation (A.4) becomes

$$\mathbb{C}\mathrm{ov}_t^{\mathbb{Q}}[r_{t+n-j}, r_{t+n}] \approx \Pr_t^{\mathbb{Q}}[s_{t+n} \geq \underline{r}]\, \mathbb{C}\mathrm{ov}_t^{\mathbb{Q}}[s_{t+n-j}, s_{t+n}].$$

Then, the second term in equation (A.1) is

$$\begin{aligned}
&\frac{1}{2}\left(\mathbb{V}\mathrm{ar}_t^{\mathbb{Q}}\left[\sum_{j=1}^{n} r_{t+j}\right] - \mathbb{V}\mathrm{ar}_t^{\mathbb{Q}}\left[\sum_{j=1}^{n-1} r_{t+j}\right]\right) \\
\approx\ & \Pr_t^{\mathbb{Q}}(s_{t+n} \geq \underline{r}) \times \frac{1}{2}\left(\mathbb{V}\mathrm{ar}_t^{\mathbb{Q}}\left[\sum_{j=1}^{n} s_{t+j}\right] - \mathbb{V}\mathrm{ar}_t^{\mathbb{Q}}\left[\sum_{j=1}^{n-1} s_{t+j}\right]\right) \\
=\ & \Phi\left(\frac{\bar{a}_n + b_n' X_t - \underline{r}}{\sigma_n^{\mathbb{Q}}}\right) \times (\bar{a}_n - a_n). \qquad \text{(A.5)}
\end{aligned}$$

Plug equations (A.2) and (A.5) to (A.1), we conclude our derivation for equation (6) with another first-order Taylor approximation:

$$\begin{aligned}
f_{n,n+1,t}^{SRTSM} &\approx \underline{r} + \sigma_n^{\mathbb{Q}} g\left(\frac{\bar{a}_n + b_n' X_t - \underline{r}}{\sigma_n^{\mathbb{Q}}}\right) + \Phi\left(\frac{\bar{a}_n + b_n' X_t - \underline{r}}{\sigma_n^{\mathbb{Q}}}\right) \times (a_n - \bar{a}_n) \\
&= \underline{r} + \sigma_n^{\mathbb{Q}} g\left(\frac{\bar{a}_n + b_n' X_t - \underline{r}}{\sigma_n^{\mathbb{Q}}}\right) + \sigma_n^{\mathbb{Q}} \frac{\partial g\left(\frac{\bar{a}_n + b_n' X_t - \underline{r}}{\sigma_n^{\mathbb{Q}}}\right)}{\partial \bar{a}_n} \times (a_n - \bar{a}_n) \\
&\approx \underline{r} + \sigma_n^{\mathbb{Q}} g\left(\frac{a_n + b_n' X_t - \underline{r}}{\sigma_n^{\mathbb{Q}}}\right). \qquad \text{(A.6)}
\end{aligned}$$

GATSM In the GATSM, the forward rate between $t+n$ and $t+n+1$ is priced as follows

$$\begin{aligned}
f_{n,n+1,t}^{GATSM} &= (n+1)y_{n+1,t} - ny_{nt} \\
&= -\log\left(e^{-s_t}\mathbb{E}_t^{\mathbb{Q}}\left[e^{-\sum_{j=1}^{n} s_{t+j}}\right]\right) + \log\left(e^{-s_t}\mathbb{E}_t^{\mathbb{Q}}\left[e^{-\sum_{j=1}^{n-1} s_{t+j}}\right]\right) \\
&= \mathbb{E}_t^{\mathbb{Q}}\left[\sum_{j=1}^{n} s_{t+j}\right] - \frac{1}{2}\mathbb{V}\mathrm{ar}_t^{\mathbb{Q}}\left[\sum_{j=1}^{n} s_{t+j}\right] - \mathbb{E}_t^{\mathbb{Q}}\left[\sum_{j=1}^{n-1} s_{t+j}\right] + \frac{1}{2}\mathbb{V}\mathrm{ar}_t^{\mathbb{Q}}\left[\sum_{j=1}^{n-1} s_{t+j}\right] \\
&= \mathbb{E}_t^{\mathbb{Q}}\left[s_{t+n}\right] - \frac{1}{2}\left(\mathbb{V}\mathrm{ar}_t^{\mathbb{Q}}\left[\sum_{j=1}^{n} s_{t+j}\right] - \mathbb{V}\mathrm{ar}_t^{\mathbb{Q}}\left[\sum_{j=1}^{n-1} s_{t+j}\right]\right) \\
&= \bar{a}_n + b_n' X_t + a_n - \bar{a}_n \\
&= a_n + b_n' X_t.
\end{aligned}$$

Appendix A.1 Approximations to variance and covariance

Define

$$\tilde{s}_{t+n} \equiv \frac{s_{t+n} - \mathbb{E}_t^{\mathbb{Q}}\left[s_{t+n}\right]}{\sigma_n^{\mathbb{Q}}} \quad \text{and} \quad \alpha_{nt} \equiv \frac{\underline{r} - \mathbb{E}_t^{\mathbb{Q}}\left[s_{t+n}\right]}{\sigma_n^{\mathbb{Q}}},$$

then $r_{t+n} = \sigma_n^{\mathbb{Q}}\tilde{r}_{t+n} + \mathbb{E}_t^{\mathbb{Q}}\left[s_{t+n}\right]$, where $\tilde{r}_{t+n} \equiv \max\left(\tilde{s}_{t+n}, \alpha_{nt}\right)$.

Variance Standard results for the truncated normal distribution states that if $x \sim N(0,1)$, then (i) $\Pr\left[x \geq \alpha\right] = 1 - \Phi\left(\alpha\right)$, (ii) $\Pr\left[x \geq \alpha\right]\mathbb{E}\left[x|x \geq \alpha\right] = \phi\left(\alpha\right)$, and (iii)$\Pr\left[x \geq \alpha\right]\mathbb{E}\left[x^2|x \geq \alpha\right] = 1 - \Phi\left(\alpha\right) + \alpha\phi\left(\alpha\right)$. Since $\tilde{s}_{t+n}$ is conditionally normally distributed with mean 0 and variance 1 under the $\mathbb{Q}$ measure,

$$\begin{aligned}
\mathbb{E}_t^{\mathbb{Q}}\left[\tilde{r}_{t+n}\right] &= \Pr_t^{\mathbb{Q}}\left[\tilde{s}_{t+n} \geq \alpha_{nt}\right]\mathbb{E}_t\left[\tilde{s}_{t+n}|\tilde{s}_{t+n} \geq \alpha_{nt}\right] + \Pr_t^{\mathbb{Q}}\left[\tilde{s}_{t+n} < \alpha_{nt}\right]\alpha_{nt} \\
&= \phi\left(\alpha_{nt}\right) + \alpha_{nt}\Phi\left(\alpha_{nt}\right), \qquad \text{(A.7)} \\
\mathbb{E}_t^{\mathbb{Q}}\left[\tilde{r}_{t+n}^2\right] &= \Pr_t^{\mathbb{Q}}\left[\tilde{s}_{t+n} \geq \alpha_{nt}\right]\mathbb{E}_t\left[\tilde{s}_{t+n}^2|\tilde{s}_{t+n} \geq \alpha_{nt}\right] + \Pr_t^{\mathbb{Q}}\left[\tilde{s}_{t+n} < \alpha_{nt}\right]\alpha_{nt}^2 \\
&= 1 - \Phi\left(\alpha_{nt}\right) + \alpha_{nt}\phi\left(\alpha_{nt}\right) + \alpha_{nt}^2\Phi\left(\alpha_{nt}\right).
\end{aligned}$$

Accordingly,

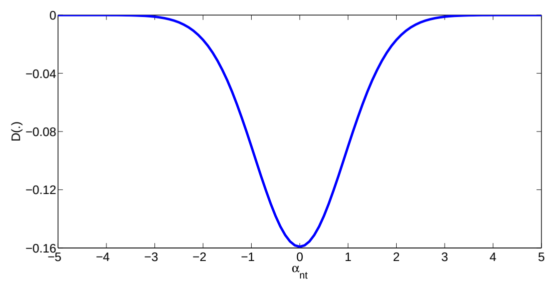
$$\begin{aligned}
\mathbb{V}\mathrm{ar}_t^{\mathbb{Q}}\left[r_{t+n}\right] &= \left(\sigma_n^{\mathbb{Q}}\right)^2\mathbb{V}\mathrm{ar}_t^{\mathbb{Q}}\left[\tilde{r}_{t+n}\right] = \left(\sigma_n^{\mathbb{Q}}\right)^2\left(\mathbb{E}_t\left[\tilde{r}_{t+n}^2\right] - \left(\mathbb{E}_t\left[\tilde{r}_{t+n}\right]\right)^2\right) \\
&= \left(\sigma_n^{\mathbb{Q}}\right)^2\left(1 - \Phi\left(\alpha_{nt}\right) + \alpha_{nt}\phi\left(\alpha_{nt}\right) + \alpha_{nt}^2\Phi\left(\alpha_{nt}\right) - \left(\phi\left(\alpha_{nt}\right) + \alpha_{nt}\Phi\left(\alpha_{nt}\right)\right)^2\right) \qquad \text{(A.8)}
\end{aligned}$$

Comparing the exact formula in equation (A.8) with the approximation in equation (A.3), or $\mathbb{V}\mathrm{ar}_t^{\mathbb{Q}}\left(r_{t+n}\right) \approx \Pr_t^{\mathbb{Q}}\left[s_{t+n} \geq \underline{r}\right]\mathbb{V}\mathrm{ar}_t^{\mathbb{Q}}\left[s_{t+n}\right] = \left(\sigma_n^{\mathbb{Q}}\right)^2\left(1 - \Phi\left(\alpha_{nt}\right)\right)$, the approximation error is

$$\begin{aligned}
&\left(\sigma_n^{\mathbb{Q}}\right)^2\left\{\left(1 - \Phi\left(\alpha_{nt}\right) + \alpha_{nt}\phi\left(\alpha_{nt}\right) + \alpha_{nt}^2\Phi\left(\alpha_{nt}\right) - \left(\phi\left(\alpha_{nt}\right) + \alpha_{nt}\Phi\left(\alpha_{nt}\right)\right)^2\right) - \left(1 - \Phi\left(\alpha_{nt}\right)\right)\right\} \\
&= -\left(\sigma_n^{\mathbb{Q}}\right)^2 g\left(\alpha_{nt}\right) g\left(-\alpha_{nt}\right) \equiv \left(\sigma_n^{\mathbb{Q}}\right)^2 D\left(\alpha_{nt}\right).
\end{aligned}$$

The first derivative of $D(\alpha_{nt})$ is $D'(\alpha_{nt}) = -g'(\alpha_{nt})\, g(-\alpha_{nt}) + g(\alpha_{nt})\, g'(-\alpha_{nt})$, and $D'(\alpha_{nt})\,|_{\alpha_{nt}=0} = 0$. Therefore $D(0)$ is a local maximum/minimum. From Figure A.1, $D(.)$ is bounded by 0 from above and achieves the global minimum at $\alpha_{nt} = 0$. Therefore, the absolute approximation error is bounded by a small number $\left(\sigma_n^{\mathbb{Q}}\right)^2 \phi(0)^2$.

Figure A.1: $D(\alpha_{nt})$



Covariance Standard results for the multivariate truncated normal distribution states that if $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}\right)$, then

$$\begin{aligned}
(i) \quad & \Pr[x_1 \geq \alpha_1, x_2 \geq \alpha_2] = F(-\alpha_1, -\alpha_2; \rho), \\
(ii) \quad & \Pr[x_1 \geq \alpha_1, x_2 \geq \alpha_2]\, \mathbb{E}[x_1 | x_1 \geq \alpha_1, x_2 \geq \alpha_2] = h(\alpha_1, \alpha_2, \rho) + \rho h(\alpha_2, \alpha_1, \rho), \\
(iii) \quad & \Pr[x_1 \geq \alpha_1, x_2 \geq \alpha_2]\, \mathbb{E}[x_1 x_2 | x_1 \geq \alpha_1, x_2 \geq \alpha_2] \\
& = \rho\left(\alpha_1 h(\alpha_1, \alpha_2; \rho) + \alpha_2 h(\alpha_2, \alpha_1; \rho) + F(-\alpha_1, -\alpha_2; \rho)\right) + \left(1 - \rho^2\right) f(\alpha_1, \alpha_2; \rho),
\end{aligned}$$

where

$$\begin{aligned}
f(x_1, x_2; \rho) & \equiv \lambda (2\pi)^{-1} exp\left\{-\frac{1}{2}\lambda^2\left(x_1^2 - 2\rho x_1 x_2 + x_2^2\right)\right\}, \\
F(\alpha_1, \alpha_2; \rho) & \equiv \int_{-\infty}^{\alpha_1} \int_{-\infty}^{\alpha_2} f(x_1, x_2; \rho) dx_1 dx_2, \\
h(\alpha_1, \alpha_2; \rho) & \equiv \phi(\alpha_1)\, \Phi(\lambda(\rho\alpha_1 - \alpha_2)), \\
\lambda & \equiv \left(1 - \rho^2\right)^{-\frac{1}{2}}.
\end{aligned}$$

Let ρ_{mnt} be the correlation between $\tilde{s}_{t+m}$ and $\tilde{s}_{t+n}$ under the $\mathbb{Q}$ measure, then,

$$\begin{aligned}
\mathbb{E}_t^{\mathbb{Q}}\left[\tilde{r}_{t+m}\tilde{r}_{t+n}\right] = \quad & \mathbb{E}_t^{\mathbb{Q}}\left[\tilde{s}_{t+m}\tilde{s}_{t+n}\right|\tilde{s}_{t+m}\geq\alpha_{mt},\tilde{s}_{t+n}\geq\alpha_{nt}]\mathrm{Pr}_t^{\mathbb{Q}}\left(\tilde{s}_{t+m}\geq\alpha_{mt},\tilde{s}_{t+n}\geq\alpha_{nt}\right)\\
& +\alpha_{mt}\mathbb{E}_t^{\mathbb{Q}}\left[\tilde{s}_{t+n}|\tilde{s}_{t+m}<\alpha_{mt},\tilde{s}_{t+n}\geq\alpha_{nt}\right]\mathrm{Pr}_t^{\mathbb{Q}}\left(\tilde{s}_{t+m}<\alpha_{mt},\tilde{s}_{t+n}\geq\alpha_{nt}\right)\\
& +\alpha_{nt}\mathbb{E}_t^{\mathbb{Q}}\left[\tilde{s}_{t+m}|\tilde{s}_{t+m}\geq\alpha_{mt},\tilde{s}_{t+n}<\alpha_{nt}\right]\mathrm{Pr}_t^{\mathbb{Q}}\left(\tilde{s}_{t+m}\geq\alpha_{mt},\tilde{s}_{t+n}<\alpha_{nt}\right)\\
& +\alpha_{mt}\alpha_{nt}\mathrm{Pr}_t^{\mathbb{Q}}\left(\tilde{s}_{t+m}<\alpha_{mt},\tilde{s}_{t+n}<\alpha_{nt}\right)\\
= \quad & \rho_{mnt}\left(\alpha_{mt}h\left(\alpha_{mt},\alpha_{nt};\rho_{mnt}\right)+\alpha_{nt}h\left(\alpha_{nt},\alpha_{mt};\rho_{mnt}\right)+F\left(-\alpha_{mt},-\alpha_{nt};\rho_{mnt}\right)\right)\\
& +\left(1-\rho_{mnt}^2\right)f(\alpha_{mt},\alpha_{nt};\rho_{mnt})\\
& +\alpha_{mt}\left(h\left(\alpha_{nt},-\alpha_{mt};-\rho_{mnt}\right)-\rho_{mnt}h\left(-\alpha_{mt},\alpha_{nt},-\rho_{mnt}\right)\right)\\
& +\alpha_{nt}\left(h\left(\alpha_{mt},-\alpha_{nt};-\rho_{mnt}\right)-\rho_{mnt}h\left(-\alpha_{nt},\alpha_{mt};-\rho_{mnt}\right)\right)\\
& +\alpha_{mt}\alpha_{nt}F\left(\alpha_{mt},\alpha_{nt};\rho_{mnt}\right).
\end{aligned}$$

With the identity $h\left(\alpha_1,\alpha_2;\rho\right)=h\left(-\alpha_1,\alpha_2;-\rho\right)$, we simplify the expression above as follows:

$$\begin{aligned}
\mathbb{E}_t^{\mathbb{Q}}\left[\tilde{r}_{t+m}\tilde{r}_{t+n}\right] \quad = \quad & \rho_{mnt}F\left(-\alpha_{mt},-\alpha_{nt};\rho_{mnt}\right)+\left(1-\rho_{mnt}^2\right)f(\alpha_{mt},\alpha_{nt};\rho_{mnt})\\
+ \quad & \alpha_{mt}h\left(\alpha_{nt},-\alpha_{mt};-\rho_{mnt}\right)+\alpha_{nt}h\left(\alpha_{mt},-\alpha_{nt};-\rho_{mnt}\right)+\alpha_{mt}\alpha_{nt}F\left(\alpha_{mt},\alpha_{nt};\rho_{mnt}\right).
\end{aligned}$$

From equation (A.7), we have

$$\mathbb{E}_t^{\mathbb{Q}}\left[\tilde{r}_{t+m}\right]\mathbb{E}_t^{\mathbb{Q}}\left[\tilde{r}_{t+n}\right] \quad = \quad \left(\phi\left(\alpha_{mt}\right)+\alpha_{mt}\Phi\left(\alpha_{mt}\right)\right)\left(\phi\left(\alpha_{nt}\right)+\alpha_{nt}\Phi\left(\alpha_{nt}\right)\right).$$

Accordingly,

$$\begin{aligned}
& \mathbb{C}\mathrm{ov}_t^{\mathbb{Q}}\left[r_{t+m},r_{t+n}\right]=\sigma_m^{\mathbb{Q}}\sigma_n^{\mathbb{Q}}\mathbb{C}\mathrm{ov}_t^{\mathbb{Q}}\left[\tilde{r}_{t+m},\tilde{r}_{t+n}\right]\\
= \quad & \sigma_m^{\mathbb{Q}}\sigma_n^{\mathbb{Q}}\left(\mathbb{E}_t^{\mathbb{Q}}\left[\tilde{r}_{t+m}\tilde{r}_{t+n}\right]-\mathbb{E}_t^{\mathbb{Q}}\left[\tilde{r}_{t+m}\right]\mathbb{E}_t^{\mathbb{Q}}\left[\tilde{r}_{t+n}\right]\right).
\end{aligned} \tag{A.9}$$

Comparing the exact formula in equation (A.9) with the approximation in equation (A.4), or $\mathbb{C}\mathrm{ov}_t^{\mathbb{Q}}\left[r_{t+m},r_{t+n}\right] \approx \mathrm{Pr}_t^{\mathbb{Q}}\left[s_{t+m}\geq\underline{r},s_{t+n}\geq\underline{r}\right]\mathbb{C}\mathrm{ov}_t^{\mathbb{Q}}\left[s_{t+m},s_{t+n}\right] = \rho_{mnt}\sigma_m^{\mathbb{Q}}\sigma_n^{\mathbb{Q}}F(-\alpha_m,-\alpha_n;\rho_{mnt})$, the approximation error is

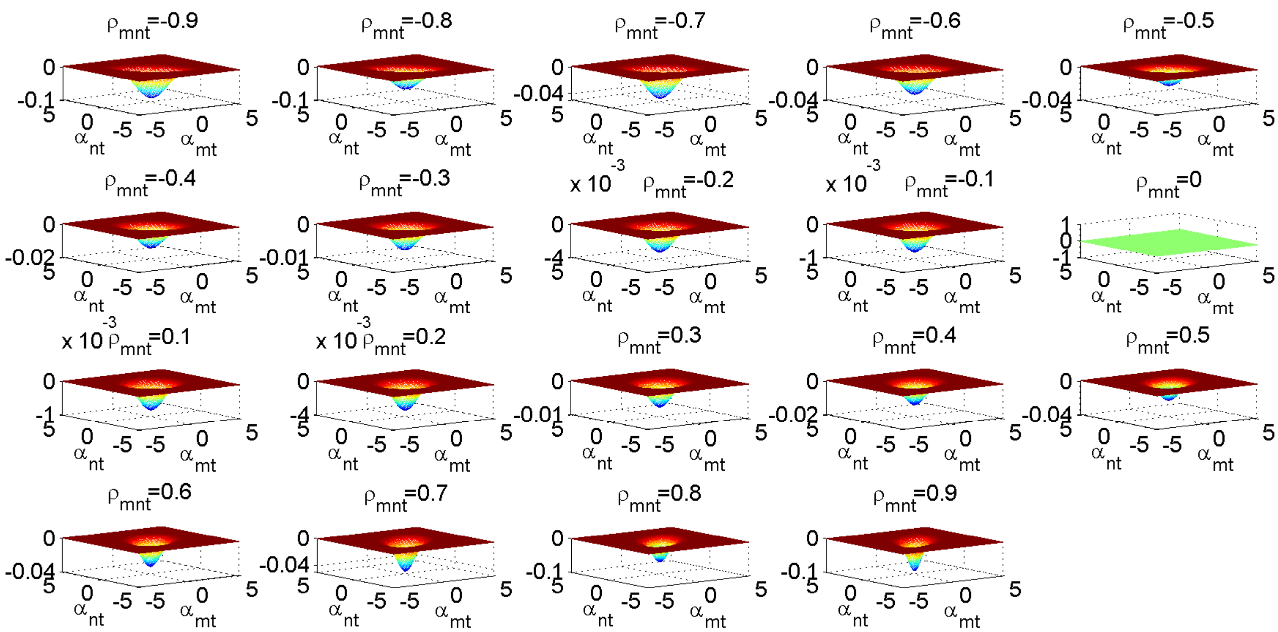
$$\begin{aligned}
& \sigma_m^{\mathbb{Q}}\sigma_n^{\mathbb{Q}}\times\Big\{\left(1-\rho_{mnt}^2\right)f(\alpha_{mt},\alpha_{nt};\rho_{mnt})+\alpha_{mt}h\left(\alpha_{nt},-\alpha_{mt};-\rho_{mnt}\right)+\alpha_{nt}h\left(\alpha_{mt},-\alpha_{nt};-\rho_{mnt}\right)\\
& +\alpha_{mt}\alpha_{nt}F\left(\alpha_{mt},\alpha_{nt};\rho_{mnt}\right)-\left(\phi\left(\alpha_{mt}\right)+\alpha_{mt}\Phi\left(\alpha_{mt}\right)\right)\left(\phi\left(\alpha_{nt}\right)+\alpha_{nt}\Phi\left(\alpha_{nt}\right)\right)\Big\}\\
\equiv \quad & \sigma_m^{\mathbb{Q}}\sigma_n^{\mathbb{Q}}D\left(\alpha_{mt},\alpha_{nt};\rho_{mnt}\right).
\end{aligned}$$

The first derivative of $D\left(\alpha_{mt},\alpha_{nt};\rho_{mnt}\right)$ with respect to α_{mt} is

$$\begin{aligned}
\frac{\partial D\left(\alpha_{mt},\alpha_{nt};\rho_{mnt}\right)}{\partial\alpha_{mt}} = \quad & -\left(\alpha_{mt}-\rho_{mnt}\alpha_{nt}\right)f\left(\alpha_{mt},\alpha_{nt};\rho_{mnt}\right)\\
& +h\left(\alpha_{nt},-\alpha_{mt};-\rho_{mnt}\right)+\lambda_{mnt}\alpha_{mt}\phi\left(\alpha_{nt}\right)\phi\left(\lambda_{mnt}\left(-\rho_{mnt}\alpha_{nt}+\alpha_{mt}\right)\right)\\
& -\alpha_{nt}\alpha_{mt}\Phi\left(\alpha_{mt}\right)\Phi\left(\lambda_{mnt}\left(-\rho_{mnt}\alpha_{mt}+\alpha_{nt}\right)\right)\\
& -\lambda_{mnt}\rho_{mnt}\alpha_{nt}\phi\left(\alpha_{mt}\right)\phi\left(\lambda_{mnt}\left(-\rho_{mnt}\alpha_{mt}+\alpha_{nt}\right)\right)\\
& +\alpha_{nt}F\left(\alpha_{mt},\alpha_{nt};\rho_{mnt}\right)+\alpha_{mt}\alpha_{nt}h\left(a_{mt},-\alpha_{nt};-\rho_{mnt}\right)\\
& -\Phi\left(\alpha_{mt}\right)\left(\phi\left(\alpha_{nt}\right)+\alpha_{nt}\Phi\left(\alpha_{nt}\right)\right),
\end{aligned}$$

where $\lambda_{mnt} = \left(1-\rho_{mnt}^2\right)^{-\frac{1}{2}}$. And $\frac{\partial D(\alpha_{mt},\alpha_{nt};\rho_{mnt})}{\partial \alpha_{mt}}|_{\alpha_{mt}=0,\alpha_{nt}=0} = \phi(0)\,\Phi(0) - \phi(0)\,\Phi(0) = 0$. Since $D(\alpha_{mt},\alpha_{nt};\rho_{mnt}) = D(\alpha_{nt},\alpha_{mt};\rho_{mnt})$, we have $\frac{\partial D(\alpha_{mt},\alpha_{nt};\rho_{mnt})}{\partial \alpha_{nt}}|_{\alpha_{mt}=0,\alpha_{nt}=0} = 0$ as well. Thus, $D(0,0;\rho_{mnt})$ is a local maximum/minimum. We plot $D(\alpha_{mt},\alpha_{nt};\rho_{mnt})$ for $\rho_{mnt} = -0.9, -0.8, ..., 0.8, 0.9$ in Figure A.2, and $D(\alpha_{mt},\alpha_{nt};\rho)$ is bounded by 0 from above and achieves the global minimum at $\alpha_{mt}=0, \alpha_{nt}=0$. Therefore, the absolute approximation error is bounded by a small number, $\sigma_m^{\mathbb{Q}}\sigma_n^{\mathbb{Q}}\left(1-\left(1-\rho_{mnt}^2\right)^{\frac{1}{2}}\right)\phi^2(0)$.

Figure A.2: $D(\alpha_{mt},\alpha_{nt};\rho_{mnt})$



Appendix B Kalman filters

Extended Kalman filter for the SRTSM The transition equation is in (3). Stack the observation equation in (8) for all 7 maturities, we get the following system:

$$F_{t+1}^o = G(X_{t+1}) + \eta_{t+1} \quad \eta_{t+1} \sim N(0, \omega I_7).$$

Approximate the conditional distribution of X_t with $X_t|F_{1:t}^o \sim N(\hat{X}_{t|t}, P_{t|t})$. Update $\hat{X}_{t+1|t+1}$ and $P_{t+1|t+1}$ as follows:

$$\begin{aligned}
\hat{X}_{t+1|t+1} &= \hat{X}_{t+1|t} + K_{t+1}(F_{t+1}^o - \hat{F}_{t+1|t}^o), \\
P_{t+1|t+1} &= \left(I_3 - K_{t+1}H_{t+1}'\right)P_{t+1|t}, \\
\hat{X}_{t+1|t} &= \mu + \rho\hat{X}_{t|t}, \\
P_{t+1|t} &= \rho P_{t|t}\rho' + \Sigma\Sigma',
\end{aligned}$$

with the matrices defined as

$$
\begin{aligned}
\hat{F}^o_{t+1|t} &= G(\hat{X}_{t+1|t}), \\
H_{t+1} &= \left(\left. \frac{\partial G(X_{t+1})}{\partial X'_{t+1}} \right|_{X_{t+1}=\hat{X}_{t+1|t}} \right)', \\
K_{t+1} &= P_{t+1|t} H_{t+1} \left(H'_{t+1} P_{t+1|t} H_{t+1} + \omega I_7 \right)^{-1},
\end{aligned}
$$

where we can obtain H'_{t+1} by stacking $\Phi\left(\frac{a_n + b'_n \hat{X}_{t+1|t} - \underline{r}}{\sigma_n^{\mathbb{Q}}}\right) \times b'_n$ for the 7 maturities. Given the initial values $\hat{X}_{0|0}$ and $P_{0|0}$, we can update $\{\hat{X}_{t|t}, P_{t|t}\}_{t=1}^T$ recursively with the above algorithm. The log likelihood is

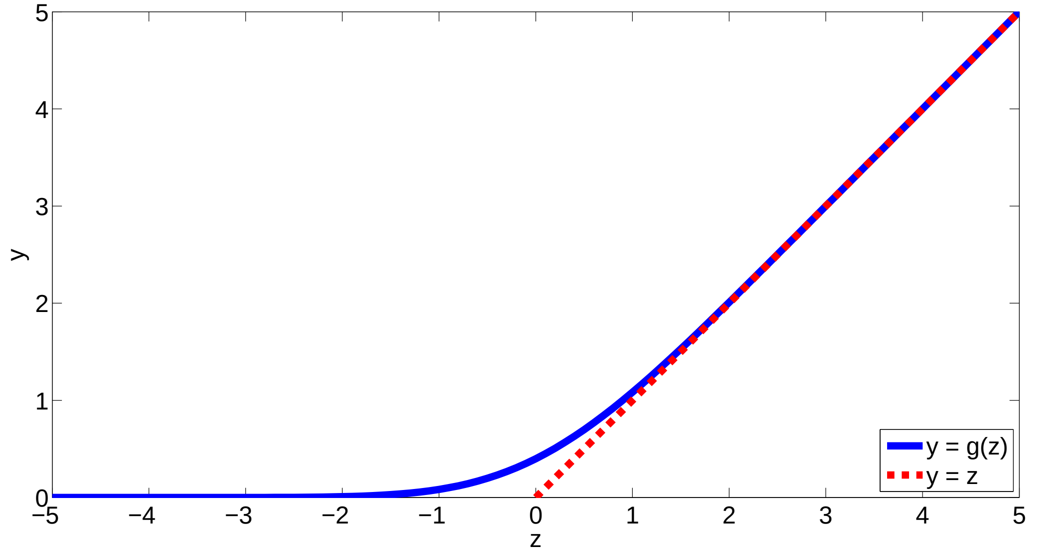
$$
\begin{aligned}
\mathcal{L} &= -\frac{7T}{2}\log 2\pi - \frac{1}{2}\sum_{t=1}^{T} \log|H'_t P_{t+1|t} H_t + \omega I_7| \\
&\quad -\frac{1}{2}\sum_{t=1}^{T} (F^o_t - G(\hat{X}_{t|t-1}))' \left(H'_t P_{t+1|t} H_t + \omega I_7 \right)^{-1} (F^o_t - G(\hat{X}_{t|t-1})).
\end{aligned}
$$

Kalman filter for the GATSM The GATSM is a linear Gaussian state space model. The $G(.)$ function stacks the linear function in equation (9). The matrix H'_{t+1} stacks b'_n for the 7 maturities. The algorithm described above collapses to a Kalman filter.

Appendix C Factor construction for the FAVAR

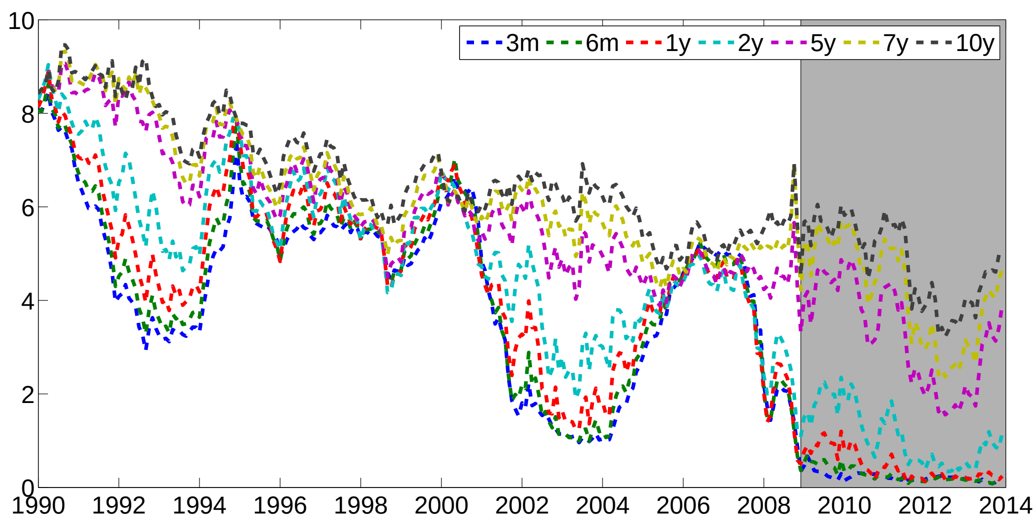
This appendix illustrates how to construct the macro factors. First, extract the first 3 principal components $\widehat{pc}_t$ from Y^m_t. Then extract first 3 principal components $\widehat{pc}^*_t$ from the slowing-moving variables indicated with "$*$" in Table 3. Normalize them to unit variance. Next, run the following regression $\widehat{pc}_t = b_{pc}\widehat{pc}^*_t + b_{pc,s} s^o_t + \eta^{pc}_t$, and construct $\hat{x}^m_t$ from $\widehat{pc}_t - \hat{b}_{pc,s} s^o_t$. We then estimate equation (11) as follows. If $Y^{m,i}_t$ is among the slow-moving variables, we regress $Y^{m,i}_t$ on a constant and $\hat{x}^m_t$ to obtain $\hat{a}_{m,i}$ and $\hat{b}_{x,i}$ and set $\hat{b}_{s,i} = 0$. For other variables, we regress $Y^{m,i}_t$ on a constant, $\hat{x}^m_t$ and s^o_t to get $\hat{a}_{m,i}$, $\hat{b}_{x,i}$ and $\hat{b}_{s,i}$.

Figure 1: The function $g(.)$



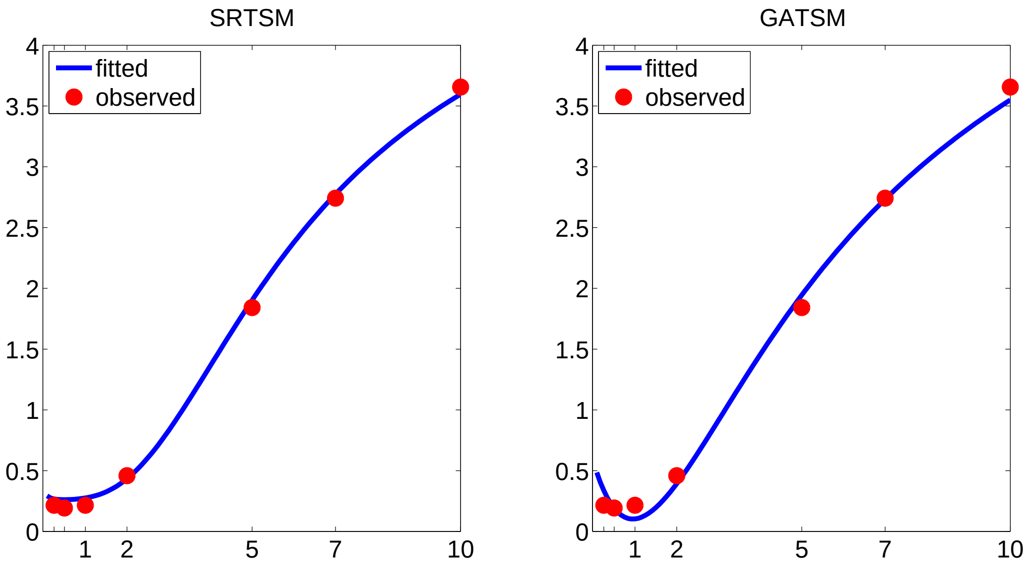
Blue curve: the function $g(z) = z\Phi(z) + \phi(z)$. Red dashed line: the 45-degree line.

Figure 2: Forward rates



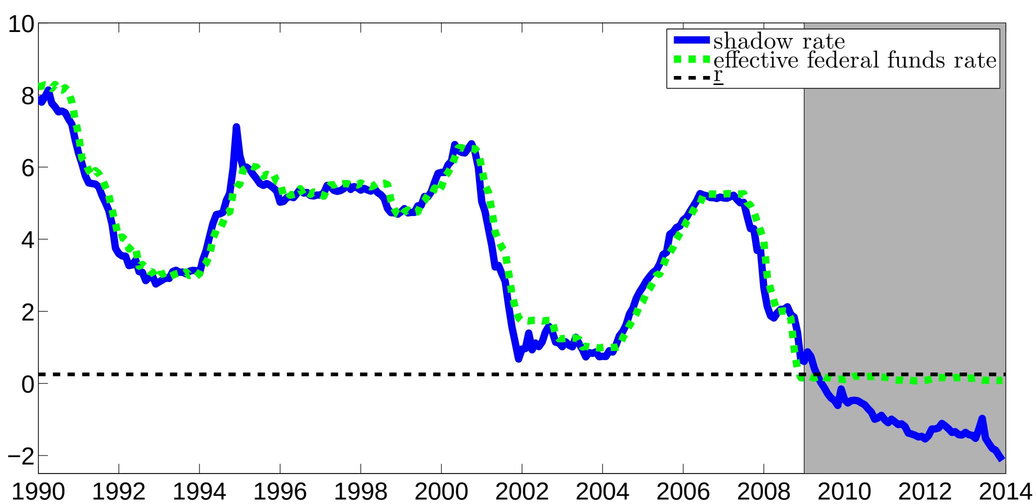
One-month forward rates monthly from January 1990 to December 2013, measured in annualized percentage points. Maturities are 3 and 6 months, 1, 2, 5, 7 and 10 years. The gray area marks the ZLB period from January 2009 to December 2013.

Figure 3: Observed and fitted forward curves



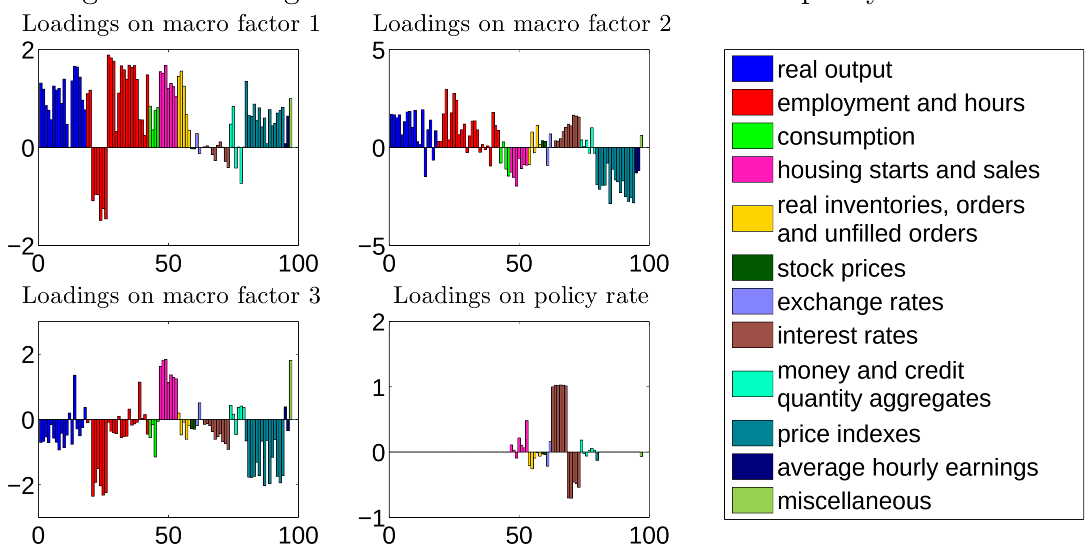
Average forward curves in 2012. Blue curves: fitted forward curves, from the SRTSM in the left panel and the GATSM in the right panel. Red dots: observed data. X-axis: maturity in years.

Figure 4: The shadow rate and effective federal funds rate



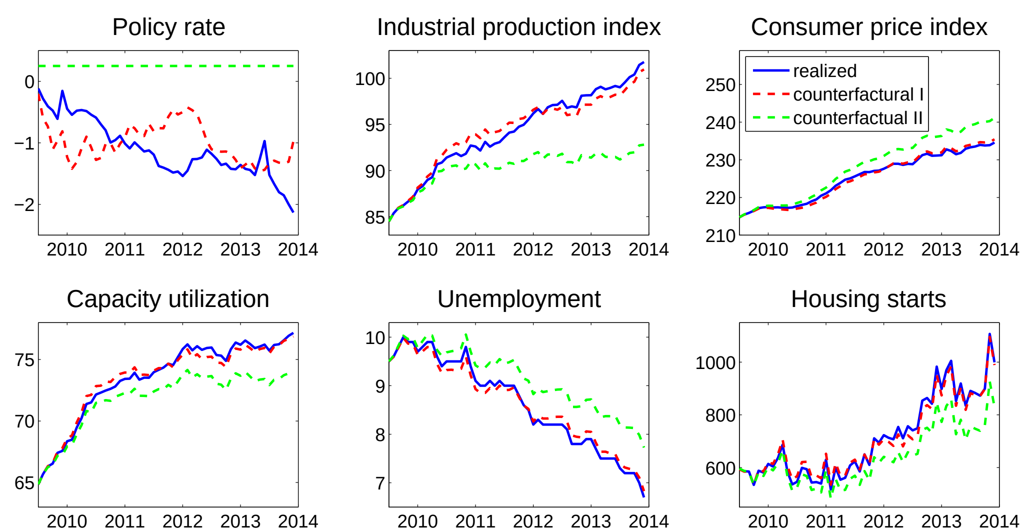
Blue line: the estimated shadow rate of the SRTSM from January 1990 to December 2013. Green line: the effective federal funds rate. Black line: lower bound $\underline{r}$. The gray area marks the ZLB period from January 2009 to December 2013.

Figure 5: Loadings on the macroeconomic factors and policy rate



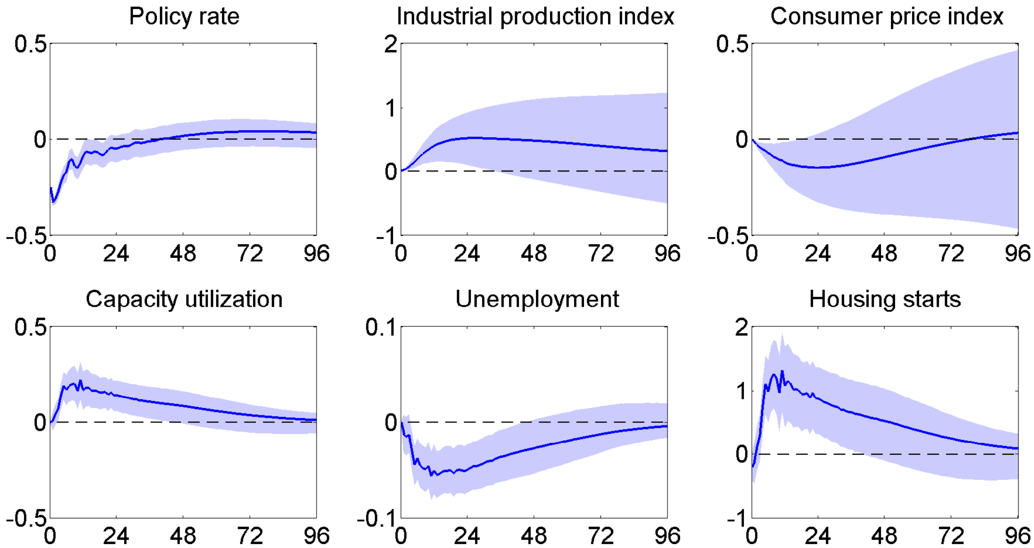
Loadings of standardized economic variables Y_t^m on the three macroeconomic factors and the standardized policy rate. X-axis: identification number for economic variables in Table 3.

Figure 6: Observed and counterfactual macroeconomic variables



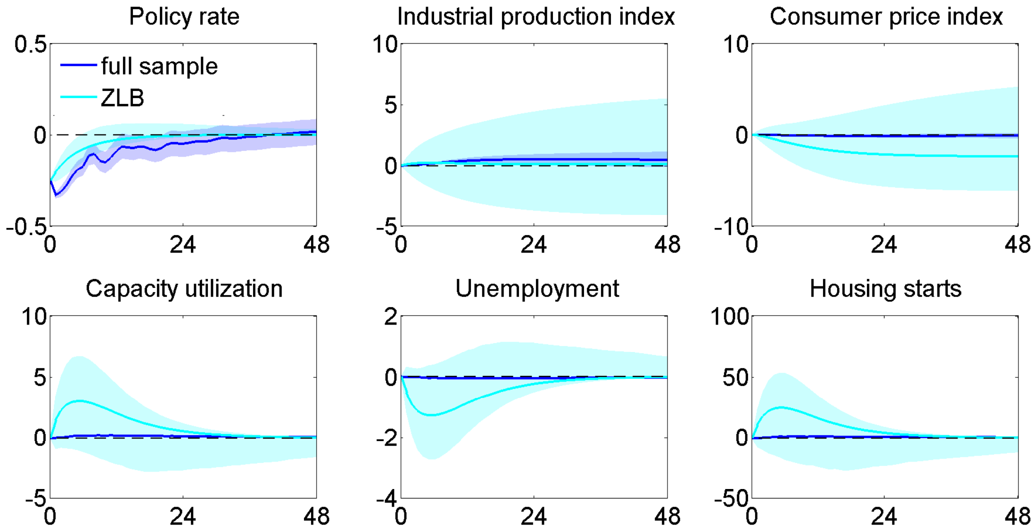
Blue lines: observed economic variables between July 2009 and December 2013. Red dashed lines: what would have happened to these macroeconomic variables, if all the monetary policy shocks were shut down. Green dashed lines: what would have happened if the shadow rate was kept at $\underline{r}$.

Figure 7: Impulse responses with full sample



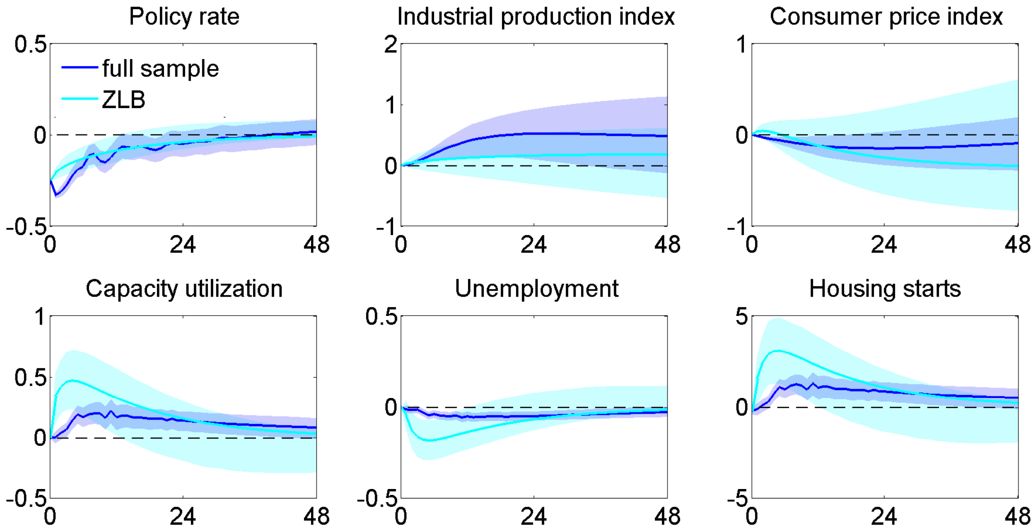
Impulse responses to a -25 basis-point shock on monetary policy. 90% confidence intervals are shaded. Sample: January 1960 - December 2013. Model: FAVAR with 3 macro factors and 13 lags. X-axis: response time in months. The policy rate is measured in annualized percentage; the industrial production index, consumer price index and housing starts are measured in percentage deviation from the steady state; the capacity utilization and unemployment rate are measured in percentage point.

Figure 8: Impulse responses (full sample vs. ZLB with EFFR)



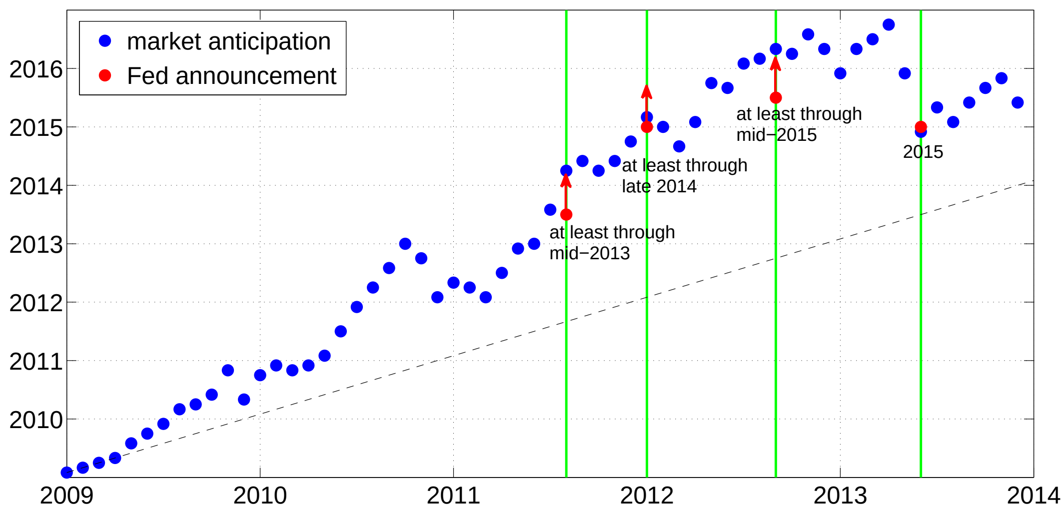
Impulse responses to a -25 basis-point shock on monetary policy. 90% confidence intervals are shaded. Blue: full sample from January 1960 to December 2013 with the policy rate in FAVAR (13). Turquoise: ZLB from July 2009 to December 2013 with the effective federal funds rate in FAVAR (1). X-axis: response time in months. The policy rate is measured in annualized percentage; the industrial production index, consumer price index and housing starts are measured in percentage deviation from the steady state; the capacity utilization and unemployment rate are measured in percentage point.

Figure 9: Impulse responses (full sample vs. ZLB with new policy rate)



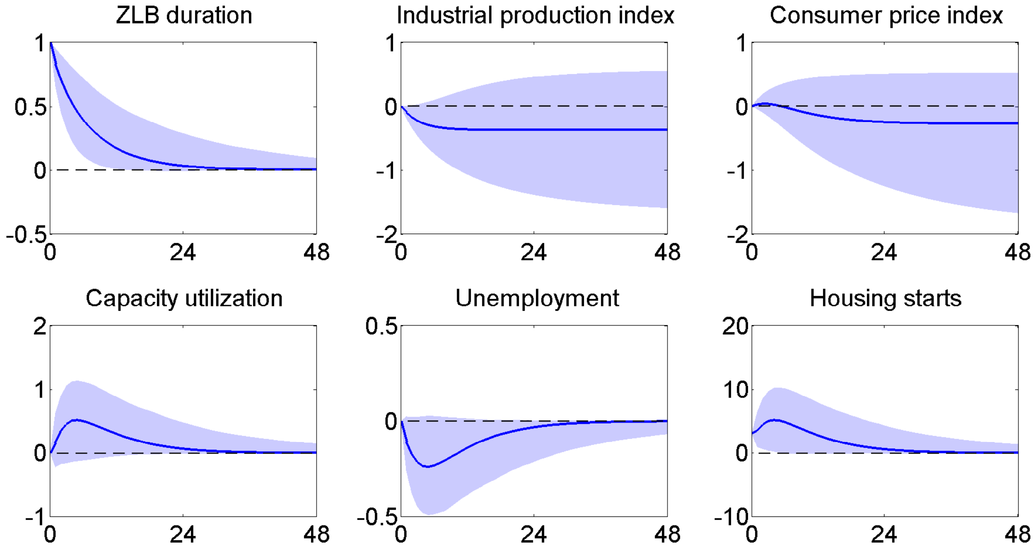
Impulse responses to a -25 basis-point shock on monetary policy. 90% confidence intervals are shaded. Blue: full sample from January 1960 to December 2013 with the policy rate in FAVAR (13). Turquoise: ZLB from July 2009 to December 2013 with the policy rate in a FAVAR (1). X-axis: response time in months. The policy rate is measured in annualized percentage; the industrial production index, consumer price index and housing starts are measured in percentage deviation from the steady state; the capacity utilization and unemployment rate are measured in percentage point.

Figure 10: the market's expected vs. Fed's announced ZLB lift-off dates



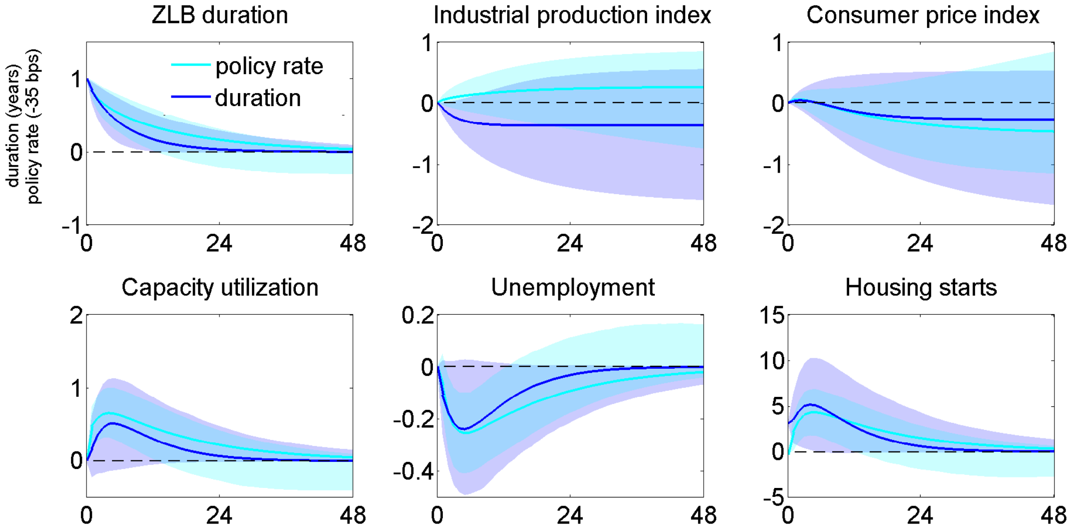
Blue dots: the market's expected lift-off dates from January 2009 to December 2013. Four green vertical lines mark the following months when forward guidance specified explicit lift-off dates for the ZLB: August 2011, January 2012, September 2012 and June 2013. The corresponding lift-off dates are in red dots. Black dashed line: the 45 degree line.

Figure 11: Impulse responses (ZLB with expected duration)



Impulse responses to a one year shock to expected ZLB duration. 90% confidence intervals are shaded. Sample: ZLB from July 2009 to December 2013. Model: FAVAR (1) with the ZLB duration as the monetary policy measure. X-axis: response time in months. The expected duration is measured in year; the industrial production index, consumer price index and housing starts are measured in percentage deviation from the steady state; the capacity utilization and unemployment rate are measured in percentage point.

Figure 12: Impulse responses at ZLB (policy rate v.s. ZLB duration)



Turquoise: impulse responses to a -35 basis-point shock on the policy rate. Blue: impulse responses to a one year shock on the ZLB duration. 90% confidence intervals are shaded. Sample: ZLB from July 2009 to December 2013. Model: FAVAR (1). X-axis: response time in months. The policy rate is measured in -35 basis points; the expected duration is measured in year; the industrial production index, consumer price index and housing starts are measured in percentage deviation from the steady state; the capacity utilization and unemployment rate are measured in percentage point.

Table 1: Maximum likelihood estimates with robust standard errors

	SRTSM			GATSM		
1200μ	-0.3035	-0.2381	0.0253	-0.2296	-0.2069	0.0185
	(0.1885)	(0.1815)	(0.0160)	(0.1464)	(0.1413)	(0.0115)
ρ	0.9638	-0.0026	0.3445	0.9676	-0.0043	0.4854
	(0.0199)	(0.0183)	(0.4821)	(0.0184)	(0.0200)	(0.5408)
	-0.0226	0.9420	1.0152	-0.0231	0.9333	1.0143
	(0.0202)	(0.0212)	(0.5111)	(0.0185)	(0.0227)	(0.5519)
	0.0033	0.0028	0.8869	0.0030	0.0028	0.8935
	(0.0018)	(0.0019)	(0.0385)	(0.0015)	(0.0020)	(0.0423)
$\mathrm{eig}(\rho)$	0.9832	0.9642	0.8452	0.9870	0.9627	0.8448
$\rho^{\mathbb{Q}}$	0.9978	0	0	0.9967	0	0
	(0.0003)			(0.0003)		
	0	0.9502	1	0	0.9503	1
		(0.0012)			(0.0012)	
	0	0	0.9502	0	0	0.9503
			(0.0012)			(0.0012)
$1200\delta_0$	13.3750			11.6760		
	(1.0551)			(0.5591)		
1200Σ	0.4160			0.4744		
	(0.0390)			(0.0497)		
	-0.3999	0.2445		-0.4589	0.2175	
	(0.0369)	(0.0233)		(0.0447)	(0.0188)	
	-0.0110	0.0033	0.0390	-0.0167	0.0013	0.0359
	(0.0069)	(0.0034)	(0.0030)	(0.0062)	(0.0029)	(0.0026)
$1200\sqrt{\omega}$	0.0893			0.0927		
	(0.0027)			(0.0027)		
Log likelihood value	855.5743			755.4587		

Maximum likelihood estimates for the three-factor SRTSM and the three-factor GATSM with robust standard errors in parentheses. Sample: January 1990 to December 2013.

Table 2: Approximation error

		3m	6m	1y	2y	5y	7y	10y
1990/01	forward rates	0.00	-0.02	-0.06	-0.09	-0.13	-0.04	0.26
	yields	-0.01	0.00	-0.02	-0.04	-0.08	-0.08	-0.03
1991/01	forward rates	0.02	0.03	0.05	0.08	-0.05	-0.01	0.04
	yields	0.00	0.01	0.02	0.05	0.03	0.01	0.01
1992/01	forward rates	0.00	-0.01	-0.02	-0.01	0.02	0.01	0.10
	yields	0.00	0.00	-0.01	-0.02	-0.01	0.00	0.02
1993/01	forward rates	-0.01	-0.01	0.03	0.03	0.04	0.12	0.29
	yields	-0.01	-0.01	0.00	0.02	0.03	0.04	0.08
1994/01	forward rates	0.03	0.03	0.04	0.01	-0.05	0.14	0.74
	yields	0.01	0.02	0.03	0.03	0.02	0.02	0.11
1995/01	forward rates	-0.02	-0.02	0.03	0.06	0.13	0.19	0.55
	yields	0.00	-0.01	0.00	0.04	0.05	0.08	0.17
1996/01	forward rates	0.00	0.01	0.02	0.04	0.02	0.36	1.22
	yields	0.00	0.00	0.00	0.02	0.03	0.07	0.26
1997/01	forward rates	0.00	0.01	0.00	-0.03	0.03	0.30	0.96
	yields	0.00	0.00	0.01	-0.01	-0.01	0.03	0.20
1998/01	forward rates	0.01	0.01	0.03	0.00	0.21	0.58	1.68
	yields	0.01	0.01	0.02	0.01	0.03	0.13	0.40
1999/01	forward rates	-0.01	-0.03	-0.05	-0.04	0.26	0.74	2.24
	yields	0.00	-0.01	-0.02	-0.03	0.03	0.16	0.53
2000/01	forward rates	-0.02	-0.02	0.00	-0.02	-0.11	0.05	0.85
	yields	-0.01	-0.02	-0.01	-0.01	-0.05	-0.06	0.07
2001/01	forward rates	0.01	0.02	0.00	-0.06	0.04	0.57	1.59
	yields	0.00	0.01	0.01	-0.01	-0.03	0.06	0.34
2002/01	forward rates	-0.01	-0.02	-0.03	-0.08	0.12	0.40	1.16
	yields	0.00	-0.01	-0.02	-0.04	-0.04	0.05	0.25
2003/01	forward rates	0.00	0.02	0.05	0.15	0.51	0.82	1.93
	yields	0.00	0.01	0.02	0.06	0.23	0.34	0.63
2004/01	forward rates	-0.01	-0.03	-0.04	0.07	0.52	0.98	1.95
	yields	0.00	-0.01	-0.02	0.00	0.16	0.32	0.65
2005/01	forward rates	0.00	-0.02	-0.05	-0.07	0.24	0.91	2.74
	yields	0.00	0.00	-0.02	-0.03	0.00	0.15	0.63
2006/01	forward rates	0.01	0.00	0.02	0.01	0.39	1.14	3.20
	yields	0.00	0.00	0.01	0.01	0.08	0.26	0.80
2007/01	forward rates	0.01	0.01	0.02	0.02	0.33	0.98	2.93
	yields	-0.01	0.00	0.01	0.01	0.05	0.21	0.70
2008/01	forward rates	-0.01	-0.02	0.03	0.11	0.84	1.41	2.86
	yields	-0.01	-0.01	0.00	0.03	0.30	0.52	0.98
2009/01	forward rates	0.01	0.02	0.08	0.41	1.52	2.02	3.25
	yields	0.00	0.01	0.03	0.12	0.66	0.97	1.45
2010/01	forward rates	0.00	0.02	0.10	0.39	1.03	1.48	2.58
	yields	0.00	0.00	0.02	0.13	0.47	0.69	1.07
2011/01	forward rates	0.00	0.00	0.09	0.66	1.70	1.99	3.16
	yields	0.00	0.00	0.01	0.17	0.86	1.13	1.54
2012/01	forward rates	0.00	0.00	0.00	0.31	4.18	6.25	9.32
	yields	0.00	0.00	0.00	0.04	1.27	2.40	3.98
2013/01	forward rates	0.00	0.00	0.01	0.36	4.05	5.78	8.77
	yields	0.00	0.00	0.00	0.05	1.31	2.33	3.78
Average	forward rates	0.01	0.02	0.04	0.13	0.69	1.14	2.26
absolute error	yields	0.00	0.01	0.01	0.04	0.24	0.42	0.78

Differences in forward rates and yields implied by equation (6) and by simulation for the 24 Januaries between 1990 and 2013. At time t, we simulate 10 million paths of s_{t+j} for $j = 1, ..., 120$ with the estimated factors X_t and $\mathbb{Q}$ parameters, and compute r_{t+j} based on equation (1). Then we compute the corresponding 10 million $y_{nt} = -\frac{1}{n}\log\left(\mathbb{E}_t^{\mathbb{Q}}[\exp(-r_t - r_{t+1} - ... - r_{t+n-1})]\right)$ and $f_{n,n+1,t} = (n+1)y_{n+1,t} - ny_{nt}$. We take the average of the 10 million draws as the simulated yield or forward rate. All numbers are measured in basis points.

Table 3: Macroeconomic data

No.	Mnemonic	Short name	Transformation
		Real output and income	
1	IPS11.M*	INDUSTRIAL PRODUCTION INDEX - PRODUCTS, TOTAL	Δln
2	IPS299.M*	INDUSTRIAL PRODUCTION INDEX - FINAL PRODUCTS	Δln
3	IPS12.M*	INDUSTRIAL PRODUCTION INDEX - CONSUMER GOODS	Δln
4	IPS13.M*	INDUSTRIAL PRODUCTION INDEX - DURABLE CONSUMER GOODS	Δln
5	IPS18.M*	INDUSTRIAL PRODUCTION INDEX - NONDURABLE CONSUMER GOODS	Δln
6	IPS25.M*	INDUSTRIAL PRODUCTION INDEX - BUSINESS EQUIPMENT	Δln
7	IPS32.M*	INDUSTRIAL PRODUCTION INDEX - MATERIALS	Δln
8	IPS34.M*	INDUSTRIAL PRODUCTION INDEX - DURABLE GOODS MATERIALS	Δln
9	IPS38.M*	INDUSTRIAL PRODUCTION INDEX - NONDURABLE GOODS MATERIALS	Δln
10	IPS43.M*	INDUSTRIAL PRODUCTION INDEX - MANUFACTURING (SIC)	Δln
11	IPS311.M*	INDUSTRIAL PRODUCTION INDEX - OIL & GAS WELL DRILLING & MANUFACTURED HOMES	Δln
12	IPS307.M*	INDUSTRIAL PRODUCTION INDEX - RESIDENTIAL UTILITIES	Δln
13	IPS10.M*	INDUSTRIAL PRODUCTION INDEX - TOTAL INDEX	Δln
14	UTL11.M*	CAPACITY UTILIZATION - MANUFACTURING (SIC)	
15	PMI.M*	PURCHASING MANAGERS' INDEX (SA)	
16	PMP.M*	NAPM PRODUCTION INDEX (PERCENT)	
17	PI001.M*	PERSONAL INCOME, BIL$, SAAR	Δln
18	A0M051.M*	PERS INCOME LESS TRSF PMT (AR BIL. CHAIN 2009 $),SA-US	Δln
		Employment and hours	
19	LHEM.M*	CIVILIAN LABOR FORCE: EMPLOYED, TOTAL (THOUS.,SA)	Δln
20	LHNAG.M*	CIVILIAN LABOR FORCE: EMPLOYED, NONAGRIC.INDUSTRIES (THOUS.,SA)	Δln
21	LHUR.M*	UNEMPLOYMENT RATE: ALL WORKERS, 16 YEARS and OVER (%,SA)	
22	LHU680.M*	UNEMPLOY.BY DURATION: AVERAGE(MEAN)DURATION IN WEEKS (SA)	
23	LHU5.M*	UNEMPLOY.BY DURATION: PERSONS UNEMPL.LESS THAN 5 WKS (THOUS.,SA)	
24	LHU14.M*	UNEMPLOY.BY DURATION: PERSONS UNEMPL.5 TO 14 WKS (THOUS.,SA)	
25	LHU15.M*	UNEMPLOY.BY DURATION: PERSONS UNEMPL.15 WKS + (THOUS.,SA)	
26	LHU26.M*	UNEMPLOY.BY DURATION: PERSONS UNEMPL.15 TO 26 WKS (THOUS.,SA)	
27	CES001.M*	EMPLOYEES, NONFARM - TOTAL NONFARM	Δln
28	CES002.M*	EMPLOYEES, NONFARM - TOTAL PRIVATE	Δln
29	CES003.M*	EMPLOYEES, NONFARM - GOODS-PRODUCING	Δln
30	CES006.M*	EMPLOYEES, NONFARM - MINING	Δln
31	CES011.M*	EMPLOYEES, NONFARM - CONSTRUCTION	Δln
32	CES015.M*	EMPLOYEES, NONFARM - MFG	Δln
33	CES017.M*	EMPLOYEES, NONFARM - DURABLE GOODS	Δln
34	CES033.M*	EMPLOYEES, NONFARM - NONDURABLE GOODS	Δln
35	CES046.M*	EMPLOYEES, NONFARM - SERVICE-PROVIDING	Δln
36	CES048.M*	EMPLOYEES, NONFARM - TRADE, TRANSPORT, UTILITIES	Δln
37	CES049.M*	EMPLOYEES, NONFARM - WHOLESALE TRADE	Δln
38	CES053.M*	EMPLOYEES, NONFARM - RETAIL TRADE	Δln
39	CES140.M*	EMPLOYEES, NONFARM - GOVERNMENT	Δln
40	CES154.M*	AVG WKLY HOURS, PROD WRKRS, NONFARM - MFG	
41	CES155.M*	AVG WKLY OVERTIME HOURS, PROD WRKRS, NONFARM - MFG	
42	PMEMP.M*	NAPM EMPLOYMENT INDEX (PERCENT)	
		Consumption	
43	PI031.M*	PERSONAL CONSUMPTION EXPENDITURES, BIL$, SAAR	Δln
44	PI032.M*	PERSONAL CONSUMPTION EXPENDITURES - DURABLE GOODS, BIL$, SAAR	Δln
45	PI033.M*	PERSONAL CONSUMPTION EXPENDITURES - NONDURABLE GOODS, BIL$, SAAR	Δln
46	PI034.M*	PERSONAL CONSUMPTION EXPENDITURES - SERVICES, BIL$, SAAR	Δln
		Housing starts and sales	
47	HSFR.M	HOUSING STARTS:NONFARM(1947-58);TOTAL FARM&NONFARM(1959-)(THOUS.,SA	ln
48	HSNE.M	HOUSING STARTS:NORTHEAST (THOUS.U.)S.A.	ln
49	HSMW.M	HOUSING STARTS:MIDWEST(THOUS.U.)S.A.	ln
50	HSSOU.M	HOUSING STARTS:SOUTH (THOUS.U.)S.A.	ln
51	HSWST.M	HOUSING STARTS:WEST (THOUS.U.)S.A.	ln
52	HS6BR.M	HOUSING AUTHORIZED: TOTAL NEW PRIV HOUSING UNITS (THOUS.,NSA)	ln
53	HMOB.M	MOBILE HOMES: MANUFACTURERS' SHIPMENTS (THOUS.OF UNITS,SAAR)	ln

No.	Mnemonic	Short name	Transformation
		Real inventories, orders and unfilled orders	
54	PMNV.M	NAPM INVENTORIES INDEX (PERCENT)	
55	PMNO.M	NAPM NEW ORDERS INDEX (PERCENT)	
56	PMDEL.M	NAPM VENDOR DELIVERIES INDEX (PERCENT)	
57	MOCMQ.M	NEW ORDERS (NET) - CONSUMER GOODS and MATERIALS, 1996 $ (BCI)	Δln
58	MSONDQ.M	NEW ORDERS, NONDEFENSE CAPITAL GOODS, IN 1996 $ (BCI)	Δln
		Stock prices	
59	FSPCOM.M	S&P'S COMMON STOCK PRICE INDEX: COMPOSITE (1941-43=10)	Δln
60	FSPIN.M	S&P'S COMMON STOCK PRICE INDEX: INDUSTRIALS (1941-43=10)	Δln
		Exchange rates	
61	EXRUK.M	FOREIGN EXCHANGE RATE: UNITED KINGDOM (CENTS PER POUND)	Δln
62	EXRCAN.M	FOREIGN EXCHANGE RATE: CANADA (CANADIAN $ PER U.S.$)	Δln
		Interest rates	
63	FYFF.M	INTEREST RATE: FEDERAL FUNDS (EFFECTIVE) (% PER ANNUM,NSA)	
64	FYGM3.M	INTEREST RATE: U.S.TREASURY BILLS,SEC MKT,3-MO.(% PER ANN,NSA)	
65	FYGM6.M	INTEREST RATE: U.S.TREASURY BILLS,SEC MKT,6-MO.(% PER ANN,NSA)	
66	FYGT1.M	INTEREST RATE: U.S.TREASURY CONST MATURITIES,1-YR.(% PER ANN,NSA)	
67	FYGT5.M	INTEREST RATE: U.S.TREASURY CONST MATURITIES,5-YR.(% PER ANN,NSA)	
68	FYGT10.M	INTEREST RATE: U.S.TREASURY CONST MATURITIES,10-YR.(% PER ANN,NSA)	
69	FYGM3.M-FYFF.M	SPREAD: FYGM3.M-FYFF.M	
70	FYGM6.M-FYFF.M	SPREAD: FYGM6.M-FYFF.M	
71	FYGT1.M-FYFF.M	SPREAD: FYGT1.M-FYFF.M	
72	FYGT5.M-FYFF.M	SPREAD: FYGT5.M-FYFF.M	
73	FYGT10.M-FYFF.M	SPREAD: FYGT10.M-FYFF.M	
		Money and credit quantity aggregates	
74	ALCIBL00.M	COML&IND LOANS OUTST IN 2009 $,SA-US	Δln
75	CCINRV.M	CONSUMER CREDIT OUTSTANDING - NONREVOLVING(G19)	Δln
76	FM1.M	MONEY STOCK: M1(CURR,TRAV.CKS,DEM DEP,OTHER CK'ABLE DEP)(BIL$,SA)	Δln
77	FM2.M	MONEY STOCK:M2(M1+O'NITE RPS,EURO$,G/P&B/D MMMFS&SAV&SM TIME DEP(BIL$,SA),	Δln
78	MBASE.M	REVISED MONETARY BASE-ADJUSTED-(FED RESERVE BANK-SAINT LOUIS),SA-US	Δln
79	MNY2.M	M2 - MONEY SUPPLY - M1 + SAVINGS DEPOSITS, SMALL TIME DEPOSITS, & MMMFS [H6],SA-US	Δln
		Price indexes	
80	PMCP.M	NAPM COMMODITY PRICES INDEX (PERCENT)	
81	PWFSA.M*	PRODUCER PRICE INDEX: FINISHED GOODS (82=100,SA)	Δln
82	PWFCSA.M*	PRODUCER PRICE INDEX:FINISHED CONSUMER GOODS (82=100,SA)	Δln
83	PWIMSA.M*	PRODUCER PRICE INDEX:INTERMED MAT.SUPPLIES & COMPONENTS(82=100,SA)	Δln
84	PWCMSA.M*	PRODUCER PRICE INDEX:CRUDE MATERIALS (82=100,SA)	Δln
85	PUNEW.M*	CPI-U: ALL ITEMS (82-84=100,SA)	Δln
86	PU83.M*	CPI-U: APPAREL & UPKEEP (82-84=100,SA)	Δln
87	PU84.M*	CPI-U: TRANSPORTATION (82-84=100,SA)	Δln
88	PU85.M*	CPI-U: MEDICAL CARE (82-84=100,SA)	Δln
89	PUC.M*	CPI-U: COMMODITIES (82-84=100,SA)	Δln
90	PUCD.M*	CPI-U: DURABLES (82-84=100,SA)	Δln
91	PUS.M*	CPI-U: SERVICES (82-84=100,SA)	Δln
92	PUXF.M*	CPI-U: ALL ITEMS LESS FOOD (82-84=100,SA)	Δln
93	PUXHS.M*	CPI-U: ALL ITEMS LESS SHELTER (82-84=100,SA)	Δln
94	PUXM.M*	CPI-U: ALL ITEMS LESS MIDICAL CARE (82-84=100,SA)	Δln
		Average hourly earnings	
95	CES277.M*	AVG HRLY EARNINGS, PROD WRKRS, NONFARM - CONSTRUCTION	Δln
96	CES278.M*	AVG HRLY EARNINGS, PROD WRKRS, NONFARM - MFG	Δln
		Miscellaneous	
97	U0M083.M	BUSINESS CYCLE INDICATORS,CONSUMER EXPECTATIONS,NSA	

This table lists the mnemonics, short names and transformations for the 97 macroeconomic series used in the paper. All series are from the Global Insights Basic Economics Database. Slow-moving variables are marked with *.

Table 4: Robustness check for structural break tests

		p-value for $\rho_1^{xs} = \rho_3^{xs}$	p-value for $\rho_1^{sx} = \rho_3^{sx}$
	Baseline	0.29	1.00
A1	estimate $\underline{r}$	0.18	1.00
A2	2-factor SRTSM	0.13	0.97
A3	Fama-Bliss	0.38	1.00
A4	5-factor FAVAR	0.70	1.00
A5	6-lag FAVAR	0.09	0.98
	7-lag FAVAR	0.19	0.97
	12-lag FAVAR	0.22	1.00

This table consists of p-values for structural break tests with alternative model specifications.

Table 5: Forward guidance quotes

Date	Quotes
12/16/2008	" ...anticipates that weak economic conditions are likely to warrant exceptionally low levels of the federal funds rate for *some time*."
03/18/2009	"...anticipates that economic conditions are likely to warrant exceptionally low levels of the federal funds rate for *an extended period*."
08/09/2011*	"...anticipates that economic conditions – including low rates of resource utilization and a subdued outlook for inflation over the medium run – are likely to warrant exceptionally low levels for the federal funds rate *at least through mid-2013*."
01/25/2012*	"...decided today to keep the target range for the federal funds rate at 0 to 1/4 percent and currently anticipates that economic conditions – including low rates of resource utilization and a subdued outlook for inflation over the medium run – are likely to warrant exceptionally low levels for the federal funds rate *at least through late 2014*."
09/13/2012*	"...decided today to keep the target range for the federal funds rate at 0 to 1/4 percent and currently anticipates that exceptionally low levels for the federal funds rate are likely to be warranted *at least through mid-2015*."
12/12/2012	"...decided to keep the target range for the federal funds rate at 0 to 1/4 percent and currently anticipates that this exceptionally low range for the federal funds rate will be appropriate at least as long as the *unemployment rate* remains above 6-1/2 percent, *inflation* between one and two years ahead is projected to be no more than a half percentage point above the Committee's 2 percent longer-run goal, and longer-term inflation expectations continue to be well anchored."
06/19/2013*	"...14 of 19 FOMC participants indicated that they expect the first increase in the target for the federal funds rate to occur in *2015*, and one expected the first increase to incur in 2016."
12/18/2013	"...anticipates, based on its assessment of these factors, that it likely will be appropriate to maintain the current target range for the federal funds rate well past the time that the unemployment rate declines below 6-1/2 percent, especially if projected *inflation* continues to run below the Committee's 2 percent longer-run goal."

This table summarizes a list of forward guidance quotes, when the Fed expected a different lift-off date or condition for the ZLB. All quotes except the one on 6/19/2013 are from FOMC statements. The quote on 6/19/2013 is from Chairman Bernanke's press conference. Asterisks mark the statements with explicit lift-off dates, with the corresponding lift-off dates in red.
Source: http://www.federalreserve.gov/monetarypolicy/fomccalendars.htm.