Stable Berk-Nash Equilibrium

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Bayesian Learning with Model Misspecification

- One agent. Infinite periods \( t = 1, 2, \ldots \).

- In each period \( t \ldots \)
  - The agent chooses an action \( a \in A \).
  - Observes a signal \( y \in Y \), with the dist \( q(a) = (q(y|a))_{y \in Y} \).
  - Payoff is \( u(a, y) \).

- Subjective model \( \theta \): \( y \) follows a dist \( q^{\theta}(a) = (q^{\theta}(y|a))_{y \in Y} \).

- Initial prior \( \mu \) on the subjective model space \( \Theta \).
  - Models are misspecified if \( q \neq q^{\theta} \) for all \( \theta \).

- Agent chooses myopically optimal action which maximizes

\[
g^{\mu^t}(a) = \sum_{\theta \in \Theta} \mu^t(\theta) \sum_{y \in Y} q(y|a)u(a, y).\]
Example: Monopolist with Unknown Demand

- Monopolist chooses a price $a$.

- The demand $y$ is given by

$$y = f(a) + \epsilon, \quad \epsilon \sim N(0, \sigma^2).$$

- In a subjective model $\theta$, the demand is

$$y = b - \theta a + \epsilon.$$

- Other examples:
  - Overconfidence. (Heidhues-Kosegi-Strack)
  - Gambler’s fallacy. (He)
What is known?

- If models are correctly specified, the agent learns the true model.
  - Given any action, the belief moves toward the true model $\theta$ on average.

- With model misspecification, the process may not converge.
  - The “drift” of the belief depends on the action.
  - Model $\theta$ may fit the true distribution for some $a$, but not for other $\tilde{a}$.

- Use relative entropy to measure fitness. (Berk)
What is known?

- Esponda-Pouzo introduce Berk-Nash equilibrium.
  - Action-belief pair \((\alpha, \mu)\) which is a fixed point of the learning process.

- Defined in a static model. Existence.
  - \(\alpha\) may be a mixed action, in many cases.

- Necessity.
  - If action converges, it must be a Berk-Nash equilibrium action.

- No sufficiency.
  - Even if a BNE exists, the learning process may not converge.
What is known?

- $|\Theta| = 2$. (General payoff function. Fudenberg-Romanyuk-Strack)
  - If there is a pure BNE, $(a, \mu)$ converges almost surely.
  - If not, $(a, \mu)$ oscillates forever almost surely.

- Continuous $\Theta$. (Examples only.)
  - HKS, He: Unique pure BNE, global convergence.
  - Nyarko: No pure BNE, action oscillates forever.

- Can mixed BNE be a log-run outcome?
  - Agent almost always chooses a pure action.
  - Action cannot converge to a mixture.
Findings

▶ Yes, the belief can converge to a mixed equilibrium belief.
  ▶ Example with continuous $\Theta$.
  ▶ Minor modification to a finite model.
  ▶ Action does not converge, but action frequency does.
  ▶ Direct foundation of mixed BNE.
  ▶ Perhaps action convergence is too demanding.

▶ General analysis for continuous $\Theta$.
  ▶ Log-concavity assumption.
  ▶ If $\Theta = [0, 1]$, the belief converges almost surely.
  ▶ If $\Theta \subseteq \mathbb{R}^k$, the belief may not converge.
Notation

- \( h = (a^T, y^T)_{T=1}^\infty \): Sample path.

- \( h^t = (a^T, y^T)_{T=1}^t \): History with length \( t \).

- A strategy is a mapping \( s : \bigcup_{t=0}^\infty H^t \rightarrow \triangle A \).

- \( P^s \): Probability measure over all sample paths \( h \), given \( s \).

- \( \mu^{t+1}(h^t) \): the posterior belief in period \( t + 1 \), given \( h^t \).

- \( \mu^{t+1}(h) = \mu^{t+1}(h^t) \), where \( h^t \) is a truncation.
Stable Beliefs

- $s$ is *myopically optimal* given an initial prior $\mu$ if for each $h^t$,

$$s(h^t) \in \arg\max g^{\mu_{t+1}}(h^t)(\alpha).$$

- Belief $\mu$ is *stable* if there is an initial prior with full support and a myopically optimal strategy $s$ such that

$$P^s \left( \lim_{t \to \infty} \mu^t(h) = \mu \right) > 0.$$  

- Use a topology of weak convergence.

- Goal: Characterize stable beliefs.
Relative Entropy

- $p(\alpha)$: True distribution of $(a, y)$ given a mixed action $\alpha$.

- $p^\theta(\alpha)$: Distribution of $(a, y)$ induced by a model $\theta$.

- Relative entropy between $p(\alpha)$ and $p^\theta(\alpha)$:

$$d(p(\alpha), p^\theta(\alpha)) = \sum_{(a,y)} p(a, y|\alpha) \log \frac{p(a, y|\alpha)}{p^\theta(a, y|\alpha)}$$

- Expected “surprise.”

- A “pseudo” metric.
  - $d(p, \tilde{p}) \geq 0$. Equality iff $p = \tilde{p}$.
  - Does not satisfy symmetry, triangle inequality.
Berk-Nash Equilibrium

► “Best” model given $\alpha$:

$$\Theta(\alpha) = \arg \min_{\theta \in \Theta} d(p(\alpha), p^\theta(\alpha)).$$

► $(\alpha^*, \mu^*)$ is a Berk-Nash equilibrium if...

(i) $\alpha^* \in \arg \max_{\alpha \in \Delta A} g^{\mu^*}(\alpha)$.

(ii) $\text{supp} \mu^* \subseteq \Theta(\alpha^*)$.

► Fixed point of the learning process.

► If the current belief is $\mu^*$, the agent must choose $\alpha^*$.

► If $\alpha^*$ is chosen forever, $\text{supp} \mu^t$ must converge to $\Theta(\alpha^*)$. (Berk)

► Stable beliefs = BNE beliefs?
Subjective models are *regular* if...

(i) $\Theta$ is finite, or a compact convex set in $\mathbb{R}^k$.
(ii) If $\Theta$ is convex, $q^\theta(y|a)$ is continuous in $\theta$.
(iii) $q^\theta(y|a) > 0$ for all $\theta$, $a$, and $y$.

Stronger than necessary...

**Proposition**

Under regularity, any stable belief $\mu$ is a BNE belief.

So if the belief converges, it is a BNE belief.

Esponda-Pouzo: Any stable action $\alpha$ is a BNE action.
Proof Idea

- Pick an arbitrary stable belief $\mu^*$. 

- Need to find $\alpha^*$ s.t. $(\alpha^*, \mu^*)$ is a BNE.

- Case 1: Optimal action is unique given $\mu^*$.
  - Let $a^*$ denote the optimal action.
  - Optimal action is $a^*$ whenever the belief is close to $\mu^*$.
  - So when the belief converges to $\mu^*$, $a^*$ is taken forever.
  - This implies $\text{supp}\mu \subseteq \Theta(a^*)$. (Berk)
  - $(a^*, \mu^*)$ is a Berk-Nash equilibrium.
Proof Idea

Case 2: Both $a^*$ and $\tilde{a}^*$ are optimal given $\mu^*$.

- Pick a sample path $h$ such that $\lim_{t \to \infty} \mu^t(h) = \mu^*$.
- When the belief approaches $\mu^*$, the agent is almost indifferent.
- But not exactly indifferent, as the belief is different from $\mu^*$.
- Action switches infinitely often.
- Let $\alpha^t(h) = \frac{1}{t} \sum_{\tau=1}^{t} s(h^{\tau-1})$ be the action frequency until period $t$.
- $\alpha^t(h) \in \triangle A$, so there is a convergent subsequence.
- Let $\alpha^*(h)$ be the limit of the subsequence.
- $\alpha^*(h)$ is optimal given $\mu$, as it is a mixture of $a^*$ and $\tilde{a}^*$.
- For “almost all” $h$, $\text{supp}\mu \subseteq \Theta(\alpha^*(h))$. 
Action Convergence vs Belief Convergence

- In general, the action cannot converge to a mixed action.
  - Optimal action is almost always unique.

- But the belief may converge to a mixed BNE belief.
  - In this case, action does not converge.
  - But action frequency can converge to a mixed action.
  - More on this later.
Proposition
Assume regularity. If \((a, \mu)\) is a pure strict BNE with \(|\Theta(a)| = 1\), then \(\mu\) is stable.

- In this case, the action also converges to the equilibrium action.
- Proof idea:
  - Suppose that the initial prior is close to \(\mu\).
  - Then the agent chooses \(a\).
  - Then her belief will be even closer to \(\mu\), on average.
Example: Two actions $a_1, a_2$. Two signals $y_1, y_2$.

In the true model, two signals are equally likely, given any action.

Two subjective models, $\theta_1$ and $\theta_2$.

$g^{\theta_1}(a_1) = g^{\theta_2}(a_2) = 1$, and $g^{\theta_1}(a_2) = g^{\theta_2}(a_1) = 0$.

Given $a_1$, the signal dist is 0.3-0.7 at $\theta_1$, 0.6-0.4 at $\theta_2$.

Given $a_2$, the signal dist is 0.4-0.6 at $\theta_1$, 0.7-0.3 at $\theta_2$. 
Mixed BNE for Finite $\Theta$

- Simpler version of FRS example.

- If the model $\theta_1$ is likely, the agent chooses $a_1$.

- But then she gets more convinced that the model is likely to be $\theta_2$.

- Then she starts to choose $a_2$.

- Now she gets more convinced that the model is likely to be $\theta_1$.

- Does this process converge?
Mixed BNE for Finite $\Theta$

- No pure-strategy BNE.

- Unique mixed BNE $(\alpha, \mu)$:
  - $\alpha$ mixes $a_1$ and $a_2$ equally.
  - $\mu$ puts prob 0.5 on both models.

- In this equilibrium...
  - Two models are equally close to the true world, given $\alpha$.
  - Two actions are indifferent, given $\mu$.

- If the belief converges, it must be the equilibrium belief.
But this equilibrium belief $\mu$ cannot be stable.

Suppose that the current belief is $\mu$.

If the current signal is $y_1$, the posterior on $\theta_1$ becomes

$$
\mu'(\theta_1) = \frac{0.5 \cdot 0.3}{0.5 \cdot 0.3 + 0.5 \cdot 0.6} = \frac{1}{3}.
$$

So the belief cannot stay around $\mu$.

Hence the belief oscillates.

In general, mixed BNE is unstable when $|\Theta| < \infty$. 
Mixed BNE for Convex \( \Theta \)

- Now modify the subjective model space: \( \Theta = [0, 1] \).
  - Given \( a_1 \), the prob of \( y_1 \) is \( 0.3(1 - \theta) + 0.6\theta \).
  - Given \( a_2 \), the prob of \( y_1 \) is \( 0.4(1 - \theta) + 0.7\theta \).

- \( g^\theta(a_1) = 1 - \theta \) and \( g^\theta(a_2) = \theta \).

- \( \theta = 0, 1 \) are identical with the models \( \theta_1, \theta_2 \) in the previous example.

- But now we have many “intermediate” models.

- Does it change the result?
There is a unique mixed BNE \((\alpha, \mu)\):
- \(\alpha\) mixes \(a_1\) and \(a_2\) equally.
- \(\mu\) puts prob one on the model \(\theta = 0.5\).

Equilibrium action is the same, but the belief is not.
- Relative entropy \(d(p, \tilde{p})\) is convex in \(\tilde{p}\).
- “Moving to the middle” improves the fitness.

This new belief is more “robust.”
- Observing one more signal does not change the posterior much.

So it can be stable.
Mixed BNE for Convex \( \Theta \)

**Proposition**

For any initial prior with full support and for any myopically optimal strategy \( s \),

\[
P^s \left( \lim_{t \to \infty} \mu^t(h) = \mu \right) = 1.
\]

- Global convergence to the mixed BNE belief.
- Action frequency also converges to the equilibrium action.
- First direct foundation of mixed BNE.
  - Esponda-Pouzo: i.i.d. payoff shock, so BNE is purified.
- New interpretation of Nyarko.
  - The belief can converge, even when the action does not.
Proof Idea

▶ Step 1: Belief concentration.

▶ Likelihood maximizer in period $t$:

$$\theta^t_{\max}(h^{t-1}) \in \arg\max_{\theta} \mu^t(h^{t-1})[\theta].$$

▶ After a long time, the posterior becomes approximately $\delta_{\theta^t_{\max}(h^{t-1})}$.

Lemma

For any smooth initial prior with full support, there is $T > 0$ and $K > 0$ such that for any $t \geq T$, for any history $h^t$, and for any $\theta$,

$$\frac{\mu^{t+1}(\theta^t_{\max})}{\mu^{t+1}(\theta)} \geq \exp \left\{ \frac{tK}{2} (\theta^t_{\max} - \theta)^2 \right\} .$$

▶ Key: Convexity of relative entropy.
Proof Idea

- **Step 2:** Show $P^s(\lim_{t \to \infty} \theta^t_{\text{max}}(h) = 0.5) = 1$.

- In the long run, BR is determined by $\theta^t_{\text{max}}$.

\[
\begin{aligned}
\theta^t_{\text{max}} &= 0 &\quad\theta^t_{\text{max}} &= 0.5 &\quad\theta^t_{\text{max}} &= 1 \\
\text{a}_1 \text{ is optimal} &\quad\text{a}_2 \text{ is optimal}
\end{aligned}
\]
Proof Idea

- Intuition is simple, but the actual proof is not.

- Reason: $\theta_{\text{max}}$ is not a sufficient statistic.
  - $\theta_{\text{max}}^{t+1}$ depends not only on $\theta_{\text{max}}^t$, but on $\mu^t$.

- Key step: For any $\varepsilon > 0$, if $a_1$ is chosen forever, $\theta_{\text{max}}$ cannot drop by $\varepsilon$ infinitely often.
Unstable Mixed BNE

- Now assume that...
  - Given $a_1$, the prob of $y_1$ is $0.3\theta + 0.4(1 - \theta)$.
  - Given $a_2$, the prob of $y_1$ is $0.6\theta + 0.7(1 - \theta)$.

- $g^\theta(a_1) = 1 - \theta$ and $g^\theta(a_2) = \theta$.

- Given $a_1$, the agent gets more convinced that $\theta = 0$ is more likely.

- Given $a_2$, the agent gets more convinced that $\theta = 1$ is more likely.
Unstable Mixed BNE

- Two pure-strategy BNE.
  - $a = a_1$ and a degenerate belief on $\theta = 0$.
  - $a = a_2$ and a degenerate belief on $\theta_1$.

- Also a mixed BNE.
  - $\alpha$ mixes two actions equally. A degenerate belief on $\theta = 0.5$.

- This mixed BNE is unstable.
Unstable Mixed BNE

Proposition
For any initial prior with full support, for any myopically optimal strategy \( s \), and for any open set \( U \) containing \( \theta = 0 \) and \( \theta = 1 \),

\[
P^s \left( \lim_{t \to \infty} \mu^t(h)[U \cap \Theta] = 1 \right) = 1.
\]

\[\theta^t_{\text{max}} = 0 \quad \theta^t_{\text{max}} = 0.5 \quad \theta^t_{\text{max}} = 1\]

\( a_1 \) is optimal \( a_2 \) is optimal
General Analysis for Continuous $\Theta$

- Previous two examples: $q^\theta(y|a)$ is linear w.r.t. $\theta$.

- Belief concentration under a more general condition.

- Signal distributions are log-concave if
  - (i) $\Theta$ is convex and
  - (ii) $q^\theta(y|a)$ is log-concave w.r.t. $\theta$.

- Many existing models satisfy this assumption.
  - Nyarko, He, Section 5 of HKS.
  - HKS’s general model does not satisfy this.
General Analysis for Continuous $\Theta$

- Strong identifiability: $q^\theta(a) \neq q^{\tilde{\theta}}(a) \ \forall a \forall \theta \forall \tilde{\theta}$.

**Proposition**

Assume log concavity and strong identifiability. Then $d(f, p^\theta(\alpha))$ is convex with respect to $\theta$ for any $\alpha$ and $f \in \triangle(A \times Y)$.

- Given an observation $f$, there is a unique “best” model $\theta$.

- Hence belief concentration.

- Also, any equilibrium belief is $\delta_\theta$ for some $\theta$. 
General Analysis for Continuous $\Theta$

- **Assume** $\Theta = [0, 1]$.

- $\theta$ is *absorbing* if there is $\varepsilon > 0$, $a_1$, and $a_2$ (possibly $a_1 = a_2$) s.t.
  - Unique optimal action is $a_1$ for any $\tilde{\theta} \in (\theta - \varepsilon, \theta)$.
  - $\Theta(a_1) > \theta$.
  - Unique optimal action is $a_2$ for any $\tilde{\theta} \in (\theta, \theta + \varepsilon)$.
  - $\Theta(a_2) < \theta$.

- $\mu$ is *absorbing* if $\mu = \delta_{\tilde{\theta}}$ for some absorbing $\theta$. 

General Analysis for Continuous $\Theta$

**Proposition**

Assume...
- $\Theta = [0, 1]$
- Regularity, log concavity, strong identifiability.
- Optimal action is unique except for finite $\theta$.

Then the belief converges almost surely to an equilibrium belief.

**Proposition**

In addition, if
- Optimal action is unique for $\Theta(a)$

then the belief converges to an absorbing belief.
General Analysis for Continuous $\Theta$

- When $\Theta \subset \mathbb{R}^k$, $\theta_{\text{max}}^t$ can be cyclic.

- So the belief may not converge, even if there is a unique pure BNE.