Assessing Point Forecast Accuracy by Stochastic Error Distance

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August 7, 2015

How Many Times Have you Ranked Forecasts' Accuracy by *RMSE*?

$$RMSE = \sqrt{\frac{1}{T}\sum_{t=1}^{T} e^2_{t/t-1}}$$

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$$RMSE = \sqrt{\frac{1}{T}\sum_{t=1}^{T} e_{t/t-1}^2}$$

What are you really *doing*?

What does it really *mean*?

What do you really *want*?

Does it matter whether you rank
using *RMSE* or other criteria like *MAE*?

Traditional Point-Forecast Accuracy Comparison: Emphasizes the Loss Function

Error: $e = y - \hat{y}$

Loss: $L(e)$, where $L(0) = 0$ and $L(e) \geq 0, \forall e$

The big three:

Absolute-error loss: $L(e) = abs(e)$

Squared-error loss: $L(e) = square(e)$

Check-error, or lin-lin, loss: $L(e) = check_\tau(e)$,

where

$$check_\tau(e) = \begin{cases} (1-\tau)|e|, & e < 0 \\ \tau|e|, & e \geq 0. \end{cases}$$

Accuracy comparison via expected loss: $E(L(e))$, e.g. $E(e^2)$

How to choose a loss function?

Does the choice matter for accuracy rankings?

This Paper's Point-Forecast Accuracy Comparison: Works Directly From First Principles

Compare:

$F(e)$ (c.d.f. of e)

vs.

$F^*(e)$ (c.d.f. of perfect forecast),

where

$$F^*(e) = \begin{cases} 0, & e < 0 \\ 1, & e \geq 0. \end{cases}$$

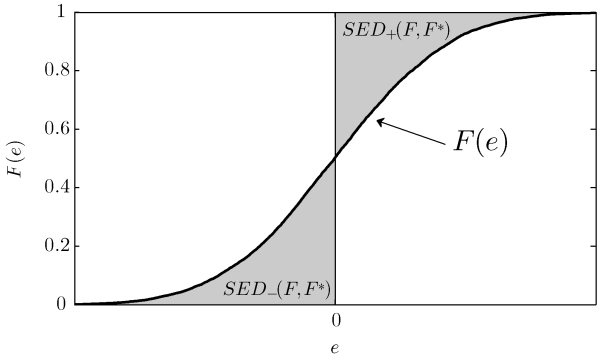
"Unit step function at zero"

Stochastic Error Distance (*SED*)

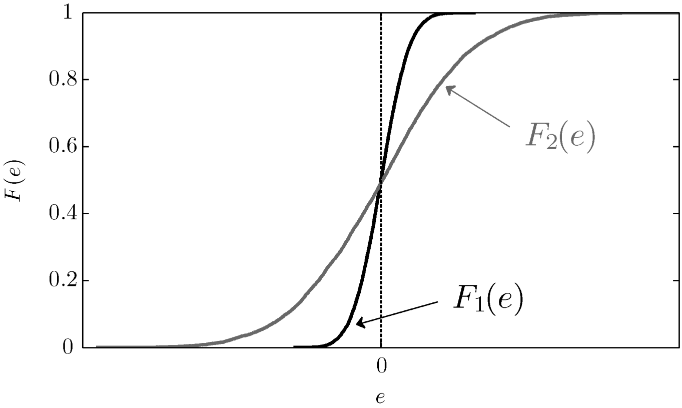
$$SED(F, F^*) = \int_{-\infty}^{\infty} |F(e) - F^*(e)| \, de$$

$$= \int_{-\infty}^{0} F(e) \, de + \int_{0}^{\infty} [1 - F(e)] \, de$$

$$= SED(F, F^*)_{-} + SED(F, F^*)_{+}$$



Example: Two Forecast Error Distributions



Under the *SED* criterion, we prefer F_1 to F_2.

SED and Expected Absolute Loss

$$SED(F, F^*) = \int_{-\infty}^{\infty} |F(e) - F^*(e)| \, de$$

Proposition (Equivalence of *SED* and Expected Absolute Loss):

If e is a forecast error with cumulative distribution function $F(e)$, such that $E(|e|) < \infty$, then SED equals expected absolute loss:

$$SED(F, F^*) = E(|e|).$$

SED accuracy evaluation is *MAE* accuracy evaluation!

Weighted Stochastic Error Distance (*WSED*)

$$WSED(F, F^*; \tau) = 2(1-\tau)SED(F, F^*)_- + 2\tau SED(F, F^*)_+,$$

where $\tau \in [0, 1]$.

WSED and Expected Lin-Lin Loss

Proposition (Equivalence of *WSED* and Expected Lin-Lin Loss):

If e is a forecast error with cumulative distribution function $F(e)$, such that $E(|e|) < \infty$, then WSED equals expected lin-lin loss:

$$\begin{aligned} WSED(F, F^*; \tau) &= 2(1-\tau)\int_{-\infty}^{0} F(e)\, de + 2\tau \int_{0}^{\infty} [1 - F(e)]\, de \\ &= 2E(L_\tau(e)), \end{aligned}$$

where $L_\tau(e)$ is the lin-lin loss function

$$L_\tau(e) = \begin{cases} (1-\tau)|e|, & e < 0 \\ \tau|e|, & e \geq 0. \end{cases}$$

Generalized Weighted Stochastic Error Distance (*GWSED*)

$$GWSED(F, F^*; p, w) = \int |F(e) - F^*(e)|^p \, w(e) \, de,$$

where $p > 0$.

SED and *WSED* are nested special cases:

- $p = 1$ and $w(e) = 1 \; \forall \; e$ produces *SED*.
- $p = 1$ and
$$w(e) = \begin{cases} 2(1-\tau), & e < 0 \\ 2\tau, & e \geq 0 \end{cases}$$
produces *WSED*.
- Other choices of p and $w(e)$?

GWSED and Expected Loss: A Complete Characterization

$$GWSED(F, F^*; p, w) = \int |F(e) - F^*(e)|^p \, w(e) \, de$$

Proposition (Equiv. of $GWSED\left(F, F^*; 1, \left|\frac{dL(e)}{de}\right|\right)$ and $E(L(e))$):

Suppose that $L(e)$ is piecewise differentiable with $dL(e)/de > 0$ for $e > 0$ and $dL(e)/de < 0$ for $e < 0$, and suppose also that $F(e)$ and $L(e)$ satisfy $F(e)L(e) \to 0$ as $e \to -\infty$ and $(1 - F(e))L(e) \to 0$ as $e \to \infty$. Then:

$$\int_{-\infty}^{\infty} |F(e) - F^*(e)| \left|\frac{dL(e)}{de}\right| de = E(L(e)).$$

Connections I: Cramér-von Mises Divergence

$GWSED(F, F^*; 2, f(e))$ is Cramér-von Mises divergence:

$$CVM(F^*, F) = \int |F^*(e) - F(e)|^2 f(e) de$$
$$= -F(0)(1 - F(0)) + \frac{1}{3}$$

$CVM(F^*, F)$ is minimized at $F(0) = \frac{1}{2}$.

That is, like $SED(F, F^*)$,
$CVM(F^*, F)$ is minimized by the conditional-median forecast.

Connections II: Kolmogorov-Smirnov Distance

$$KS(F, F^*) = \sup_e \left| F(e) - F^*(e) \right| = max\left(F(0), 1 - F(0)\right)$$

$KS(F, F^*)$ is minimized at $F(0) = \frac{1}{2}$,
as is $CVM(F^*, F)$.

That is, like $SED(F, F^*)$,
$KS(F, F^*)$ is minimized by the conditional-median forecast.

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Practical Implications

Switch from *RMSE* to *MAE* for forecast accuracy rankings.

– But is it really important to make the switch?

– That is, will rankings really change?

– In general, yes!

MSE vs. *MAE* Rankings

In general, *MSE* and *MAE* rankings differ.

Simplest Gaussian environment:

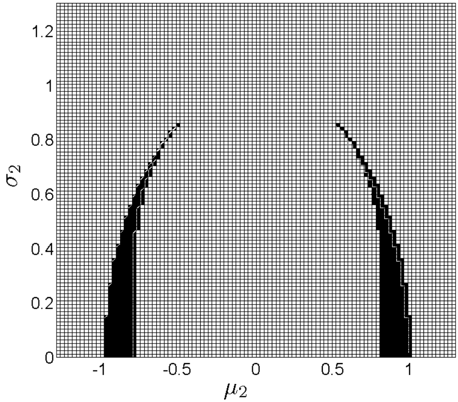
$$e \sim N\left(\mu, \sigma^2\right)$$

$$\implies \quad E(|e|) = \sigma\sqrt{2/\pi}\exp\left(-\frac{\mu^2}{2\sigma^2}\right) + \mu\left[1 - 2\Phi\left(-\frac{\mu}{\sigma}\right)\right]$$

Unbiased case ($\mu = 0$): $E(|e|) \propto \sigma$
MAE and *MSE* rankings must be identical

Biased case ($e_1 \sim N(0,1)$ and $e_2 \sim N(\mu_2, \sigma_2^2)$):
MAE and *MSE* rankings can diverge, even under normality.

MSE and *MAE* Divergence Regions, Gaussian Case



$$e_1 \sim N(0,1),\; e_2 \sim N(\mu_2, \sigma_2^2)$$

Conclusions

We have:

1. Approached forecast accuracy comparison from first principles. (*SED*.)
2. Arrived inescapably at *MAE* loss.
3. Clarified what it means to "select a loss function."
   (Select a $w(e)$ function in *GWSED*.)
4. Compared *SED* to *CVM* and *KS*.
   (Each is minimized by the conditional-median forecast.)
5. Shown that *MSE* forecast rankings do *not* match those of *SED*/*MAE* in general.