

MODELING THE PERSISTENCE OF CONDITIONAL VARIANCES: A COMMENT

Professors Engle and Bollerslev have delivered an excellent blend of "forest" and "trees"; their important extensions of the basic ARCH(q) model nicely complement the comprehensive survey. In these comments I will focus on some general issues which perhaps received insufficient attention, as well as on specific issues related to the authors' new results. In the former category are the statistical versus economic motivations for GARCH, the effects of GARCH on other standard diagnostics (in particular, tests for linear and nonlinear serial correlation), and multivariate GARCH modeling. In the latter category are specification caveats regarding integrated variance models, and the effects of temporal aggregation on the unconditional distribution of GARCH processes.

The statistical motivation for the ARCH(q) model, as well as all of the extensions introduced by Engle and Bollerslev, is best seen by recalling the fundamental Wold decomposition: every covariance stationary stochastic process may be written as the sum of 1) a linearly deterministic component, and 2) a linearly indeterministic component with one-sided MA representation:

$$(1) \quad \sum_{j=0}^{\infty} b_j \epsilon_{t-j}, \text{ where}$$

$$(2) \quad \sum_{j=0}^{\infty} b_j^2 < \infty, \text{ and}$$

$$E(\epsilon_t \epsilon_\tau) = \sigma^2 < \infty \text{ if } t = \tau, 0 \text{ otherwise.}$$

Notice that, while the innovations are uncorrelated, they need not be independent since neither their conditional nor unconditional density need be Gaussian. Furthermore, it is only the unconditional innovation variance $E(\epsilon_t^2)$ which must be constant and finite; this is a restriction of constancy and finiteness on the expectational integral of the conditional innovation variance (i.e., on $E_{\Omega}(E_{\epsilon/\Omega} \epsilon_t^2)$), but it in no way requires constancy of the conditional variance ($E_{\epsilon/\Omega}(\epsilon_t^2)$) itself. In summary, then, (1) allows for a time-varying conditional mean while (2) allows for a time-varying conditional variance. Although all stationary ARMA models satisfy (1) and (2), the converse is not true. Similarly, while all GARCH models satisfy (1) and (2), there are many conditional variance structures not captured by the models considered by Engle and Bollerslev. The search for a parsimonious and descriptively accurate subclass of conditional-mean models has largely ended; we now routinely consider only models with rational spectral densities (e.g. ARMA models). The Engle-Bollerslev paper makes great progress in the search for a similar "consensus" conditional variance specification. I believe that the GARCH(p,q) model, with Student's-t conditional density, is the major contribution of the paper due to the improvements in descriptive accuracy and parsimony which it facilitates relative to the conditionally normal ARCH(q) model. The nonlinear conditional variance models will probably prove less useful, unless as-yet-nonexistent evidence is uncovered indicating that the linear specification is an inadequate approximation for economic phenomena.

The statistical considerations sketched above indicate that the presence of GARCH is possible, but is it probable in economic terms? My guess is yes. First, the results of Clark (1973), as extended by Stock's (1984) important work on time deformation, indicate that inappropriate use of a "calendar" time scale may lead to GARCH-type volatility clustering. Consider, for example, an economic variable evolving at the rate of one step per unit of some

non-calendar time (e.g., "business-cycle time" or "information-arrival time"). Then, relative to calendar time, the process actually evolves more quickly in some periods than in others, with associated movements in prediction error variances. Stock's work also provides a good example of the association of continuous-time phenomena and GARCH effects, as stressed in Professor Sims' comments. The linkage between time deformation and GARCH needs further study, presumably leading to testable restrictions between the two.

Assuming that there exist economic time series not subject to time deformation but nevertheless displaying GARCH, we need a truly economic theory leading to GARCH in an equilibrium model of optimizing agents. Because economic theory usually does not generate testable implications for higher-ordered moments of economic variables, this appears an arduous task, but some progress may not be too far off.

To the extent that GARCH is present in observed economic time series or in model disturbances, it invalidates other important diagnostics, such as tests for serial correlation. Domowitz and Hakkio (1984) develop an LM test for serial correlation which is robust to heteroskedasticity of unknown form, while Diebold (1986a) develops serial correlation tests specifically robust to ARCH. The relative power of the two approaches depends on the accuracy of the ARCH approximation; further work on "robustifying" other model diagnostics is needed as well.

While all covariance-stationary stochastic processes have a Wold representation (1)-(2), it is not necessarily the most efficient form for prediction; a well-known example is the class of bilinear models. GARCH effects in the residuals of linear models may indicate the superiority of a nonlinear conditional mean specification, leading to improved point prediction relative to a linear model with GARCH disturbances. Some initial progress in distinguishing GARCH from bilinearity (and in combining the two

models) has been made by Weiss (1986), but further research is needed.

Multivariate GARCH, which is only briefly discussed in the paper, may prove particularly useful in the modeling of time-varying risk premia and general disturbance-related economic phenomena; the main problem so far has been the huge number of parameters which must be estimated. Even the "diagonal" model in which conditional variances depend only on own lags and squared innovation lags, and conditional covariances depend only on own lags and innovation cross products, is tractable only in low-dimensional cases. Furthermore, the diagonality restriction sometimes appears invalid, as in Engle, Granger and Kraft (1984). A factor-analytic model with GARCH effects may prove to be a particularly attractive alternative. Consider the N-dimensional time series $\{y_t\}$, given by:

$$(3) \quad y_{it} = \lambda_i F_t + \varepsilon_{it}, \quad i = 1, \dots, N, \quad t = 1, \dots, T,$$

where:

$$E F_t = E \varepsilon_{it} = 0, \quad \text{for all } i \text{ and } t$$

$$E F_t F_\tau = 0, \quad \text{all } t \neq \tau, \quad \text{and } E F_t \varepsilon_{i\tau} = 0, \quad \text{all } i, t, \tau$$

$$F_t / \Omega_{t-1} \sim (0, \sigma_t^2)$$

$$\sigma_t^2 = \omega + \sum_{i=1}^q \alpha_i F_{t-1}^2 + \sum_{i=1}^p \beta_i \sigma_{t-1}^2$$

$$E \varepsilon_{it} \varepsilon_{j\tau} = \theta_i \quad \text{if } i = j \text{ and } t = \tau, \quad 0 \text{ otherwise.}$$

In obvious matrix notation, we write the model as:

$$(4) \quad y_t = \lambda F_t + \varepsilon_t.$$

Immediately, then,

$$(5) \quad \Sigma_t \equiv \text{cov}_{t-1}(y_t) = \sigma_t^2 \lambda \lambda' + \theta,$$

where $\theta = \text{cov}(\varepsilon_t)$.

The substantive motivation of such an approach is quite strong. The "common factor," F , represents general influences which tend to affect all variables, albeit with different strengths captured by the λ_i 's. The "unique factors," represented by

the ε_i 's, are uncorrelated variable-specific shocks. The rich (and testable) conditional variance-covariance structure of the observed variables arises from their joint dependence on the common factor F ; this leads to commonality in temporal volatility movements across economic variables, which is frequently observed. For example, Diebold and Nerlove (1986) find strong evidence of one common GARCH-factor in a seven-variable system of dollar exchange rates and provide estimates of the corresponding multivariate model.

In closing, I have a few remarks concerning integrated variance models and temporal aggregation. While the integrated variance model may prove very useful, it should be considered "dangerous" at this preliminary stage. First, the authors correctly note that all of the problems which plague conditional mean unit root tests may carry over to the conditional variance. In particular, it may prove very difficult in specific cases to determine whether a trend or a unit root (or both) is operative, in spite of the fact that the models have very different properties and implications. Second, the problem of initial conditions becomes very important for the estimation of integrated variance models, which are nonstationary. Third, while one motivation for the use of the GARCH approximation is that appropriate variance "forcing" variables are rarely known in the time-series context, this argument is less convincing when the variance appears integrated. Integration implies persistent movements in variance, in which case we should search harder for some economic explanation of the movements. For example, while interest-rate equations may appear to have integrated-variance disturbances, it may be due to a failure to include monetary-regime dummies for the conditional variance intercept, ω . This would correspond to stationary GARCH movements within regimes, with an unconditional "jump" occurring between regimes.

Finally, in work complementary to the authors' results on the effects of temporal aggregation on the conditional density, Diebold

(1986b) has shown that temporal aggregation of GARCH processes leads to unconditional normality, in spite of the fact that successive observations are not independent. Thus, as a distributional model for asset returns, GARCH leads to high frequency unconditional leptokurtosis, volatility clustering, and convergence to normality under temporal aggregation, all of which are observed in the data.

Francis X. Diebold
Federal Reserve Board

ADDITIONAL REFERENCES

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