Cut-Elimination with the Ackermann Function

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For our purposes we'll work in PA together with a fresh ternery predicate symbol A, denoting (the relational version of) the Ackermann function. Specifically, we want to work with the properties:

- $\forall y\, A(0, y, y + 1)$,
- $\forall x, y\, A(x, 1, y) \to A(x + 1, 0, y)$,
- $\forall x, y, z, w\, A(x + 1, y, z) \to A(x, z, w) \to A(x + 1, y + 1, w)$.

For convenience we'll write Λ for the disjunction of the negation of these properties:

$$\exists y\, \neg A(0, y, y + 1), \exists x, y\, A(x, 1, y) \land \neg A(x + 1, 0, y), \exists x, y, z, w\, A(x + 1, y, z) \land A(x, z, w) \land \neg A(x + 1, y + 1, w).$$

We're interested in a proof of the sequent

$$\Lambda, \forall x, y \exists z\, A(x, y, z).$$

$$\dfrac{\dfrac{d(0)\ \dfrac{\dfrac{\dfrac{\neg A(0,y,y+1),\, A(0,y,y+1)}{\Lambda,\, A(0,y,y+1)}}{\Lambda,\, \exists z\, A(0,y,z)}}{\Lambda,\, \forall y\exists z\, A(0,y,z)} \qquad d(x+1)\ \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A(x,1,z),\, \neg A(x,1,z) \qquad \neg A(x+1,0,z),\, A(x+1,0,z)}{A(x,1,z)\land\neg A(x+1,0,z),\, \neg A(x,1,z),\, A(x+1,0,z)}}{\Lambda,\, \neg A(x,1,z),\, A(x+1,0,z)}}{\Lambda,\, \neg A(x,1,z),\, \exists z\, A(x+1,0,z)}}{\Lambda,\, \forall z\, \neg A(x,1,z),\, \exists z\, A(x+1,0,z)}}{\Lambda,\, \exists y\forall z\, \neg A(x,y,z),\, \exists z\, A(x+1,0,z)} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{*}{\Lambda,\, \neg A(x,z,w),\, \neg A(x+1,y,z),\, A(x+1,y+1,w)}}{\Lambda,\, \neg A(x,z,w),\, \neg A(x+1,y,z),\, \exists z\, A(x+1,y+1,z)}}{\Lambda,\, \forall w\, \neg A(x,z,w),\, \neg A(x+1,y,z),\, \exists z\, A(x+1,y+1,z)}}{\Lambda,\, \exists y\forall w\, \neg A(x,y,w),\, \neg A(x+1,y,z),\, \exists z\, A(x+1,y+1,z)}}{\Lambda,\, \exists y\forall w\, \neg A(x,y,w),\, \forall z\, \neg A(x+1,y,z),\, \exists z\, A(x+1,y+1,z)}}{\dfrac{\Lambda,\, \exists y\forall w\, \neg A(x,y,w),\, \exists z\, A(x+1,y,z)}{\Lambda,\, \exists y\forall w\, \neg A(x,y,w),\, \forall y\exists z\, A(x+1,y,z)}}}{\Lambda,\, \forall y\exists z\, A(x,y,z)}}{\Lambda,\, \forall x, y\exists z\, A(x,y,z)}$$

$*$ is a short deduction from axioms similar to the other two, which I'm omitting for space.

For convenience, there are some named deductions here. In particular, $d(x+1)$ is a deduction of

$$\Lambda, \exists y \forall w\, \neg A(x, y, w), \forall y \exists z\, A(x+1, y, z),$$

with x understood as a parameter other than 0. By $d(3)$, for instance, we mean the same deduction with every instance of x replaced by 3. (By $d(0)$ we mean the deduction of $\Lambda, \forall y \exists z\, A(0, y, z)$.)

Now suppose we plug in some not-completely-trivial values of x and y, say $x = 3$, $y = 2$, and apply cut-elimination. We start by unraveling the induction; depending on our exact algorithm, we get something like the following.

We end up with a deduction of $\Lambda, \forall y, \exists z\, A(2, y, z)$:

Unraveling the induction, we get a deduction like the following (depending on our exact algorithm):

$$\text{Cut}\ \dfrac{\text{Cut}\ \dfrac{d(0)\ \dfrac{\vdots}{\Lambda, \forall y \exists z\, A(0, y, z)} \qquad d(1)\ \dfrac{\vdots}{\Lambda, \exists y \forall w\, \neg A(0, y, w), \forall y \exists z\, A(1, y, z)}}{\Lambda, \forall y \exists z\, A(1, y, z)} \qquad d(2)\ \dfrac{\vdots}{\Lambda, \exists y \forall w\, \neg A(1, y, w), \forall y \exists z\, A(2, y, z)}}{\Lambda, \forall y \exists z\, A(2, y, z)}$$

We'll call this $p(2)$. More interestingly, we get the following deduction:

$$\text{Cut}\ \dfrac{\text{Cut}\ \dfrac{\dagger\ \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A(2,1,z), \neg A(2,1,z) \qquad \neg A(3,0,z), A(3,0,z)}{A(2,1,z) \wedge \neg A(3,0,z), \neg A(2,1,z), A(3,0,z)}}{\Lambda, \neg A(2,1,z), A(3,0,z)}}{\Lambda, \neg A(2,1,z), \exists z\, A(3,0,z)}}{\Lambda, \forall z \neg A(2,1,z), \exists z\, A(3,0,z)}}{\Lambda, \exists y \forall z\, \neg A(2,y,z), \exists z\, A(3,0,z)} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{*}{\Lambda, \neg A(2,z,w), \neg A(3,0,z), A(3,1,w)}}{\Lambda, \neg A(2,z,w), \neg A(3,0,z), \exists z\, A(3,1,z)}}{\Lambda, \forall w \neg A(2,z,w), \neg A(3,0,z), \exists z\, A(3,1,z)}}{\Lambda, \exists y \forall w\, \neg A(2,y,w), \neg A(3,0,z), \exists z\, A(3,1,z)}}{\Lambda, \exists y \forall w\, \neg A(2,y,w), \forall z\, \neg A(3,0,z), \exists z\, A(3,1,z)}}{\Lambda, \exists y \forall w\, \neg A(2,y,w), \exists z\, A(3,1,z)} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{*}{\Lambda, \neg A(2,z,w), \neg A(3,1,z), A(3,2,w)}}{\Lambda, \neg A(2,z,w), \neg A(3,1,z), \exists z\, A(3,2,z)}}{\Lambda, \forall w \neg A(2,z,w), \neg A(3,1,z), \exists z\, A(3,2,z)}}{\Lambda, \exists y \forall w\, \neg A(2,y,w), \neg A(3,1,z), \exists z\, A(3,2,z)}}{\Lambda, \exists y \forall w\, \neg A(2,y,w), \forall z\, \neg A(3,1,z), \exists z\, A(3,2,z)}}{\Lambda, \exists y \forall w\, \neg A(2,y,w), \exists z\, A(3,2,z)}$$

We'll call this a. Our unraveled deduction is the result of cutting these together:

$$\text{Cut}\ \dfrac{p(2)\ \dfrac{\vdots}{\Lambda, \forall y \exists z\, A(2, y, z)} \qquad a\ \dfrac{\vdots}{\Lambda, \exists y \forall w\, \neg A(2, y, w), \exists z\, A(3, 2, z)}}{\Lambda, \exists z\, A(3, 2, z)}$$

The cut-elimination procedure says we should follow the existential side of the cuts up to an actual $\exists$ introduction rule; in this case it's the inference rule in a marked with †. We permute the introduction rule down to match its cut, which in this case is all the way to the bottom.

$$
\dfrac{
 p(2)\;\dfrac{\vdots}{\Lambda, \forall y \exists z\, A(2,y,z)}
 \qquad
 \dfrac{
  \text{Cut}\;\dfrac{
   \text{Cut}\;\dfrac{
    \dfrac{\dfrac{\dfrac{\dfrac{A(2,1,z), \neg A(2,1,z) \qquad \neg A(3,0,z), A(3,0,z)}{A(2,1,z) \wedge \neg A(3,0,z), \neg A(2,1,z), A(3,0,z)}}{\Lambda, \neg A(2,1,z), A(3,0,z)}}{\Lambda, \neg A(2,1,z), \exists z\, A(3,0,z)}}{\Lambda, \forall z \neg A(2,1,z), \exists z\, A(3,0,z)}
    \qquad
    \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{*}{\Lambda, \neg A(2,z,w), \neg A(3,0,z), A(3,1,w)}}{\Lambda, \neg A(2,z,w), \neg A(3,0,z), \exists z\, A(3,1,z)}}{\Lambda, \forall w \neg A(2,z,w), \neg A(3,0,z), \exists z\, A(3,1,z)}}{\Lambda, \exists y \forall w\, \neg A(2,y,w), \neg A(3,0,z), \exists z\, A(3,1,z)}}{\Lambda, \exists y \forall w\, \neg A(2,y,w), \forall z\, \neg A(3,0,z), \exists z\, A(3,1,z)}
   }{\Lambda, \forall z \neg A(2,1,z), \exists y \forall w\, \neg A(2,y,w), \exists z\, A(3,1,z)}
   \qquad
   \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{*}{\Lambda, \neg A(2,z,w), \neg A(3,1,z), A(3,2,w)}}{\Lambda, \neg A(2,z,w), \neg A(3,1,z), \exists z\, A(3,2,z)}}{\Lambda, \forall w \neg A(2,z,w), \neg A(3,1,z), \exists z\, A(3,2,z)}}{\Lambda, \exists y \forall w\, \neg A(2,y,w), \neg A(3,1,z), \exists z\, A(3,2,z)}}{\Lambda, \exists y \forall w\, \neg A(2,y,w), \forall z\, \neg A(3,1,z), \exists z\, A(3,2,z)}
  }{\Lambda, \forall z \neg A(2,1,z), \exists y \forall w\, \neg A(2,y,w), \exists z\, A(3,2,z)}
 }{\Lambda, \exists y \forall w\, \neg A(2,y,w), \exists z\, A(3,2,z)}
}{\Lambda, \exists z\, A(3,2,z)}
$$

Notice that this will tend to make ordinals smaller—the $\exists$ introduction rule adds one to the ordinal, and when that happens higher in the proof, the $+1$ gets included in more exponents. (We don't really care about a decrease, what's important is that the ordinal doesn't increase when we do this.)

Note that, by substituting 1 for y in $p(2)$ we obtain a proof $p(2,1)$ of $\Lambda, \exists z\, A(2,1,z)$, and when we eliminate the cut, we get:

$$
\dfrac{
 p(2,1)\;\dfrac{\vdots}{\Lambda, \exists z\, A(2,1,z)}
 \qquad
 \dfrac{
  p(2)\;\dfrac{\vdots}{\Lambda, \forall y \exists z\, A(2,y,z)}
  \qquad
  \text{Cut}\;\dfrac{
   \text{Cut}\;\dfrac{
    \dfrac{\dfrac{\dfrac{\dfrac{A(2,1,z), \neg A(2,1,z) \qquad \neg A(3,0,z), A(3,0,z)}{A(2,1,z) \wedge \neg A(3,0,z), \neg A(2,1,z), A(3,0,z)}}{\Lambda, \neg A(2,1,z), A(3,0,z)}}{\Lambda, \neg A(2,1,z), \exists z\, A(3,0,z)}}{\Lambda, \forall z \neg A(2,1,z), \exists z\, A(3,0,z)}
    \qquad
    \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{*}{\Lambda, \neg A(2,z,w), \neg A(3,0,z), A(3,1,w)}}{\Lambda, \neg A(2,z,w), \neg A(3,0,z), \exists z\, A(3,1,z)}}{\Lambda, \forall w \neg A(2,z,w), \neg A(3,0,z), \exists z\, A(3,1,z)}}{\Lambda, \exists y \forall w\, \neg A(2,y,w), \neg A(3,0,z), \exists z\, A(3,1,z)}}{\Lambda, \exists y \forall w\, \neg A(2,y,w), \forall z\, \neg A(3,0,z), \exists z\, A(3,1,z)}
   }{\Lambda, \forall z \neg A(2,1,z), \exists y \forall w\, \neg A(2,y,w), \exists z\, A(3,1,z)}
   \qquad
   \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{*}{\Lambda, \neg A(2,z,w), \neg A(3,1,z), A(3,2,w)}}{\Lambda, \neg A(2,z,w), \neg A(3,1,z), \exists z\, A(3,2,z)}}{\Lambda, \forall w \neg A(2,z,w), \neg A(3,1,z), \exists z\, A(3,2,z)}}{\Lambda, \exists y \forall w\, \neg A(2,y,w), \neg A(3,1,z), \exists z\, A(3,2,z)}}{\Lambda, \exists y \forall w\, \neg A(2,y,w), \forall z\, \neg A(3,1,z), \exists z\, A(3,2,z)}
  }{\Lambda, \forall z \neg A(2,1,z), \exists y \forall w\, \neg A(2,y,w), \exists z\, A(3,2,z)}
 }{\Lambda, \forall z \neg A(2,1,z), \exists z\, A(3,2,z)}
}{\Lambda, \exists z\, A(3,2,z)}
$$

It's not immediately obvious that this is a simpler proof—we replaced the old cut with a new instance of the *same* cut, and an *extra* cut. Recall that the deduction of $\Lambda, \forall z \neg A(2,1,z), \exists z\, A(3,2,z)$ has smaller ordinal than the old deduction of $\Lambda, \exists z\, A(3,2,z)$ because we shortened the proof when we removed the existential introduction rule, and because the new cut at the bottom is lower rank, this removal more than makes up for the addition of an extra cut.

When we apply the next stage of cut-elimination, we'll follow the existential branch into $p(2,1)$; since the conclusion of $p(2,1)$ is itself Σ_1, we'll save ourselves some effort and carry out that elimination off screen. Ultimately, it ends up substituting the value 5 for z in our deduction:

$$
\dfrac{
 p(2)\;\dfrac{\vdots}{\Lambda, \forall y \exists z\, A(2,y,z)}
 \qquad
 \text{Cut}\;\dfrac{
  \text{Cut}\;\dfrac{
   \dfrac{\dfrac{\dfrac{\dfrac{\vdots}{\Lambda, A(2,1,5)} \qquad \neg A(3,0,5), A(3,0,5)}{\Lambda, A(2,1,5) \wedge \neg A(3,0,5), A(3,0,5)}}{\Lambda, A(3,0,5)}}{\Lambda, \exists z\, A(3,0,z)}
   \qquad
   \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{*}{\Lambda, \neg A(2,z,w), \neg A(3,0,z), A(3,1,w)}}{\Lambda, \neg A(2,z,w), \neg A(3,0,z), \exists z\, A(3,1,z)}}{\Lambda, \forall w \neg A(2,z,w), \neg A(3,0,z), \exists z\, A(3,1,z)}}{\Lambda, \exists y \forall w\, \neg A(2,y,w), \neg A(3,0,z), \exists z\, A(3,1,z)}}{\Lambda, \exists y \forall w\, \neg A(2,y,w), \forall z\, \neg A(3,0,z), \exists z\, A(3,1,z)}
  }{\Lambda, \exists y \forall w\, \neg A(2,y,w), \exists z\, A(3,1,z)}
  \qquad
  \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{*}{\Lambda, \neg A(2,z,w), \neg A(3,1,z), A(3,2,w)}}{\Lambda, \neg A(2,z,w), \neg A(3,1,z), \exists z\, A(3,2,z)}}{\Lambda, \forall w \neg A(2,z,w), \neg A(3,1,z), \exists z\, A(3,2,z)}}{\Lambda, \exists y \forall w\, \neg A(2,y,w), \neg A(3,1,z), \exists z\, A(3,2,z)}}{\Lambda, \exists y \forall w\, \neg A(2,y,w), \forall z\, \neg A(3,1,z), \exists z\, A(3,2,z)}
 }{\Lambda, \exists y \forall w\, \neg A(2,y,w), \exists z\, A(3,2,z)}
}{\Lambda, \exists z\, A(3,2,z)}
$$

The next cut to consider is over $\exists z\, A(3,0,5)$, which is again a Σ_1 formula, and therefore not so exciting. (We'll expand $*$ to see what's happening, though.) We get:

$$
\dfrac{p(2)\;\dfrac{\vdots}{\Lambda, \forall y \exists z\, A(2,y,z)} \qquad \text{Cut}\;\dfrac{\dagger\;\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\neg A(2,5,w), A(2,5,w) \qquad \dfrac{\vdots}{\Lambda, A(3,0,5)}}{\Lambda, A(3,0,5) \land A(2,5,w), \neg A(2,5,w)} \qquad \neg A(3,1,w), A(3,1,w)}{\Lambda, A(3,0,5) \land A(2,5,w) \land \neg A(3,1,w), \neg A(2,5,w), A(3,1,w)}}{\Lambda, \neg A(2,5,w), A(3,1,w)}}{\Lambda, \neg A(2,5,w), \exists z\, A(3,1,z)}}{\Lambda, \forall w \neg A(2,5,w), \exists z\, A(3,1,z)}}{\Lambda, \exists y \forall w\, \neg A(2,y,w), \exists z\, A(3,1,z)} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{*}{\Lambda, \neg A(2,z,w), \neg A(3,1,z), A(3,2,w)}}{\Lambda, \neg A(2,z,w), \neg A(3,1,z), \exists z\, A(3,2,z)}}{\Lambda, \forall w \neg A(2,z,w), \neg A(3,1,z), \exists z\, A(3,2,z)}}{\Lambda, \exists y \forall w\, \neg A(2,y,w), \neg A(3,1,z), \exists z\, A(3,2,z)}}{\Lambda, \exists y \forall w\, \neg A(2,y,w), \forall z\, \neg A(3,1,z), \exists z\, A(3,2,z)}}{\Lambda, \exists y \forall w\, \neg A(2,y,w), \exists z\, A(3,2,z)}}{\Lambda, \exists z\, A(3,2,z)}
$$

The next cut to be eliminated is the one corresponding the existential quantifier at †, which is being cut with the copy of $p(2)$. Something exciting has finally happened, so we should take note: the value for y we're about to substitute into $p(2)$ is the value 5, which is itself the value of the Ackermann function $A(2,1) = 5$. If we think of $p(2)$ as a function $y \mapsto A(2,y)$, we're now iterating this function.

Once we start iterating functions—or equivalently, eliminating the same cut we already eliminated ones—we should be worried about non-termination. But remember what happened when we eliminated the cut over $\forall y \exists z\, A(2,y,z)$ the first time: we created a new copy of the same cut, but *with a lower ordinal*. For thoroughness, let's do this one more time; we permute the $\exists$ introduction down:

$$
\dfrac{p(2)\;\dfrac{\vdots}{\Lambda, \forall y \exists z\, A(2,y,z)} \qquad \dfrac{\text{Cut}\;\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\neg A(2,5,w), A(2,5,w) \qquad \dfrac{\vdots}{\Lambda, A(3,0,5)}}{\Lambda, A(3,0,5) \land A(2,5,w), \neg A(2,5,w)} \qquad \neg A(3,1,w), A(3,1,w)}{\Lambda, A(3,0,5) \land A(2,5,w) \land \neg A(3,1,w), \neg A(2,5,w), A(3,1,w)}}{\Lambda, \neg A(2,5,w), A(3,1,w)}}{\Lambda, \neg A(2,5,w), \exists z\, A(3,1,z)}}{\Lambda, \forall w \neg A(2,5,w), \exists z\, A(3,1,z)} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{*}{\Lambda, \neg A(2,z,w), \neg A(3,1,z), A(3,2,w)}}{\Lambda, \neg A(2,z,w), \neg A(3,1,z), \exists z\, A(3,2,z)}}{\Lambda, \forall w \neg A(2,z,w), \neg A(3,1,z), \exists z\, A(3,2,z)}}{\Lambda, \exists y \forall w\, \neg A(2,y,w), \neg A(3,1,z), \exists z\, A(3,2,z)}}{\Lambda, \exists y \forall w\, \neg A(2,y,w), \forall z\, \neg A(3,1,z), \exists z\, A(3,2,z)}}{\Lambda, \forall w \neg A(2,5,w), \exists y \forall w\, \neg A(2,y,w), \exists z\, A(3,2,z)}}{\Lambda, \exists y \forall w\, \neg A(2,y,w), \exists z\, A(3,2,z)}}{\Lambda, \exists z\, A(3,2,z)}
$$

and then eliminate the cut:

$$\dfrac{\neg A(2,5,w),\ A(2,5,w) \qquad \dfrac{\vdots}{\Lambda,\ A(3,0,5)}}{\Lambda,\ A(3,0,5) \wedge A(2,5,w),\ \neg A(2,5,w)}$$

$$\dfrac{\Lambda,\ A(3,0,5) \wedge A(2,5,w),\ \neg A(2,5,w) \qquad \neg A(3,1,w),\ A(3,1,w)}{\dfrac{\Lambda,\ A(3,0,5) \wedge A(2,5,w) \wedge \neg A(3,1,w),\ \neg A(2,5,w),\ A(3,1,w)}{\dfrac{\Lambda,\ \neg A(2,5,w),\ A(3,1,w)}{\dfrac{\Lambda,\ \neg A(2,5,w),\ \exists z\, A(3,1,z)}{\Lambda,\ \forall w \neg A(2,5,w),\ \exists z\, A(3,1,z)}}}}$$

$$\dfrac{*}{\dfrac{\Lambda,\ \neg A(2,z,w),\ \neg A(3,1,z),\ A(3,2,w)}{\dfrac{\Lambda,\ \neg A(2,z,w),\ \neg A(3,1,z),\ \exists z\, A(3,2,z)}{\dfrac{\Lambda,\ \forall w \neg A(2,z,w),\ \neg A(3,1,z),\ \exists z\, A(3,2,z)}{\dfrac{\Lambda,\ \exists y \forall w\, \neg A(2,y,w),\ \neg A(3,1,z),\ \exists z\, A(3,2,z)}{\Lambda,\ \exists y \forall w\, \neg A(2,y,w),\ \forall z\, \neg A(3,1,z),\ \exists z\, A(3,2,z)}}}}}$$

$$\text{Cut}\ \dfrac{\Lambda,\ \forall w \neg A(2,5,w),\ \exists z\, A(3,1,z) \qquad \Lambda,\ \exists y \forall w\, \neg A(2,y,w),\ \forall z\, \neg A(3,1,z),\ \exists z\, A(3,2,z)}{\Lambda,\ \forall w \neg A(2,5,w),\ \exists y \forall w\, \neg A(2,y,w),\ \exists z\, A(3,2,z)}$$

$$\dfrac{p(2)\ \dfrac{\vdots}{\Lambda,\ \forall y \exists z\, A(2,y,z)} \qquad \Lambda,\ \forall w \neg A(2,5,w),\ \exists y \forall w\, \neg A(2,y,w),\ \exists z\, A(3,2,z)}{\Lambda,\ \forall w \neg A(2,5,w),\ \exists z\, A(3,2,z)}$$

$$\dfrac{p(2,5)\ \dfrac{\vdots}{\Lambda,\ \exists z\, A(2,5,z)} \qquad \Lambda,\ \forall w \neg A(2,5,w),\ \exists z\, A(3,2,z)}{\Lambda,\ \exists z\, A(3,2,z)}$$

Again, we've made a new copy of $p(2)$, meaning we'll get to plug in a third value for y, but we've pushed the cut further up in the proof (by eliminating the $\exists$ introduction down), meaning it has a smaller ordinal, and therefore we've made progress.

Going back to the beginning of our construction, let's consider what happened. We think of $p(2)$, the proof of $\Lambda, \forall y \exists z\, A(2,y,z)$, as corresponding to having the function $y \mapsto A(2,y)$. We cut it with a proof of $\Lambda, \exists y \forall w\, \neg A(2,y,w), \exists z\, A(3,n,z)$ for some n. A single lower-rank induction axiom over n means that we introduced $\exists y \forall w\, \neg A(2,y,w)$ n separate times (phrased differently, the proof of $\Lambda, \exists y \forall w\, \neg A(2,y,w), \forall y \exists z\, A(3,y,z)$ morally contains ω introductions of $\exists y \forall w\, \neg A(2,y,w)$; note that we do mean the ordinal ω, not the cardinal $\aleph_0$—the introductions are still placed in a well-founded ordering). We then have to eliminate the cut over $\forall y \exists z\, A(2,y,z)$ n times, which corresponds to iteration: $A(2, A(2, \cdots))$, n times.